



Dynamical model of mesoscales in z -coordinates

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Abstract

Using the equations of the dynamical mesoscale model developed previously [Ocean Modell. (2004) 8, 1–30], we derive a mesoscale model in z -coordinates to be used in coarse resolution OGCMs. We present a model for the eddy-induced velocities, mesoscale diffusivity, eddy kinetic energy, eddy potential energy, residual diapycnal flux, velocity across mean isopycnals, Reynolds stresses, E-P, PV and RV fluxes.

Specifically, in the *mean buoyancy equation*, mesoscales give rise to two terms: an eddy-induced velocity $\mathbf{u}_M = (\mathbf{u}^+, w^+)$ and a residual diapycnal flux Σ . While \mathbf{u}_M has received much attention, Σ has always been taken to be zero. Physical and numerical arguments are presented to show that Σ is not zero. We present the model results for both \mathbf{u}_M and Σ . The new expression for \mathbf{u}^+ contains four terms, the first of which has the structure of the GM model while the remaining three terms are new. Several interpretations of the new terms are given. The boundary conditions at $z = -H, 0$ are satisfied by \mathbf{u}_M and Σ , thus avoiding the need for “tapering schemes” employed thus far to amend the failure of models for \mathbf{u}_M to satisfy the proper boundary conditions. We also present the model results for the mesoscale diffusivity and show that the predicted magnitude and z -dependence are in accord with recent numerical models. The residual flux Σ gives rise to a mesoscale-induced diapycnal diffusivity which in the ACC is larger than the diabatic one. The resulting “velocity across mean isopycnals” may thus significantly affect the dynamics of the thermocline (e.g., the Munk–Wunsch advective–diffusive model). The predicted magnitude of this velocity is in an accord with recent results from eddy-resolving codes.

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The presence of a non-zero residual flux Σ in the mean buoyancy equation implies that the effect of mesoscales in the T–S equations is not fully accounted for with only the eddy-induced velocity, as generally done. Additional Σ -like diapycnal fluxes must be added to the T–S equations.

Mean momentum equations. Since the down-gradient model used in most OGCMs does not represent mesoscales, the latter have not yet been accounted for. A model for the divergence of the Reynolds stresses, the Eliassen–Palm fluxes, the PV (potential vorticity) and RV (relative vorticity) fluxes is presented. We show that the Sverdrup vorticity balance is modified by mesoscales. In particular, while the standard Sverdrup relation does not allow meridional currents to cross the equator, the presence of mesoscales allows such a possibility.

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1. Introduction

A satisfactory representation of the very energetic, ~ 100 km size ocean mesoscale eddies in coarse resolution OGCMs (e.g., Griffies et al., 2000) is still an open problem. Significant progress was achieved when it was realized that mesoscale models had to reflect the fact that there is a huge reservoir of mean potential energy MPE (Lueck and Reid, 1984; Huang, 2004) which, through baroclinic instabilities, feeds mesoscales. It was therefore natural to model such process with a down-gradient expression to represent the transfer of energy from MPE to eddy potential energy EPE. The GM model (Gent and McWilliams, 1990, cited as GM) that was devised to explicitly capture this important physical process, considerably improved the performance of coarse resolution OGCMs (Gent and McWilliams, 1990; Danabasoglu et al., 1994; Böning et al., 1995; Hirst and McDougall, 1996, 1998). These improvements have rekindled the interest in mesoscale modeling and at the same time they revealed the complexity of the physical processes characterizing mesoscales.

It is fair to say that since the appearance of the GM model the bulk of the work is represented by numerical studies (both eddy resolving and coarse resolution ocean codes), as well as by heuristic models (Böning et al., 1995; Beckmann et al., 1994; Danabasoglu and McWilliams, 1995; Gille and Davis, 1999; Drijfhout and Hazeleger, 2001; McDougall and McIntosh, 2001; Karsten and Marshall, 2002; Radko and Marshall, 2004a,b; Ferreira and Marshall, submitted; Olbers and Visbeck, in press). While eddy-resolving codes have shed much light on the physics of mesoscales, it is difficult to translate such information into a model usable in OGCMs. At the same time, heuristic models cannot determine important parameters such as the mesoscale diffusivity and its z -dependence. In this context, it is of interest the suggestion by Muller and Garrett (2002) that any parameterization “must be specified as formulae rather than just numerical values”. This requirement acquires particular relevance in the case of mesoscales since eddy fluxes at a given depth are functions of large scale fields not only at the same depth, but at all other depths (Muller and Garrett, 2002), a non-local feature that is quite hard to model phenomenologically, as models for the PBL (planetary boundary layer) have shown over the years (Cheng et al., 2002). Thus, there is a need to construct a mesoscale model based on the dynamic equations governing the mesoscales. In contrast to the rather large literature of numerical simulations and heuristic models,

