

## A VIRIAL THEOREM INVESTIGATION OF MAGNETIC VARIATIONS IN THE SUN

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### ABSTRACT

The magnetic virial theorem is applied here to a long-standing astrophysical problem, namely, the sign of the Sun's radius change during the solar activity cycle. The solar radius is theoretically found to decrease around the time of maximum magnetic field strength, in agreement with the best available observational evidence. This theoretical prediction, although simply based, instills some confidence by explicitly satisfying the conservation of total energy.

*Subject headings:* magnetic fields — Sun: interior

### 1. INTRODUCTION

The virial theorem has long been a powerful tool for studying the global energy properties of astrophysical systems. Despite its simplicity, it remains a useful device, both as an illuminator of the results of detailed numerical simulations of various systems and as a means of investigating systems that have stayed too complex to simulate numerically in a realistic way.

In the present Letter, we apply the magnetic virial theorem (Chandrasekhar & Fermi 1953) to a complex astrophysical system for which numerical simulations have so far proven to be inadequate. The problem concerns the radius changes in the Sun induced by magnetic variations during the course of solar activity.

### 2. VIRIAL THEOREM

The mean equilibrium state of a star at any given time will obey the reduced magnetic virial theorem (Chandrasekhar & Fermi 1953):

$$2K + W + E_{\text{mag}} = 0, \quad (1)$$

where  $K$  is the sum of all types of thermal energy (gas kinetic energy, turbulence, convective flows, rotation, and pulsation),  $W$  is the gravitational potential energy, and  $E_{\text{mag}}$  is the magnetic energy. The radiant energy content is ignored here as being comparatively small. Total energy must be conserved:

$$K + W + E_{\text{mag}} = C, \quad (2)$$

$C$  being a constant. Not included in  $C$  are the radiant energy losses at the stellar surface, because we will be considering only times short compared to a star's cooling (Helmholtz-Kelvin) time.

In general,  $2K + W + E_{\text{mag}} = 0$  (for equilibrium states), and  $K + W + E_{\text{mag}} = C$  (for all states). We take first a simple case. Whenever  $E_{\text{mag}} = 0$ , we have  $2K_0 + W_0 = 0$  and  $K_0 + W_0 = C$ . If the internal changes are not dynamically fast, we may combine all four equations, obtaining  $K = K_0$  and  $W - W_0 = -E_{\text{mag}}$ . This implies that the magnetic energy, which decays into Joule heat, ultimately gets converted into gravitational potential energy. The total thermal energy is unchanged, because  $K =$

$K_0$ . More generally, for *any* change  $\delta E_{\text{mag}}$ , we have, differentially from equations (1) and (2),

$$\delta K = 0, \quad \delta E_{\text{mag}} = -\delta W. \quad (3)$$

Including the radiant energy losses at the surface by writing  $C = C(t_0) + \int L dt$ , where  $L$  is luminosity, we would have had  $\delta K = \delta \int L dt$  and  $\delta E_{\text{mag}} = -\delta W - 2\delta \int L dt$ . Since  $t - t_0$  and  $\delta L$  are both assumed to be small, the term  $\delta \int L dt$  is of second order and therefore can be neglected, justifying our original neglect of the surface losses.

From  $W = -\int GM(r)r^{-1} dM(r)$ , we have  $\delta W/W = -\xi \delta R/R$ , where  $0 < \xi \leq 1$ . In the case of a homologous change of the star's structure,  $\xi = 1$ . In general,

$$\delta E_{\text{mag}}/|W| = -\xi \delta R/R. \quad (4)$$

Like the various kinds of thermal pressure, the magnetic stresses support the star against gravity. Therefore, if one removes the magnetic field, the star expands to a larger equilibrium radius. It is possible that for a highly *nonhomologous* radius perturbation,  $\xi$  could be negative (specifically if the surface layers were perturbed in an opposite direction from the much more massive interior layers); in that case, we could write  $\delta W/W = -\xi \langle \delta r/r \rangle$  for a suitable mass-averaged radius perturbation  $\langle \delta r/r \rangle$ , and  $\xi$  would then be positive, as expected.

