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GENERALIZED ANALYTIC STELLAR STABILITY CRITERIA WITH APPLICATIONS TO LUMINOUS STELLAR ENVELOPES

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ABSTRACT

Baker's one-zone model of a radiative stellar envelope is generalized here to include additional forces that can be represented as a function of only the stellar radius. The criteria for dynamical, secular, and pulsational stability against radial perturbations are derived and expressed in simple, general analytic forms. Applications are made to the outer envelopes of luminous blue variables (LBVs). The acceleration of stellar-wind mass loss has no effect on the stability criteria, but axial rotation and slowly adapting convective turbulence produce more complicated effects, depending on whether the envelope is dynamically stable or not. On the other hand, rotation and turbulence are probably very weak in most LBV outer envelopes.

Subject headings: stars: mass loss — stars: oscillations — stars: rotation — stars: variables: other — turbulence

1. INTRODUCTION

Linear nonadiabatic stability criteria for a simple model of a star have been derived by Jeans (1929), Baker (1966), Stellingwerf & Gautschy (1988), and others, who considered the effects of the gravitational force and the force due to the thermal pressure. Centrifugal force due to axial rotation and the Lorentz force due to tangled magnetic fields were added later (Stothers 1981), while the turbulent pressure force, which is not so straightforward to treat, has been studied only in part (Cowling 1935; Unno & Kamijo 1966; Unno 1967; Gough 1967; Stellingwerf 1986). Antonello (1982) has considered a very generalized force law that acts radially and contains the gravitational force; however, he applied his results only to the already studied cases of axial rotation and tangled magnetic fields and got the same answers.

Some of these forces, even if they are not strictly conservative, can, under certain assumptions, be represented as functions of only the stellar radius. These include the forces due to gravity, stellar-wind mass loss, axial rotation, and slowly adapting convective turbulence. It seems worthwhile to study the effects of these particular forces explicitly on the threefold problem of stellar stability—dynamical, pulsational, and secular. The fourth type of stability—radiative—involves calculating the generalized Eddington luminosity limit, which is a very different problem that has been recently treated elsewhere (Stothers 2003b and references therein). The present results have potential implications for observed instabilities in luminous blue variables (LBVs), where the acceleration of mass loss becomes significant, even though axial rotation and turbulent pressure are less important factors.

2. PHYSICAL ASSUMPTIONS

The additional forces can be conveniently expressed in the form of a generalized acceleration $f(r, t) \propto r(t)^b$, under the assumption of spherical symmetry of the star. The gravitational acceleration, $g = GM(r)/r^2$, is then replaced by the effective gravity,

$$g_{\rm eff} = g - f = g(1 - \psi), \tag{1}$$

where $\psi = f/g$; the effective gravity appears in the hydrodynamic equation of motion. Only the stellar outer envelope is considered here, in a one-zone model that employs the approximation of complete radiative equilibrium. Although the outer envelopes of luminous blue supergiants are in part convectively unstable, the convective flux is very small compared to the radiative flux, except deep within the iron convection zone, which is formed by an opacity bump at temperatures around 2×10^5 K. The turbulent pressure, however, can occasionally be quite high, especially in the bluest, most luminous objects.

The additional forces considered here are represented by the following expressions for ψ (Stothers 1974, 2002, 2003a):

$$\text{axial rotation,} \quad \psi_{\rm rot} = (2/3)\Omega^2/\sigma_0^2, \quad b = -3; \tag{2}$$

$$\text{mass loss,} \quad \psi_{\rm loss} = \left(h\dot{M}\right)^2/(\sigma_0 \Delta M)^2, \quad b = -2; \tag{3}$$

$$\text{turbulent pressure,} \quad \psi_{\rm turb} = -\left(\frac{4\pi r}{\sigma_0^2}\right) dP_{\rm turb}/dM(r), \quad b = 2. \tag{4}$$

Here $\sigma_0^2 = GM(r)/r^3$, ΔM is the mass contained in a single zone, Ω is the angular velocity of rotation, $\dot{M}$ is the mass-loss rate, h is a constant of the order of unity, and $P_{\rm turb}$ is the turbulent pressure, taken to be constant in time.