theoretically based models of mesoscales are relatively few (Killworth, 1997; Smith, 1999; Dukowicz and Greatbatch, 1999), a reflection of the difficulty of the problem.

Thus far, it has also been assumed that mesoscales affect the mean buoyancy field and thus the T–S equations, only via the eddy-induced velocities \mathbf{u}_M . In the language of several authors (Andrews and McIntyre, 1976, 1978; Andrews et al., 1987; McDougall and McIntosh, 1996; Greatbatch, 1998; Ferrari and Plumb, 2003) this is equivalent to assuming that the buoyancy flux \mathbf{F} is fully represented by its component along isopycnal (*skew flux*) while the flux orthogonal to it (*residual flux*) is zero. This explains why most of the models focused on finding a suitable expression for \mathbf{u}_M . However, the following questions must be posed:

- Is the GM model for $\mathbf{F}(\text{skew})$ complete?
- Is the residual flux $\mathbf{F}(\text{residual})$ zero?
- How are the mean momentum equations affected by mesoscales?

The answer to the first two questions is negative. Using an eddy-resolving ocean code, Bryan et al. (1999) showed that in addition to the GM term, additional terms must be present that are “unrelated to thickness sources or sinks”. Furthermore, Gille and Davis (1999) and McDougall (2004) used eddy-resolving codes to show that the residual flux is as important as the skew flux. This translates into a subset of problems which we now discuss.

First, consider the buoyancy skew flux $\mathbf{F}(\text{skew})$ and the eddy-induced velocity $\mathbf{u}_M = (\mathbf{u}^+, w^+)$ that represents its divergence, $\nabla \cdot \mathbf{F}(\text{skew}) = \mathbf{u}_M \cdot \nabla \bar{b}$.

- w^+ must vanish at $z = -H, 0$. Since \mathbf{u}_M is divergence free, it follows that the divergence of the column integral of $\mathbf{u}^+(z)$ must vanish. This is usually interpreted as a reflection of the fact that mesoscales are not supposed to exert a mean stress on the ocean but only to distribute the external stresses that are applied to the surface. Since present models for \mathbf{u}_M do not satisfy this condition, the problem has been dealt with using different tapering schemes (Danabasoglu and McWilliams, 1995; Large et al., 1997; Visbeck et al., 1997; Gerdes et al., 1999; Killworth, 2001; McDougall and McIntosh, 2001) which affect the outcome of climate studies since they modify the heat exchange between ocean and atmosphere,
- the mesoscale diffusivity $k_m(z)$ is known to exhibit a distinct z -dependence. Visbeck et al. (1997), Karsten and Marshall (2002), Ferreira and Marshall (submitted) and Olbers and Visbeck (in press) have suggested heuristic models for $k_m(z)$ but no theoretical derivation within a dynamical model has thus far been presented,
- extant mesoscale models are local in that the eddy fluxes are expressed in terms of the gradients of local variables. However, advection may also be caused by eddies that remain coherent over distances much larger than those over which the gradients are constant. A non-local mesoscale model is thus needed.

Second, consider $\mathbf{F}(\text{res}) = \mathbf{e}_z \Sigma$. Thus far, it has always been assumed that $\Sigma = 0$ (Treguier et al., 1997; Karsten and Marshall, 2002). However, since $-\Sigma$ is the source of eddy potential energy, it cannot be zero on physical grounds, a conclusion in accord with the results of several eddy-permitting/resolving ocean codes (Böning and Budich, 1992; Beckmann et al., 1994; Gille and Davis, 1999; McDougall, 2004). Furthermore, since $N^{-2} \partial \Sigma / \partial z$ acts as an adiabatic diapycnal velocity to

be added to the diabatic part, mesoscales may affect diffusion–advection models like the Munk–Wunsch model. The presence of a non-zero residual flux Σ in the mean buoyancy equation implies that the effect of mesoscales in the T–S equations is not fully accounted for by adding only the eddy-induced velocities, as generally done. Additional Σ -like terms must be present in the T–S equations.