A different approach, using the dynamical form of the reduced virial theorem (Chandrasekhar & Fermi 1953),

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2K + W + E_{\text{mag}}, \quad (5)$$

was adopted by Gough (1981) in his study of the Sun. The new quantity,  $I = \int r^2 dM(r)$ , is the total moment of inertia about the center. Gough explicitly computed  $\frac{1}{2} d^2 I/dt^2$  rather than  $\delta E_{\text{mag}}$  for successive states of quasi-equilibrium. Therefore, he was unable to determine the relative amounts of thermal energy and gravitational potential energy that were exchanged with the magnetic energy. He could establish only the initial sign of the radial acceleration,  $d^2 R/dt^2$ , caused by a magnetic change.

## 3. SUN

The most obvious target for application of the foregoing results is the Sun. We start by reviewing earlier theoretical studies of magnetically induced variations of the solar radius.

Previous models of magnetic changes in the Sun have almost entirely focused on the outer convection zone. It is within this zone (possibly at its base or in the strongly superadiabatic region near the surface) that the observed magnetic field is believed to be generated. Ulrich (1975) argued that the known magnetic cycles alter the convective efficiency throughout this zone and thereby also the photospheric radius. He applied his idea, however, only to the much longer timescales that are characteristic of the cooling time of the whole convective envelope,  $2 \times 10^5$  yr. Although the total amount of energy in the convection zone was changed very slowly in his calculations, the total energy of the star was not conserved, and his results are therefore not rigorously correct. Since the derived radius shift is very small, its correct sign—positive or negative—still remains in doubt (Gilliland 1982).

Dearborn & Newman (1978) later treated the cases of both slow and fast changes of the Sun's convective efficiency. All subsequent work has considered only fast changes. Sofia et al. (1979) introduced the concept of localized perturbations of the convective efficiency. The effect on the solar radius of the presence of magnetic flux tubes in the convection zone was studied by Thomas (1979), while Spiegel & Weiss (1980) proposed that the buoyant magnetic flux tubes are generated by dynamo action in a layer at the base of the convection zone (but see, originally, Parker 1975). Dearborn & Blake (1980, 1982) followed this up by examining the effects of changes in the local magnetic pressure and in the amount of spot area at the surface (see also Spruit 1982). Many studies since 1981, cited by Endal et al. (1985) and Spruit (2000), have elaborated and refined these ideas. In each case, the induced expansion or contraction of the Sun is very small, while the convective envelope remains out of thermal equilibrium owing to the short timescale of solar activity ( $\sim 11$  yr) that is involved. Although hydrostatic equilibrium is essentially maintained at all times since the hydrodynamical timescale is only  $\sim 1$  hr, and heat is rapidly transported because the advective timescale for convective turnover of the envelope is only  $\sim 1$  month, full thermal equilibrium requires for its attainment a much longer time, the Helmholtz-Kelvin or cooling time.

In probably all of these studies that were based on assuming relatively fast changes of the convective efficiency, total energy was not conserved (Gough 1981; Däppen 1983). Perhaps it is not surprising that the results differ in many cases. If magnetic energy is supplied without any convective feedback, all layers of the envelope expand, as does the photospheric radius (e.g., Lydon & Sofia 1995). On the other hand, if the convective efficiency is lowered because of an increasing (but not explicitly included) magnetic field, the photospheric radius shrinks (e.g., Balmforth et al. 1996). The clear need to introduce an explicit physical interaction between turbulent convection and magnetic fields has been stressed by Li et al. (2003). However, we would emphasize the still more important need to conserve the total energy of the star, which is a more fundamental constraint and does not require knowledge of how the various kinds of energy are exchanged with each other.

Although our new equation (4) is based very simply on the virial theorem, it conserves total energy. It unambiguously predicts a shrinkage of the photospheric radius whenever the mag-

netic energy is increased, so that a minimum solar radius should occur around the time of maximum solar activity. Thus, it supports the earlier published results based on changing the convective efficiency.

Except for the *sign* of our new result, however, a more quantitative prediction cannot be made because the magnitude of  $E_{\text{mag}}$  is so poorly constrained by models of the solar convective envelope. If the magnetic flux is produced near the base of the envelope, then  $E_{\text{mag}} \sim 10^{39}$  ergs (Spiegel & Weiss 1980), but if it is generated in the superadiabatic region near the surface,  $E_{\text{mag}} \sim 5 \times 10^{35}$  ergs (Dearborn & Blake 1982). In comparison, we know that  $|W| \sim 10^{48}$  ergs. Although, for the solar envelope,  $\xi$  must be very small, we do not know how small it actually is.

