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ANALYTIC CRITERIA FOR THE MECHANICAL AND THERMAL STABILITY OF MAGNETIC STARS WITH FINITE ELECTRICAL CONDUCTIVITY

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ABSTRACT

The usual assumption of infinite electrical conductivity has been relaxed in a new analytic treatment of the magnetohydrodynamics of a variable magnetic star in the simple one-zone approximation. For a not too low electrical conductivity, the magnetic pressure changes with the mass density as roughly $\rho^{4/3}$, and the Joule heating rate goes as roughly ρ. The magnetic effective adiabatic exponent of about 4/3 proves to have only a small influence on the criteria for dynamical, secular, and pulsational stability, but the Joule heating rate directly affects the secular and pulsational stability criteria. Thus, a finite electrical conductivity tends to stabilize a magnetic star secularly and to destabilize it pulsationally. These specific results apply, however, only to purely radial perturbations of the star's upper radiative layers.

Subject headings: instabilities — MHD — stars: magnetic fields — stars: oscillations — stars: variables: other

1. INTRODUCTION

Magnetic fields are ubiquitous in nature. Detected in varying strengths at the surfaces of many classes of ordinary stars, they doubtless pervade their interiors as well and, if strong enough, could affect the structure, evolution, and stability of these stars. The Sun is the best-studied example of how magnetic fields interact with the bulk motions of a star, but the solar problem, which deals with fast nonradial oscillations and slow dynamo-related radial adjustments, is complicated by the presence of vigorous turbulent convection in the envelope. This physical complication also affects the interpretation of the RR Lyrae stars and other cool variable stars, most of which are classical radial pulsators. It may be that some of the radially and nonradially pulsating β Cephei and δ Scuti stars in common with other early-type stars (Babcock 1958; Kochukhov & Bagnulo 2006) possess strong magnetic fields and that the bipolar shapes of the nebulae surrounding some of the possibly dynamically and secularly unstable luminous blue variables (LBVs or S Doradus variables) may also have a magnetic explanation (Stothers 2004). In these classes of relatively hot variable stars, the destabilized envelope is mostly radiative and convection probably plays only a small role there. All these stars contain radiative interiors below the unstable upper layers, and the more massive ones among them also have convective central cores.

In any case, it is worth studying theoretically how magnetic fields affect the radial stability of a star's near-surface radiative layers, where the observed variability is mainly produced. Such magnetic fields probably consist of both poloidal and toroidal components in order to survive for an observably long time (Prendergast 1956; Tayler 1980; Braithwaite & Spruit 2004; Braithwaite & Nordlund 2006), and the most recent numerical simulations suggest that the field lines may be fairly well tangled at depth, although observations of the surfaces of many such stars reveal large-scale, ordered, dipole-like fields. The full mathematical problem of the mechanical and thermal stability of the radiative layers (apart from the MHD stability of the twisted magnetic field lines themselves) is too difficult to solve analytically or numerically at present, and so recourse must be had to the use of approximate stellar models and methods.

One fruitful, simple approach has been to adopt a one-zone model of a star, whose nonmagnetic properties were first studied by Jeans (1927, 1929) and Baker (1966). The magnetic case with infinite electrical conductivity has already been solved exactly, and the criteria for dynamical, secular, and pulsational radial stability turn out to be the same as in the absence of a magnetic field (Stothers 1981). Although the electrical conductivity is undoubtedly very high at great depth in the stellar interior, its value near the surface is much more modest (Cowling 1953), and hence its finiteness has to be taken into account, for example, in numerical simulations of the magnetized convective envelope of the Sun where the ohmic decay time is very short (e.g., Tobias et al. 2001; Stein & Nordlund 2006). This is the case whenever the conductivity, which depends on the temperature as $T^{3/2}$, is $\sim(vl)^{-1}$ or smaller, where v is the plasma velocity and l is the length scale of the spatial variation of the magnetic field. Recently, Saio (2005) has studied the radial and nonradial pulsational stability of a magnetized radiative envelope, intended to represent the rapidly oscillating Ap stars, but he assumed for simplicity a purely dipole magnetic field and an infinite electrical conductivity.