3. STABILITY ANALYSIS OF THE ONE-ZONE MODEL

For the linearized radial stability analysis, we adopt Baker's (1966) one-zone model of a stellar envelope. Small perturbations are made in the form of a radial displacement multiplied by exp (st), where s is a complex temporal frequency. Linearizing all of the constitutive equations leads to a fully nonadiabatic dispersion relation, given by Baker's equation (29):

$$s^3 + K\sigma_0 A s^2 + \sigma_0^2 B s + K\sigma_0^3 D = 0. \tag{5}$$

Here $K = (2\rho L)/(P\delta\sigma_0\Delta M)$, δ is defined below, and all of the other physical symbols have their usual meanings. The parameter K represents the degree of nonadiabaticity, which is approximately equal to the ratio of the free-fall collapse time, σ_0^{-1}, to the thermal timescale, $E_{\rm th}/L$.

The three critical coefficients are given here by

$$A = -(\alpha\Gamma_1 - 1)\left[\delta^{-1}\alpha(\kappa_T - 4) + \kappa_P\right], \tag{6}$$

$$B = 3\Gamma_1\theta_1 - 4\theta_2, \tag{7}$$

$$D = (\alpha\Gamma_1 - 1)\left[\delta^{-1}(4\alpha\theta_2 - 3\theta_1)(\kappa_T - 4) + 4\theta_2\kappa_P + 4\theta_1\right], \tag{8}$$

where

$$\alpha = (\partial \ln \rho/\partial \ln P)_T, \quad \delta = -(\partial \ln \rho/\partial \ln T)_P, \tag{9}$$

$$\kappa_P = (\partial \ln \kappa/\partial \ln P)_T, \quad \kappa_T = (\partial \ln \kappa/\partial \ln T)_P, \tag{10}$$

$$\theta_1 = 1 - \psi, \quad \theta_2 = 1 - \psi(2 - b)/4, \tag{11}$$

and Γ_1 is the first generalized adiabatic exponent (Cox & Giuli 1968). A useful combination of these coefficients is

$$AB - D = -3\theta_1\Gamma_1(\alpha\Gamma_1 - 1) \times \left[(\delta\Gamma_1)^{-1}(\alpha\Gamma_1 - 1)(\kappa_T - 4) + \kappa_P + 4/(3\Gamma_1)\right]. \tag{12}$$

Stability conditions for the envelope follow from consideration of the nature of the three roots of equation (5) and are as follows:

$$\text{dynamical stability}, \quad B > 0; \tag{13}$$

$$\text{secular stability}, \quad D > 0; \tag{14}$$

$$\text{pulsational stability}, \quad AB - D > 0. \tag{15}$$

4. APPLICATIONS WITH K = 0

In the adiabatic case, $K = 0$, equation (5) reduces to a simple quadratic, with the solution

$$s = \pm iB^{1/2}\sigma_0. \tag{16}$$

If $B > 0$, this implies an adiabatic oscillation, while if $B < 0$, exponential growth occurs, implying dynamical instability, in conformity with equation (13).

The stability condition $B > 0$ can be written

$$\Gamma_1 > (4/3)(\theta_2/\theta_1). \tag{17}$$

In this form it is easy to see that the acceleration of mass loss has no effect on the standard criterion, $\Gamma_1 > 4/3$. Axial rotation, however, tends to stabilize the envelope. As for $P_{\rm turb}$, if $dP_{\rm turb}/dM(r) > 0$ (as in the deeper layers of a convection zone), slowly adapting turbulent pressure likewise tends to stabilize, but if $dP_{\rm turb}/dM(r) < 0$ (as in the upper layers of a convection zone), the effect is reversed. These results have already been found in substance in previous studies focused specifically on dynamical stability (Ledoux 1945; Stothers 1981, 2002, 2003a). However, the present integrated approach is new and yields a transparently simple result, equation (17).