Third, consider the *mean momentum equations*. In this case, the modeling efforts have made less progress. In fact, most OGCMs only employ a down-gradient of momentum which is not meant to represent mesoscales but processes at smaller scales. For example, McWilliams (1996) has noted that “the default sub-grid mesoscale parameterization form of horizontal momentum diffusion is inadequate for it fails to represent the qualitatively important feature of wind-driven gyres”. This means that to all practical purposes mesoscales have not yet been accounted for in the mean momentum equations. In addition, the models that have been suggested (Lee and Leach, 1996; Gent and McWilliams, 1996; Greatbatch, 1998; Wardle and Marshall, 2000; Plumb and Ferrari, 2004) were formulated in terms that cannot be readily used in z -coordinates OGCMs. However, irrespectively of any specific mesoscale model, the non-linearity in the original momentum equations implies that mesoscales momentum flux must exhibit the horizontal gradient of the mesoscales kinetic energy that observations show to be large (Schmitz, 1976; Wyrтки et al., 1976, Fig. 4; Wunsch, 1981).

In a recent paper (Canuto and Dubovikov, 2004, cited as OM1), a dynamical model for mesoscale eddies was presented in isopycnal coordinates. The hope was that such an approach would catch the major features of mesoscales described above without the need of empirical adjustments. Since the exact dynamic equations for mesoscales are not solvable analytically, approximations had to be made at the outset to render the equations solvable analytically. The dynamical equations were considered in the small Froude and Rossby numbers regime, a physically acceptable framework to treat baroclinic instabilities that were the primary focus of OM1. Shear instabilities responsible for purely barotropic modes were not included in OM1 (nor in this paper) and their addition is left to future work. The key new ingredient of this model vis a’ vis a linear treatment (Killworth, 1997) is the presence of non-linear interactions for which we employed a model (Canuto and Dubovikov, 1996, Canuto et al., 1999 and references therein) that had been previously tested on a wide variety of flows. Under these conditions, in OM1 it was shown that the eddy field equations reduce to a vertical eigenvalue problem for the Bernoulli function representing mesoscales. Knowledge of the solutions allows us to construct the eddy fluxes in terms of the large scale, resolved fields. The model results will be tested against a large set of data to assess their reliability before being used in OGCMs.

2. Structure and organization of the paper

Since the key features of the mesoscale model were presented in detail in OM1, there is no need to repeat them here. However, in z -coordinates new problems arise which are of both mathematical and physical nature. To the first category belongs the fact that the transition from isopycnal to z -coordinates is not simply a mathematical transformation since the density field itself is a random variable, a subject that requires special handling and for that reason it is discussed in Appendix A. The more physically interesting new feature is the appearance of a

“residual buoyancy flux”. The physical origin, interpretation, closure and properties thereof are a major part of the main text because of their direct oceanic implications. The derivation of many results is on average a rather laborious process and often times it has no direct physical interpretation. For that reason, the emphasis in this paper has been on first presenting the problem we plan to discuss, followed by the model results with mathematical details given in the Appendices. The physical interpretation of the model results then follows together with an assessment of the model results, a process that uses data from eddy-resolving codes, empirical relations and direct measurements.

In Section 3 we discuss the mean buoyancy equation; in Section 4 we discuss the *residual buoyancy flux* and offer several different viewpoints on its significance; in Section 5, we present the model results and a variety of implications and assessments we have made of the model results; in Sections 6 and 7, we present the model results for the Reynolds stresses in the mean momentum equation; in Section 8, we present the model results for the Eliassen–Palm fluxes, the PV (potential vorticity) and RV (relative vorticity) fluxes, derive a generalized Eliassen–Palm theorem and discuss previous models; in Sections 9 and 10, we present a detailed derivation of the model results. Finally, in Section 11 we present some conclusions.