The fact that our simple model assumes complete thermal equilibrium of the Sun is not crucial, because a timescale of  $\sim 11$  yr is at least long enough compared to the advective timescale of the envelope for our equation (4) to remain approximately valid. On very long timescales that approach or exceed the envelope cooling time, the neglected term  $\int L dt$  becomes comparable to or exceeds  $|W|$ , in which case a reduction of the convective efficiency turns out to require an *expansion* of the envelope to maintain thermal (secular) stability (e.g., Schwarzschild 1958, Fig. 24.2).

What do solar radius observations tell us? The long-term observational record, from the late seventeenth century to the 1970s, consists of comparatively crude measurements of the solar diameter. This record, however, covers several cycles of the  $\sim 80$  yr Gleissberg solar variation, which may have a larger amplitude than does the  $\sim 11$  yr Schwabe solar variation. For the Gleissberg cycle, the published observations disagree with each other. In one modern investigation, the maximum diameter was judged to have occurred around the time of maximum surface activity (Parkinson et al. 1980; Parkinson 1983); in other investigations, the diameter increased around the time of minimum surface activity (Dunham et al. 1980; Gilliland 1981; Ribes et al. 1987), while further studies have found no significant diameter change at all (Shapiro 1980; Morrison et al. 1988; Toulmonde 1997). The historical results suggest only that  $|\delta R/R| < 10^{-3}$ .

More accurate, and therefore more trustworthy, are the measurements of the solar diameter during the past few decades. The time coverage in these studies ranges from a fraction of one  $\sim 11$  yr cycle up to a few such cycles. As in the case of the  $\sim 80$  yr cycle, however, the results vary. The diameter appears to have changed either in phase with surface activity (Ulrich & Bertello 1995; Basu 1998; Emilio et al. 2000; Noël 2004) or in antiphase with surface activity (Gilliland 1981; Sofia et al. 1994; Laclare et al. 1996; Li & Sofia 2001; Reis Neto et al. 2003; Thuillier et al. 2005; Egidi et al. 2006). Some studies have detected no significant change at all (LaBonte & Howard 1981; Brown & Christensen-Dalsgaard 1998; Wittmann 2003; Kuhn et al. 2004; Badache-Damiani & Rozelot 2006).

An indirect method of deriving the solar diameter can be applied by first measuring both the Sun's effective temperature and its irradiance on the Earth, and then by employing the Stefan-Boltzmann law  $L/4\pi R^2 = \sigma T_e^4$ . Even if the light from the Sun is not emitted wholly isotropically, we need use only annual averages of  $T_e$  and  $L$ . Then over a complete  $\sim 11$  yr solar cycle,  $\delta R/R = 0.5\delta L/L - 2\delta T_e/T_e$ . Adopting irradiance data from Fröhlich & Lean (1998) and effective temperature data from Gray & Livingston (1997), both displayed as annual

means by Li & Sofia (2001), we find  $\delta R/R = -2 \times 10^{-4}$ , i.e., a radius change in antiphase with solar activity. This agrees precisely with the Solar Disk Sextant (SDS) measurements of Egidi et al. (2006). The long-running CERGA astrolabe measurements (Laclare et al. 1996) give essentially the same result,  $\delta R/R \approx -1 \times 10^{-4}$ .

If these three empirical results are roughly correct, they support, at least in sign, our present conclusions based on the theoretical global solar considerations. Helioseismological observations may eventually be able to provide the needed detailed information about the radial run of  $\delta r/r$  through the envelope (Dziembowski & Goode 2005; Sofia et al. 2005; Lefebvre & Kosovichev 2005) and hence about the unknown structure of  $\xi$ .

#### 4. CONCLUSION

Some success has attended the present application of the magnetic virial theorem to a long-standing astrophysical problem. Specifically, the radius of the Sun has been theoretically predicted to *decrease* around the time of maximum solar activity. The best available observational evidence indicates that this is indeed the case, although the total amount of radius shrinkage cannot be accurately modeled theoretically at present. This apparent success ought to have some validity, because it is based explicitly on the conservation of total energy. It leads us to suspect that analogous applications to other stars on the lower main sequence, to pre-main-sequence stars, and perhaps even to interstellar clouds might prove to be fruitful.

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