5. APPLICATIONS WITH SMALL, NONZERO K

5.1. *Dynamical Stability*

For any value of K other than infinite K, the criterion for dynamical stability is given by equation (13). The discussion of § 4, therefore, also pertains to the present, *nonadiabatic* case. This conclusion has been confirmed by numerical hydrodynamic simulations at least for highly nonadiabatic LBV envelope models with $\psi = 0$ (Stothers 1999a).

5.2. *Secular Stability*

Secular stability for small K requires $D/B > 0$ (Baker 1966). If $B > 0$, this criterion reduces simply to $D > 0$.

Although all previous analytic studies using the one-zone model assumed $B > 0$, there is now semiempirical evidence that at least some LBVs are dynamically unstable (Stothers 1999b; de Jager et al. 2001). In this case the criterion for secular stability is opposite in sign from before and becomes $D < 0$.

In either case, equation (8) for D is so complicated a function of the additional force terms that in order to achieve any useful insight we must consider the simplifying conditions inside an LBV envelope. In such an envelope, radiation pressure is so high compared to gas pressure that we can assume $\alpha = 1/\beta$ and $\delta = (4 - 3\beta)/\beta$, where $\beta = P_{\rm gas}/(P_{\rm rad} + P_{\rm gas})$. Furthermore, $\Gamma_1 \approx 4/3$. Since the opacity lies close to the constant electron-scattering limit, we also have $|\kappa_T| \ll 1$ and $|\kappa_P| \ll 1$. With these simplifications,

$$D \approx 16(3\beta)^{-1}(\theta_1 - \theta_2) = -4(3\beta)^{-1}(b + 2)\psi. \tag{18}$$

The acceleration of mass loss clearly has no effect on D. Axial rotation, however, leads to a larger value of D and therefore enhances secular stability if the envelope is dynamically stable but diminishes it if the envelope is dynamically unstable. Slowly adapting turbulent pressure raises D if $dP_{\rm turb}/dM(r) > 0$ but lowers D if $dP_{\rm turb}/dM(r) < 0$. The consequences for secular stability in this case depend, again, on whether the envelope is dynamically stable or unstable.

To first order in K, Baker (1966) showed that

$$s = -(D/B)K\sigma_0. \tag{19}$$

Therefore, the secular timescale is $\sim(K\sigma_0)^{-1}$. In LBV envelopes, quasi-static stellar evolutionary calculations indicate that this timescale is only about 1 order of magnitude greater than the pulsation period (Stothers & Chin 1997). To see why it is so short, consider the fact of the rough constancy of density ρ throughout the LBV envelope, such that $\Delta M \approx (4/3)\pi R^3\rho$. Then using $L = \pi R^2 acT_e^4$, we find $K\sigma_0 \approx (9/8)\beta(c/R)(T_e/T)^4$. Although the radius R is large and β is small, the effective temperature T_e is quite high with respect to the envelope mean temperature T. Consequently, K turns out to be fairly large; i.e., the envelope is highly nonadiabatic, and therefore the secular timescale becomes short.

5.3. *Pulsational Stability*

To have pulsational stability, the necessary condition for it, $(AB - D)/B > 0$, assuming K is small, hinges on the sign of B, as in the secular stability case. On the other hand, the numerator, $AB - D$, is completely independent of the additional force terms, apart from a positive multiplicative factor, θ_1. This result first appeared in some specific applications (Stothers 1981) and was later proven in general (Antonello 1982).

Let us again take up the case of an LBV outer envelope. Since the additional force terms are wholly irrelevant pulsationally (except for determining the sign of B), we retain κ_T and κ_P, however small they may be. Then

$$AB - D \approx -4(3\beta)^{-1}(\kappa_T + 4\kappa_P)\theta_1. \tag{20}$$

The sign of $AB - D$ is thus determined only by κ_T and κ_P. Numerical hydrodynamic simulations of highly nonadiabatic LBV envelope models with either a constant electron-scattering opacity or a more realistic variable opacity demonstrate that pulsational instability does in fact depend only on the opacity variations (Stothers 1999a). Observationally, many LBVs show small, rapid quasi-periodic light variations (van Genderen 2001) that could reflect a radial pulsational instability.