3. Mean buoyancy equation

In OM1 (Canuto and Dubovikov, 2004) we presented a dynamical mesoscale model in isopycnal coordinates (an overbar and a prime were used to denote average and fluctuating components). In z -coordinates, we employ a double overbar and a double prime $\bar{\bar{A}}, A''$ to denote average and fluctuating components of a given variable. For the mean flow we employ the notation $\bar{\bar{\mathbf{U}}} = \bar{\bar{\mathbf{u}}} + \bar{\bar{w}}\mathbf{e}_z$ while for the eddy field we use $\mathbf{U}'' = \mathbf{u}'' + w''\mathbf{e}_z$; $\nabla = \nabla_H + \mathbf{e}_z\partial_z$, \mathbf{e}_z is the unit vector in the z -direction and $a_x = \partial a/\partial x$. In the adiabatic limit, the equation for the mean of the buoyancy field $b = -g(\rho - \rho_0)\rho_0^{-1}$ is

$$\bar{\bar{b}}_t + \bar{\bar{\mathbf{U}}} \cdot \nabla \bar{\bar{b}} + \nabla \cdot \mathbf{F}_b = 0 \quad (1a)$$

where the mesoscale *buoyancy flux* is defined as

$$\mathbf{F}_b \equiv \overline{\mathbf{U}''b''} = \mathbf{F}_H + \mathbf{e}_z F_V \quad \mathbf{F}_H = \overline{\mathbf{u}''b''} \quad F_V = \overline{w''b''} \quad (1b)$$

where \mathbf{F}_H and F_V are the horizontal and vertical components. The following exact equation (Treguier et al., 1997) can be derived from (1a):

$$\bar{\bar{b}}_t + (\bar{\bar{\mathbf{u}}} + \mathbf{u}^+) \cdot \nabla_H \bar{\bar{b}} + (\bar{\bar{w}} + w^+) \bar{\bar{b}}_z = -\frac{\partial \Sigma}{\partial z} \quad (1c)$$

where $\mathbf{u}_M = (\mathbf{u}^+, w^+)$ is the *eddy-induced velocity* defined as ($\bar{\bar{b}}_z = N^2$)

$$\mathbf{u}^+ = -\partial_z(N^{-2}\mathbf{F}_H), \quad w^+(z) = -\nabla_H \cdot \int_{-H}^z \mathbf{u}^+(z')dz' \quad (1d)$$

while the residual flux Σ and the slope of the isopycnal surfaces \mathbf{L} are defined as

$$\Sigma = F_V - \mathbf{F}_H \cdot \mathbf{L} \quad \mathbf{L} = -N^{-2}\nabla_H \bar{\bar{b}} \quad (1e)$$

Thus, a mesoscale model must provide both $\mathbf{u}_M = (\mathbf{u}^+, w^+)$ and the residual flux Σ which we now discuss in order. In the adiabatic approximation all fluxes across true isopycnals (which vary with time) can be shown to vanish. The vanishing is due to the cancellation between the flux across frozen isopycnals and the one due to the variation in time of the isopycnals themselves. This is not the case for fluxes across mean isopycnals in z -coordinates (e.g., residual flux Σ) since, within the eddy's characteristic dynamic time scale, mean isopycnals can be considered frozen in time (Dubovikov and Canuto, submitted).

From the boundary conditions $w^+(0) = 0$, $w^+(-H) = 0$, it follows that the vertically integrated $\mathbf{u}^+(z)$ must have zero horizontal divergence. While in principle a theory could be developed in which such a flow is non-zero, it is not obvious how this would interact with lateral boundary conditions on the vertical walls. Instead, in common with all other formulations of which we are aware of, we take the simple solution

$$\int_{-H}^0 dz \mathbf{u}^+(z) = 0 \quad (1f)$$

which trivially satisfies all requirements (see Killworth (1997) for a discussion on this point). Eq. (1f) is also interpreted as a reflection of the fact that mesoscales are not supposed to exert a mean stress on the ocean but only to distribute the external stresses that are applied to the surface. Since previous models did not satisfy (1f), the problem was dealt with by adopting tapering schemes (Danabasoglu and McWilliams, 1995; Large et al., 1997; Visbeck et al., 1997; Gerdes et al., 1999; Killworth, 2001; McDougall and McIntosh, 2001; Ferrari and McWilliams, 2004). However, different schemes affect the outcome of climate studies differently since they modify the heat exchange between ocean and atmosphere. The model results for $\mathbf{u}_M = (\mathbf{u}^+, w^+)$ and $\Sigma(z)$ presented below satisfy (1f) and the other boundary condition