We instead inquire here what qualitative effects a finite electrical conductivity has on the mechanical and thermal radial stability of the upper envelope of an idealized radiative magnetic star. It turns out that a simple approximation for the complicated term including the electrical conductivity in the induction equation leads to an analytically tractable set of equations that may also be useful in future work. The results at present shed light on the conditions for stability in the types of magnetic stars enumerated above and show how the finiteness of the conductivity influences these conditions This work should also prove useful in interpreting the results of the detailed numerical simulations that may be performed in the future for these stars.

2. BASIC EQUATIONS

The basic equations for the present problem include equations representing the conservation of mass, momentum, and energy, together with a suitable form of Faraday's equation of electromagnetic induction and Gauss's equations of magnetic-field divergence and gravitational-field divergence (e.g., Cowling 1953):

$$\frac{d\rho}{dt} + \rho\nabla\cdot\boldsymbol{v} = 0, \tag{1}$$

$$\rho\frac{d\boldsymbol{v}}{dt} = -\nabla\boldsymbol{P} + \rho\boldsymbol{g} + \frac{1}{4\pi}(\nabla\times\boldsymbol{H})\times\boldsymbol{H}, \tag{2}$$

$$\frac{dU}{dt} + \frac{Pd}{dt}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho}\nabla\cdot\boldsymbol{F}_{\rm rad} + \frac{1}{(4\pi)^2\sigma_E\rho}(\nabla\times\boldsymbol{H})\cdot(\nabla\times\boldsymbol{H}), \tag{3}$$

$$\frac{\partial\boldsymbol{H}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{H}) + \frac{1}{4\pi\sigma_E}\nabla^2\boldsymbol{H}, \tag{4}$$

$$\frac{d\boldsymbol{H}}{dt} = \frac{\partial\boldsymbol{H}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{H}, \tag{5}$$

$$\nabla\cdot\boldsymbol{H} = 0,$$
$$\nabla\cdot\boldsymbol{g} = -4\pi G\rho. \tag{6}$$

In the above equations, $\boldsymbol{v}$ is velocity, ρ is density, P is pressure, U is specific internal energy, $\boldsymbol{g}$ is gravitational acceleration, $\boldsymbol{F}_{\rm rad}$ is radiative flux, $\boldsymbol{H}$ is magnetic intensity, and σ_E is electrical conductivity, assumed here to be a constant. Nuclear energy generation (Jeans 1927; Ledoux 1963) as well as rotation, slowly adapting turbulence, and mass loss (Stothers 2006) have been treated elsewhere. The left-hand side of equation (3) contains the standard expression for the heat derivative, assumed to be unmodified by the magnetic field (as justified in the Appendix).

To proceed further, spherical symmetry is assumed, the equations are reduced to mean radial form, and only radial motions are considered. Trasco (1970) has defined a mean radial form of the Lorentz force:

$$f(r) = \frac{1}{4\pi}\int\int\left[\frac{1}{4\pi}(\nabla\times\boldsymbol{H})\times\boldsymbol{H}\right]\cdot\hat{\boldsymbol{r}}\sin\theta d\theta d\phi, \tag{7}$$

where $\hat{\boldsymbol{r}}$ is a unit vector in the r-direction. It is thus possible to express $f(r)$ in terms of a mean squared magnetic intensity,

$$\langle H^2\rangle = \frac{1}{2}\int\boldsymbol{H}\cdot\boldsymbol{H}\sin\theta d\theta, \tag{8}$$

for the axially symmetric case. To simplify equation (7), Trasco assumed that the mean radial component satisfies $\langle H_r^2\rangle = \frac{1}{3}\langle H^2\rangle$, which would certainly hold for a small-scale random magnetic field but is not necessarily restricted to this case. Then equation (7) reduces to

$$f(r) = -\frac{d}{dr}\frac{\langle H^2\rangle}{8\pi n} \tag{9}$$

with $n = 3$. For greater generality we allow n to be a free parameter that permits equation (9) to represent $f(r)$ for any type of magnetic field (e.g., Chandrasekhar 1961, p. 148; Parker 1979, p. 59).