6. APPLICATIONS WITH VERY LARGE K

If K is extremely large, equation (5) becomes approximately a quadratic, whose solution is

$$s = \pm i(D/A)^{1/2}\sigma_0. \tag{21}$$

This result was first obtained by Buchler & Regev (1982). If $D/A > 0$, a completely nonadiabatic oscillation occurs, but if $D/A < 0$, exponential growth takes place, which can be regarded as a dynamical-like secular instability. As long as K is less than infinite and both B and D/A are positive, two oscillatory modes of comparable period can therefore coexist, an ordinary mode (based on B) and a strange mode (based on D/A), as has been noted elsewhere (Stothers & Chin 1997, Appendix). This strange mode, of course, is only one type of strange mode among several types that can occur in stellar envelopes.

Turning to the example of an LBV envelope, we assume, as before, $|\kappa_T| \ll 1$ and $|\kappa_P| \ll 1$, in order to evaluate the influence of the additional force terms. Under this assumption, $A \approx 4(3\beta)^{-1}$; hence, in view of equation (18),

$$D/A \approx -(b+2)\psi. \tag{22}$$

Thus, the effects of the additional force terms on D/A are identical to their effects on D, which have already been discussed in § 5.2.

7. CONCLUSION

The main results obtained in this paper are as follows:

1. A unified approach has been used to analytically study the effects of a certain class of forces (those that can be expressed as a function of only the stellar radius) on the stability of radiative stellar envelopes, adopting Baker's (1966) one-zone model.

2. Very simple formulae emerge for the criteria for dynamical stability and for pulsational stability, although these two results had already been worked out elsewhere in a somewhat less explicit fashion. The present result for the criterion for secular stability is new.

3. Applications to the outer envelopes of LBVs, where the physical conditions are such that simplifying mathematical approximations can be made, lead to easily diagnosable expressions for the criteria for dynamical, secular, and pulsational stability. Whether or not the envelope is subject to secular and pulsational instability critically depends on whether or not it is also dynamically unstable.

4. The acceleration of stellar-wind mass loss has no effect on the stability criteria. However, axial rotation and slowly adapting turbulence have significant, and also complicated, effects on the criteria. If the envelope is dynamically unstable, the unperturbed configuration used for the stability analysis should ideally be assumed to be not a structure in hydrostatic equilibrium but rather one undergoing dynamical outflow. This can be easily accomplished by introducing the acceleration of stellar-wind mass loss in addition to, rather than separately from, rotation and turbulence. The revised criteria turn out, however, to be unchanged, since the acceleration of mass loss has been found to have no effect on the criteria.

Some checks on the one-zone model have already been published. A comparison of the results of numerical hydrodynamic simulations for highly nonadiabatic LBV envelopes, omitting all additional forces (Stothers 1999a), with the results predicted by the one-zone model has shown surprisingly close agreement, at least in the two cases of dynamical instability and pulsational instability. (Secular instability could not be checked owing to its longer timescale.) The reason is that in LBV envelopes the radial gradients of the relative perturbations of all the physical quantities are nearly zero, just as they are assumed to be in the one-zone model. Because of this close agreement of the detailed models and the simple model, it is likely that the predicted effects of the various additional forces have been correctly portrayed by the one-zone model.

Other published numerical hydrodynamic simulations for LBVs have been more restricted in scope and in applicability. L. Dessart (2000, private communication) has confirmed the existence of strong convective motions deep in the envelope and a rapidly accelerated outward mass flux over one dynamical response time for a model that was dynamically unstable. Guzik et al. (1999), Glatzel et al. (1999), and Dorfi & Gautschy (2000) found pulsations in models that were probably dynamically stable. Some other hydrodynamic simulations have been performed solely for the outer atmospheres of LBVs, in particular for the wind of η Car. Consequently, our present analytic results cannot yet be checked further, but they could prove to be useful as predictions and interpretive guides for future work.

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