$$\Sigma(-H, 0) = 0 \quad (1g)$$

4. Diapycnal residual flux Σ

Since this term has thus far been taken to be zero while it is not, it is important to discuss its physical meaning. Consider the following variables: eddy potential, eddy kinetic energy and mean kinetic energy, $EPE = (2N^2)^{-1} \overline{b'^2}$, $EKE = 1/2 \overline{\mathbf{U}'' \cdot \mathbf{U}''}$, $MKE = 1/2 \overline{\mathbf{U}}^2$; eddy energy $TEE = EKE + EPE$, total potential energy $TPE = g\rho z$, mean potential energy $MPE = TPE - EPE$ and mean energy $ME = MPE + MKE$. Neglecting terms not directly related to the main discussion and using the relations $D/Dt = \partial/\partial t + \overline{\mathbf{U}} \cdot \nabla$, $Dz/Dt = w$ and $Db/Dt = 0$ (adiabaticity), we obtain

$$\frac{D}{Dt} EPE = -\Sigma + \dots = \mathbf{F}_H \cdot \mathbf{L} - F_v + \dots, \quad \frac{D}{Dt} EKE = F_v + \dots = \mathbf{F}_H \cdot \mathbf{L} + \Sigma + \dots \quad (2a)$$

$$\frac{D}{Dt} TEE = \mathbf{F}_H \cdot \mathbf{L} + \dots \quad (2b)$$

$$\frac{D}{Dt} TPE + \overline{wb} = -\Sigma - \mathbf{F}_H \cdot \mathbf{L} + \dots \quad (2c)$$

$$\frac{D}{Dt} \text{MPE} + \overline{\overline{wb}} = -\mathbf{F}_H \cdot \mathbf{L}, \quad \frac{D}{Dt} \text{MKE} = \overline{\overline{wb}} \quad (2d)$$

Several considerations are in order. Eqs. (2d) and (2b) show that $\mathbf{F}_H \cdot \mathbf{L}$ represents the rate of loss of ME and the rate of gain of TEE. How is the TEE partitioned between EPE and EKE? The EPE does not get the full $\mathbf{F}_H \cdot \mathbf{L}$ but only part of it, specifically $\mathbf{F}_H \cdot \mathbf{L} - F_v$ since F_v represents the flux that goes from EPE to EKE, as Eqs. (2a) show. To be a source of EPE, $\mathbf{F}_H \cdot \mathbf{L} - F_v$ must be positive which means that

$$\Sigma < 0 \quad (2e)$$

This has been confirmed by eddy-resolving codes (Böning and Budich, 1992; Gille and Davis, 1999; McDougall, 2004). There is another physical consideration that leads to (2e). Comparing (2d) and (2b), one concludes that in an adiabatic regime (no sinks), TEE cannot decrease since it is continuously fed by MPE. On the other hand, since the ratio EKE/EPE has been found to be constant (Böning and Budich, 1992, Table 2; Wunsch, 1999), an increase of TEE implies an increase of EPE. Thus, the rhs of the first of (2a) cannot be zero and (2e) follows. Alternatively, in an adiabatic regime one cannot have a stationary state since the absence of sinks means that the variable under consideration can only increase with time: indeed, the lhs of Eqs. (2) must be viewed as the “growth rates” of the corresponding variables. As a corollary, the solution $\text{DEPE}/Dt = 0$ and $\Sigma = 0$ is a mathematical rather than a physical solution.

Finally, consider the dynamic equation for the mean thickness ($h = \partial\rho/\partial z$) in isopycnal coordinates (see OM1). With $\hat{\mathbf{u}} = \bar{\mathbf{u}} + \mathbf{u}_*$, where \mathbf{u}_* is the bolus velocity, we have

$$\bar{h}_t + \hat{\mathbf{u}} \cdot \nabla \bar{h} = -\bar{h} \nabla \hat{\mathbf{u}} \quad (2f)$$

If we compare (2f) with (1c) we notice that the “source of mean thickness” or the bulging of the isopycnals represented by the rhs of (2f), is the physical analog of the rhs term in (1c), a physical process represented in z -coordinates by a diapycnal flux.