Consequently, equation (2) can be written

$$\frac{\rho d^2r}{dt^2} = -\frac{dP}{dr} - GM(r)\frac{\rho}{r^2} - \frac{d}{dr}\frac{\langle H^2\rangle}{8\pi n}, \tag{10}$$

where the quantity $\langle H^2\rangle/8\pi n$ is equivalent to an isotropic magnetic pressure and where $M(r)$, the mass contained within radius r, is given by

$$\frac{dM(r)}{dr} = 4\pi r^2\rho. \tag{11}$$

Likewise reduced to a mean radial form, equation (3) becomes

$$\frac{dU}{dt} + \frac{Pd}{dt}\left(\frac{1}{\rho}\right) = -\frac{dL(r)}{dM(r)} + \frac{1}{(4\pi)^2\sigma_E\rho}|\nabla\times\boldsymbol{H}|^2, \tag{12}$$

where the last term represents Joule heating, and the radiative flux is given by

$$\frac{L(r)}{4\pi r^2} = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}, \tag{13}$$

where T is temperature, κ is opacity, a is the radiation density constant, and c is the speed of light. The rest of the vector equations will be retained as they are until after the variables have been perturbed by introducing small radial vector displacements of position.

The equilibrium state of the star is perturbed by setting $\boldsymbol{r} = \boldsymbol{r}_0 + \delta\boldsymbol{r}\exp(st)$ (and similarly for the other variables) and then by linearizing the set of differential equations. Equation (1) immediately becomes

$$\frac{\delta\rho}{\rho_0} = -\nabla\cdot\delta\boldsymbol{r}. \tag{14}$$

Equations (4)–(6) for the magnetic field, however, require some study and simplification. In writing equation (4), it has been necessary to assume that σ_E is constant. Under realistic stellar conditions, σ_E is also so large that the second term on the right-hand side of equation (4) will be much smaller than the first term. We choose to write the second term as

$$\frac{1}{4\pi\sigma_E}\nabla^2\boldsymbol{H} = -\frac{\partial\boldsymbol{R}}{\partial t}, \tag{15}$$

as if it were a small correction to $\partial\boldsymbol{H}/\partial t$. Then equations (4)–(6) yield

$$\delta\boldsymbol{H} = (\boldsymbol{H}_0\cdot\nabla)\delta\boldsymbol{r} - \boldsymbol{H}_0(\nabla\cdot\delta\boldsymbol{r}) - \delta\boldsymbol{R}. \tag{16}$$

For convenience, we can arbitrarily put

$$\delta\boldsymbol{R} = \frac{\epsilon}{2}\boldsymbol{H}_0(-\nabla\cdot\delta\boldsymbol{r}) \tag{17}$$

with ϵ being a small positive or negative quantity. Then

$$\delta\langle\boldsymbol{H}\cdot\boldsymbol{H}\rangle = \int\boldsymbol{H}_0\cdot\delta\boldsymbol{H}\sin\theta d\theta = \left(\frac{4}{3}+\epsilon\right)\langle H_0^2\rangle(-\nabla\cdot\delta\boldsymbol{r}). \tag{18}$$

Using equation (14), we finally get

$$\frac{\delta\langle H^2\rangle}{\langle H_0^2\rangle} = \left(\frac{4}{3}+\epsilon\right)\frac{\delta\rho}{\rho_0}. \tag{19}$$

Defining a magnetic effective adiabatic exponent $\gamma_M = (4/3) + \epsilon$, we arrive at a generalization of the simple result for the case of infinite conductivity ($\epsilon = 0$) considered previously (Stothers 1979).