In Fig. 1 we present the energy diagram borrowed from Böning and Budich (1992, Fig. 8) in which the loss by MPE is called T_2 while the gain by EKE is called T_3 .

Because of (2e), there is a considerable cancellation between $\mathbf{F}_H \cdot \mathbf{L} > 0$ and $\Sigma < 0$ with the result that the transfer term T_3 is smaller than T_2 . It is therefore useful to view the ratio T_3/T_2 as an “efficiency” or “rate of leakage” factor given by

$$\xi = \frac{F_v}{\mathbf{F}_H \cdot \mathbf{L}} = 1 - \frac{|\Sigma|}{\mathbf{F}_H \cdot \mathbf{L}} \quad (2g)$$

If both numerator and denominator are integrated over z , the results of Böning and Budich (1992, Fig. 9) indicate that $\xi \approx 1/5$, a value that we discuss in Section 5. This indicates that the leakage from EPE to EKE is about 20%.

Thus far, it has always been assumed that $\Sigma = 0$. What are implications of such a choice? It implies that there is no source of EPE, as the first of equation (2a) shows, and that all the MPE goes into EKE, an assumption not supported by detailed energy balance analyses (Böning and Budich, 1992; Beckmann et al., 1994). In Section 3 we shall present the model results for Σ and $\mathbf{F}_H \cdot \mathbf{L}$ and numerical results for ξ .

A non-zero residual flux Σ has also implications concerning whether baroclinic instabilities of the mean flow only extract energy from the mean flow, as the term $\mathbf{F}_H \cdot \mathbf{L}$ in (2d) indicates, or

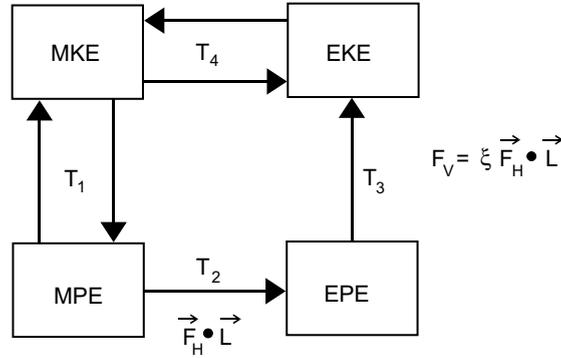


Fig. 1. The energy flow among different components. MKE and MPE stand for mean kinetic and mean potential energies; EPE and EKE stand for mean potential and eddy energies. The fluxes T 's between the different blocks follow the notation of Böning and Budich (1992) and Böning and Budich (1992). In the subpolar gyre ($46\text{--}60^\circ\text{N}$ – $(25\text{--}60)^\circ\text{E}$), the input is MPE (T_4 is directed to the left while T_1 is upward). In the sub-tropical gyre ($10\text{--}30^\circ\text{N}$ – $(25\text{--}60)^\circ\text{E}$), the input is MKE (T_4 is directed to the right while T_1 is downward).

whether they also give rise to mixing. Wunsch and Ferrari (2004, WF4) concluded that mesoscale eddies cannot increase the TPE which they write, “is the requirement for vertical mixing processes”. Consider their Eq. (28) which we rewrite as

$$\overline{g\rho w} = -G(\text{mean}) - G(\text{mesoscale eddies}) - G(\text{turb}) \quad (3a)$$

The term in question is the second one which in our notation is the rhs of Eq. (2c)

$$G(\text{mesoscale eddies}) = \mathbf{F}_H \cdot \mathbf{L} + \Sigma \quad (3b)$$

Since $\mathbf{F}_H \cdot \mathbf{L}$ is positive, its contribution to (3a) is negative, a sink that represents extraction of energy, a process ultimately responsible for the “flattening of the isopycnal surfaces”, a decrease of baroclinic energy and ultimately, a lowering of the center of mass. Such a term produces no mixing. However, in (3b) there is second term not accounted for by WF4. Since Σ is negative, its contribution to (3a) is positive, it raises the center of mass and leads to vertical mixing. *In what follows we present the model results for $\mathbf{u}_M = (\mathbf{u}^+, w^+)$ and Σ .*

5. Model results

Since the derivation of the results entails a certain level complexity and since the original model was already discussed in OM1, we deem it useful to first present the results with a description and interpretation of their physical content. The detailed derivation is then presented in Sections 9, 10 and in the Appendices.