It is now possible to calculate the perturbation of the magnetic field in the Joule heating term:

$$\begin{aligned}\delta|\nabla \times \boldsymbol{H}|^2 &= 2(\nabla \times \boldsymbol{H}_0) \cdot (\nabla \times \delta\boldsymbol{H}) \\ &= (\nabla \times \boldsymbol{H}_0) \cdot [(2-\epsilon)(\nabla \times \boldsymbol{H}_0) \\ &\quad \times (-\nabla \cdot \delta\boldsymbol{r}) + 2\nabla \times (\boldsymbol{H}_0 \cdot \nabla)\delta\boldsymbol{r}]. \qquad (20)\end{aligned}$$

To assess the relative magnitudes of the two terms in the square brackets, consider a homologous displacement, $\delta\boldsymbol{r} = \eta\boldsymbol{r}$, with η being a constant. In view of the fact that in this representation $\nabla \cdot \delta\boldsymbol{r} = 3\eta$ and $\nabla \times \delta\boldsymbol{r} = 0$, we find that the second term in the brackets must be much smaller than the first term and thus can be neglected. Hence,

$$\frac{\delta|\nabla \times \boldsymbol{H}|^2}{|\nabla \times \boldsymbol{H}_0|^2} = (2-\epsilon)\frac{\delta\rho}{\rho_0}. \qquad (21)$$

3. STABILITY ANALYSIS

For the case of spherical symmetry and radial motions only, the basic equations reduce to equations (10)–(13). This set of equations will now be linearized by employing as auxiliary relations the standard thermodynamic functions for gas and radiation (see the Appendix), as well as the magnetic perturbations approximated by equations (19) and (21) with ϵ set equal to a constant.

The procedure followed here hews closely to that presented by Baker (1966) for the case of no magnetic field. We consider a one-zone model of a star in which the spatial derivatives of all the perturbed quantities, except the luminosity, are set equal to zero. This is equivalent to assuming homologous radial displacements. Baker's approximation consists of writing for the luminosity

$$\frac{d}{dM(r)}\frac{\delta L}{L_0} = \frac{2}{\Delta M}\frac{\delta L}{L_0}, \qquad (22)$$

where δL is the mean value of the luminosity perturbation within the single zone of mass ΔM.

After considerable reduction, the final result for s, the complex temporal frequency, is the solution of a cubic equation:

$$s^3 + (K_T + K_M)\sigma_0 A s^2 + \sigma_0^2 B s + (K_T + K_M)\sigma_0^3 D = 0, \qquad (23)$$

where $\sigma_0^2 = GM(r)/r_0^3$ and

$$A = -(\alpha\Gamma_1 - 1)[\delta^{-1}\alpha(\kappa_T - 4) + \kappa_P], \qquad (24)$$

$$B = (1-\mu)(3\Gamma_1 - 4) + \mu(3\gamma_M - 4), \qquad (25)$$

$$\begin{aligned}D = (\alpha\Gamma_1 - 1)(1-\mu)\{&\delta^{-1}[4\alpha - 3 - \alpha\mu(1-\mu)^{-1} \\ &\times(3\gamma_M - 4)](\kappa_T - 4) + [4 - \mu(1-\mu)^{-1} \\ &\times(3\gamma_M - 4)]\kappa_P + 4 + 3N\}, \qquad (26)\end{aligned}$$

$$K_T = \frac{2\rho_0 L_0}{P_0\delta\sigma_0\Delta M},$$
$$K_M = \frac{|\nabla \times \boldsymbol{H}_0|^2}{(4\pi)^2\sigma_E P_0\delta\sigma_0}, \qquad (27)$$

$$N = \frac{(1-\epsilon)K_M}{K_T + K_M},$$
$$\mu = -\frac{4\pi r_0^4}{GM(r)}\frac{d}{dM(r)}\frac{\langle H^2\rangle}{8\pi n}, \qquad (28)$$

$$\alpha = \left(\frac{\partial \ln \rho}{\partial \ln P}\right)_T,$$
$$\delta = -\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P, \qquad (29)$$

$$\kappa_P = \left(\frac{\partial \ln \kappa}{\partial \ln P}\right)_T,$$
$$\kappa_T = \left(\frac{\partial \ln \kappa}{\partial \ln T}\right)_P. \qquad (30)$$