5.1. Eddy-induced velocity

The eddy-induced velocity defined in Eq. (1d) was derived to have the following form in terms of the mean fields:

$$\begin{aligned}
 \mathbf{u}^+ &= -k_m \boldsymbol{\psi} \\
 \boldsymbol{\psi} &= \partial_z \mathbf{L} + \boldsymbol{\psi}(\text{new}) \\
 \boldsymbol{\psi}(\text{new}) &= -\langle \partial_z \mathbf{L} \rangle - (1 + \sigma_t^{-1}) f^{-1} r_d^{-2} \mathbf{e}_z \times (\bar{\mathbf{u}} - \langle \bar{\mathbf{u}} \rangle)
 \end{aligned} \tag{4a}$$

The first term in $\boldsymbol{\psi}$ is of the GM-type while the second term is new. The mesoscale diffusivity $k_m(z)$ was derived to have the form

$$k_m = 1.7s^{1/2} r_d K(z)^{1/2} \tag{4b}$$

where the average $\langle \dots \rangle$ is defined as

$$\langle A \rangle \equiv \int_{-H}^0 A(z) K^{1/2}(z) dz \left[\int_{-H}^0 K^{1/2}(z) dz \right]^{-1} \tag{4c}$$

The notation is as follows. The vector \mathbf{L} is defined in Eq. (1e), f is the Coriolis parameter, r_d the Rossby radius, σ_t is the turbulent Prandtl number and s is a filling factor. The last two variables are discussed below.

Several considerations are in order. *First*, in OM1 it was shown that in computing the rate of production of the total eddy energy given by the integral of $\mathbf{F}_H \cdot \mathbf{L}$, see Eqs. (2b) and (6c) below, the first two terms in $\mathbf{u}^+(z)$ yield a positive contribution implying a drain of energy from the mean fields while the last two terms yield a negative contribution indicating a back-scatter of some of the energy to the mean field. The latter can be viewed as reducing the efficiency of the “flattening of the isopycnals” produced by the first two terms. *Second*, we recall that using an eddy-resolving ocean code, Bryan et al. (1999) concluded that the GM model was not sufficient to represent the data and that “additional terms, not related to the gradient of thickness”, were needed. To interpret $\boldsymbol{\psi}(\text{new})$ in this context, we rewrite it in the alternative form

$$\boldsymbol{\psi}(\text{new}) = A \mathbf{e} \times (\mathbf{u}_d - \bar{\mathbf{u}}) - f^{-1} \boldsymbol{\beta} \tag{4d}$$

where

$$A = (1 + \sigma_t^{-1})(f r_d^2)^{-1} \tag{4e}$$

$$\mathbf{u}_d = \langle \bar{\mathbf{u}} \rangle + (1 + \sigma_t)^{-1} \mathbf{c}_R + A^{-1} \mathbf{e}_z \times \langle \partial_z \mathbf{L} \rangle, \quad \mathbf{c}_R = \sigma_t r_d^2 \mathbf{e} \times \boldsymbol{\beta} \tag{4f}$$

whose physical interpretation is as follows. The term (4d) has a dynamical meaning since it depends on the difference between the mean velocity $\bar{\mathbf{u}}$ and \mathbf{u}_d which, as we show below, is the mesoscale “drift velocity” that characterizes the eddy motion as a whole since “eddies move through the background water at speeds and direction inconsistent with background flow” (Richardson, 1993). The drift velocity is the same throughout the entire vertical eddy’s structure and is z -independent. Finally, \mathbf{c}_R is the velocity of barotropic Rossby waves. It follows from (4a, d–f) that a mesoscale model that uses only the first, down-gradient term (e.g., a GM-type model) is valid only if

$$\bar{\mathbf{u}}(z) = \mathbf{u}_d, \quad \boldsymbol{\beta} = 0 \tag{4g}$$

Eq. (4g) requires that at all points in the z -column the velocity of the mean flow be equal to the drift velocity and that there are no $\boldsymbol{\beta}$ effects. While the latter may be an acceptable approximation,