Note that Γ_1 is the standard first generalized adiabatic exponent for the gas and radiation, defined as $\Gamma_1 = (d \ln P/d \ln \rho)_S$ at constant entropy S. Combining terms,

$$\begin{aligned}AB - D = &-3\Gamma_1(\alpha\Gamma_1 - 1)(1-\mu) \\ &\times\left[(\delta\Gamma_1)^{-1}(\alpha\Gamma_1 - 1)(\kappa_T - 4)\right. \\ &\left. + \kappa_P + \frac{4}{3\Gamma_1} + \frac{N}{\Gamma_1}\right]. \qquad (31)\end{aligned}$$

Stability depends on the nature of the three roots of equation (23). If $\delta > 0$, the stability criteria are

$$\text{dynamical stability, } B > 0; \qquad (32)$$

$$\text{secular stability, } D > 0; \qquad (33)$$

$$\text{pulsational stability, } AB - D > 0. \qquad (34)$$

Dynamical stability refers to the condition of hydrostatic equilibrium of the star, secular stability to the star's thermal equilibrium, and pulsational stability to the lack of growing oscillations. The dynamical and pulsational timescales are both very fast, $\sim(G\rho)^{-1/2}$, while the secular timescale is slow, $\sim E_{\rm th}/L$, where $E_{\rm th}$ is the thermal energy content of the unstable layers.

4. INFINITE CONDUCTIVITY

If $\sigma_E = \infty$, the stability criteria are the same as if there were no magnetic field present (provided that $\mu < 1$). This result has long been known (Stothers 1981). It is due to the fact that the magnetic field behaves like a gas with adiabatic exponent 4/3. Although Joule heating is absent, magnetic induction does occur, with $\gamma_M = 4/3$.

In the strictly adiabatic case, both K_T and K_M are zero, since the thermal properties of the oscillations decouple from the mechanical behavior. Consequently, the eigenvalue equation (23) has the simple solution $s = \pm iB^{1/2}\sigma_0$, with $B = (1-\mu)(3\Gamma_1-4)$. This adiabatic solution was originally derived many years ago (Chandrasekhar & Limber 1954). The star becomes dynamically unstable when $\Gamma_1 < 4/3$ (again if $\mu < 1$). The solution also shows that the dynamical instability can occur even when $\Gamma_1 > 4/3$ if μ, the ratio of the magnetic pressure force to gravity, exceeds unity. Thus, there is a limit on the permissible gradient of magnetic field through a hydrostatically stable layer. In the form of the solution given by Chandrasekhar & Limber (1954) as an integral over all layers of the star, the factor μ is replaced by

$E_{\mathrm{mag}}/|E_{\mathrm{grav}}|$, the ratio of the magnetic energy to the gravitational potential energy; in this case, $E_{\mathrm{mag}} < |E_{\mathrm{grav}}|$ for stability.

Real stars that could be dynamically unstable in their upper layers include very cool red giants (Tuchman et al. 1978), as well as yellow hypergiants and some highly evolved LBVs (de Jager et al. 2001). In these disparate objects the average value of Γ_1 drops below 4/3 as a result of the extensive ionization zones of hydrogen and helium. Therefore, the case of $\Gamma_1 < 4/3$ is not a purely academic one.

5. FINITE CONDUCTIVITY

In general, B is given by equation (25). Note that B involves a weighted mean of the thermal "gamma" and the magnetic "gamma." As long as $\mu < 1$, magnetic fields tend to stabilize or destabilize dynamically, depending on whether $\gamma_M > 4/3$ or $<4/3$, respectively.

If $B > 0$, secular stability depends only on the value of D, as given by equation (26). Magnetism then acts to stabilize secularly through the positive quantity N. The effects on D of the opacity derivatives, κ_P and κ_T, are weakened or strengthened, depending on whether $\gamma_M > 4/3$ or $<4/3$, respectively. In the limit of extreme nonadiabaticity, K_T is infinite. Equation (23) then yields the solution $s = \pm i(D/A)^{1/2}\sigma_0$. The star can avoid a dynamical-like secular instability if $D/A > 0$ (see Buchler & Regev [1982] for the nonmagnetic case).

Finally, pulsational instability, if $B > 0$, is governed by the sign of the mixed quantity $AB–D$, as displayed in equation (31). The consequence of magnetism is to pulsationally destabilize the star through the quantity N. This destabilization is not surprising because Joule heating increases with the density just as nuclear energy generation does. It has been known since Jeans's time (Jeans 1927) that subatomic energy release tends to destabilize a star pulsationally. The direct effect of γ_M on pulsational stability, however, is very small since $N \propto (7/3-\gamma_M)$ and γ_M differs little from 4/3.

6. CONCLUSION

Reduced to simple mean radial form, the full set of MHD equations have been here linearized and then solved in the case of the one-zone model of a radiative stellar envelope. Approximate expressions, however, have been developed for the perturbations $\delta\langle \boldsymbol{H}\cdot\boldsymbol{H}\rangle$ and $\delta|\nabla\times\boldsymbol{H}|^2$ in terms of small corrections to the exact forms known for the case of infinite electrical conductivity. The small correction parameter ϵ in the equation of electromagnetic induction has only a slight influence on the magnetic effective adiabatic exponent γ_M and a wholly negligible effect on the Joule heating rate, which is influenced much more directly by the finite electrical conductivity itself. Therefore, the extreme assumptions that we have made about the form of ϵ, and in particular about its assigned constancy, are not very critical. It is found that the magnetic pressure changes with mass density as $\rho^{4/3+\epsilon}$ and the Joule heating rate as $\rho^{1-\epsilon}$.

Although all three types of stability—dynamical, secular and pulsational—show either a positive or a negative dependence on γ_M depending on whether $\gamma_M > 4/3$ or $<4/3$, the actual influence of γ_M is found to be relatively small. As long as the electrical conductivity is not too low, the familiar criterion for dynamical stability, $\Gamma_1 > 4/3$, should still hold very closely. Magnetic fields, however, tend to stabilize the star secularly and to destabilize it pulsationally through the quantity N, which is, roughly speaking, the ratio of the thermal timescale to the timescale for converting the magnetic energy to heat (and vice versa), as can be seen from an inspection of equation (3). Consequently, the Joule heating rate, which is inversely proportional to the electrical conductivity, appears to be the paramount factor in determining the effect of a magnetic field on the secular and pulsational radial stability of a star.

In any star that lies at or near the limit of either dynamical stability or radiative stability, small changes of the structure can have quite exaggerated effects on the mechanical and thermal stability. Although the formal stability criteria remain the same, the underlying unperturbed structure of the star is different. The case of magnetism's effect on the Eddington luminosity limit for radiative stability has been discussed elsewhere (Stothers 2004). Here we simply remark that the formal criteria for secular and pulsational stability will, in general, change their sign depending on whether the star is dynamically stable or not. Applications to real stars, therefore, will be complicated and tricky. It has been our intention in this paper only to formally derive the effect of a magnetic field on the stability criteria themselves.

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APPENDIX

MAGNETOTHERMODYNAMICS

Several approaches have been used in the past to include a magnetic field in the thermodynamics of a stellar plasma. The standard approach, as here, is to ignore the magnetic field entirely, regarding its thermodynamic effects as being negligibly small. A different approach has been to treat the magnetic field's energy and pressure in the same manner as for the gas and radiation, or else to regard the magnetic field as having arisen from a prior process of magnetization of the whole thermodynamic system. These alternative approaches sometimes give different thermodynamic results from the standard approach. Which of these approaches is correct?

A1. MECHANICAL APPROACH

Cowling (1952) has stated, without giving any proof, that the adiabatic relations between the pressure and density of an isotropic ionized gas threaded by a magnetic field are simply the usual ones for the gas alone. He has remarked of the magnetic pressures that "any work done by these is done at the expense of the magnetic energy, not of the heat." Adoption of the usual adiabatic relations was later made, again, by Cowling (1976, p. 4) as well as by Chandrasekhar & Limber (1954), Spitzer (1962, p. 17), Alfvén & Fälthammar (1963, p. 76), Goedbloed & Poedts (2004, p. 134), and many others. Parker (1979, p. 55) has shown on the basis of a mechanical treatment that the work done by the gas against the magnetic stresses is just equal to the increase of the magnetic energy.

A2. DIRECT THERMODYNAMIC APPROACH

Tutukov & Ruben (1974) and Mollikutty et al. (1989), on the other hand, assigned arbitrary forms to $P_{\rm mag}$ and $U_{\rm mag}$, added them to P and U for the gas and radiation, and entered the sums into a new expression for the first law of thermodynamics:

$$TdS = d(U_{\rm gas} + U_{\rm rad} + U_{\rm mag}) + (P_{\rm gas} + P_{\rm rad} + P_{\rm mag})\,dV, \tag{A1}$$

where $V(=1/\rho)$ is the specific volume. By setting $TdS = 0$, they obtained expressions for the adiabatic gradients and related quantities. These contained nonvanishing dependences on the magnetic field and are clearly unphysical in view of § A1.

A3. MAGNETIZATION APPROACH

Callen (1960, p. 242) has treated the magnetization of an initially unmagnetized thermodynamic system that is placed inside a current-carrying solenoid. His result for the first law, with the magnetization $\boldsymbol{M}$ here set equal to zero as appropriate for a stellar plasma, is

$$TdS = dU' + PdV. \tag{A2}$$

Callen defined U' as "the total energy contained within the solenoid relative to the state in which the system is removed to its field free fiducial state and the solenoid is left with the field." Since the total energy within the solenoid is the sum of the system's internal (in our case, gas and radiation) energy and the energy of the magnetic field that threads the system, U' must be equal to $(U_{\rm gas} + U_{\rm rad} + U_{\rm mag}) - U_{\rm mag} = U_{\rm gas} + U_{\rm rad}$. Accordingly, the magnetic field does not appear explicitly in the first law of thermodynamics if $\boldsymbol{M} = 0$. This result has also been derived in many other textbooks of thermodynamics, probably most thoroughly and clearly by Carrington (1994, § 8.2). It is consistent with § A1 above.

Lydon & Sofia (1995, eq. [19]), however, assumed that

$$TdS = d(U_{\rm gas} + U_{\rm rad}) + PdV + d\chi, \tag{A3}$$

where $-PdV[= -(P_{\rm gas} + P_{\rm rad})dV]$ is the nonmagnetic work and $d\chi(= dU_{\rm mag})$ is the magnetic work. In a later study by Li et al. (2006, eq. [13]), this was tacitly changed to

$$TdS = d(U_{\rm gas} + U_{\rm rad}) + PdV - d\chi \tag{A4}$$

in order to ostensibly conform with Callen's more rigorous derivation (Callen 1960). Li et al. have misinterpreted Callen's U' as being equal to $(U_{\rm gas} + U_{\rm rad}) - U_{\rm mag}$. Both equations (A3) and (A4) are therefore unphysical.

Thus, the various thermodynamic derivatives such as specific heats and adiabatic exponents reduce to simply those derived for a nonmagnetic system composed of ionized gas and radiation alone. Although Alfvén & Fälthammar (1963, p. 77) have pointed out that the gas velocity distribution can become anisotropic in the presence of a strong magnetic field, thereby affecting the thermodynamic derivatives indirectly, Cowling (1953) determined that in the Sun (at least below the photosphere) collisions remain frequent enough to maintain a high electrical conductivity. Therefore, the velocity distribution will be nearly isotropic in the stellar interior for all but the strongest magnetic fields. As a consequence, the system of gas and radiation can be considered to be effectively in thermodynamic equilibrium.

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