the first requirement in (4g) is clearly impossible to satisfy since \mathbf{u}_d is a barotropic, z -independent variable while the mean velocity $\bar{\mathbf{u}}(z)$ is z -dependent. Though the first of (4g) may be satisfied at some point, it cannot be true in general. Two more physical considerations are in order concerning the origin and interpretation of \mathbf{u}_d . In solving the mesoscale eigenvalue problem discussed in OM1, the space and time variables (\mathbf{r}, t) were transformed to (\mathbf{k}, ω) variables. The solution of the eigenvalue problem led to the relation

$$\omega = \mathbf{k} \cdot \mathbf{u}_d \quad (4h)$$

where \mathbf{u}_d is given by (4f). To interpret (4h), consider the general Doppler transformation $\omega = \omega_0 + \mathbf{k} \cdot \mathbf{V}$ where \mathbf{V} represents the relative velocity between two frames. Choosing the initial frame to be the system at rest of the eddy's center and assuming the eddy to be axisymmetric, we have $\omega_0 = 0$. Comparing the resulting Doppler relation with (4h), we obtain $\mathbf{u}_d = \mathbf{V}$, that is, \mathbf{u}_d is the velocity of the eddy as a whole which can be called *drift velocity*. Actually, as shown in OM1, the first two relations in Eq. (4a) with (4d–f) were the original solution of the eigenvalue problem which was then transformed to acquire the form of ψ (new) presented in (4a).

5.2. Boundary conditions

Since the mesoscale diffusivity is proportional to $K^{1/2}(z)$, which is the weighing factor in the definition (4c), the eddy-induced velocity $\mathbf{u}^+(z)$ automatically satisfies the relation

$$\int_{-H}^0 \mathbf{u}^+(z) dz = 0 \quad w^+(-H, 0) = 0 \quad (4i)$$

In carrying out the integration in (4i), it can be easily seen that the first two terms in (4a) cancel each out as do the last two terms. This means that in (4a) and/or (4d) the $\langle \dots \rangle$ terms (which are z -independent and thus “barotropic”) are indispensable to satisfy (4i).

5.3. Comparison with the GM model

Since the GM model was primarily intended to account for the energy sink of MPE, it employs only the first term in the second relation (4a), that is

$$\mathbf{u}^+(\text{GM}) = -\partial_z k_m \mathbf{L} \quad (4j)$$

Because of its heuristic nature, within the GM model the mesoscale diffusivity may in principle be inside or outside the derivative. GM chose the form (4j) so as to satisfy the first of (4i) by imposing that

$$k_m(-H, 0) = 0 \quad (4k)$$

However, (4k) is not confirmed by the present model that yields (4b) which has its maximum value at the surface where $K(z)$ is the largest. Recent numerical simulation studies discussed below (Ferreira and Marshall, submitted, Fig. 13; Olbers and Visbeck, in press, Fig. 6) do not confirm (4k) but (4b).

5.4. Non-locality

Muller and Garrett (2002) have pointed out that the effect of mesoscales may not be entirely representable as a local effect with eddy fluxes proportional to local mean gradients. As Eqs. (4a) show, only the first term is proportional to the local gradient while ψ (new) contains the $\langle \dots \rangle$ terms that “sample” the entire water column which represents a non-local effect. Thus, the present model can be viewed as non-local in the vertical.

5.5. Scaling laws

We use the geostrophic approximation to estimate the various terms in the new expression for the eddy-induced velocity, Eqs. (4a). It is clear that the first and the second terms are of the same order and so are the third and the fourth. We further derive that $|\mathbf{L}| \sim O(f\bar{u}/HN^2)$ and $\psi \sim Ro/r_d$ where $Ro = \bar{u}/fr_d$ and $r_d = NH/f$ are the Rossby number and deformation radius. The ratio of the first to the third term is $(fr_d/NH)^2 \sim O(1)$ and thus the new terms in (4a) are of the same order as the first term.

5.6. Mesoscale kinetic energy

The model yields the following result:

$$K(z) = K_t \Gamma(z) \quad (5a)$$

with K_t is the surface eddy kinetic energy (the subscript t stands for top) to be discussed below. The dimensionless function $\Gamma(z)$ is given by

$$\Gamma(z) = (|B_1(z)|^2 + |a_0|^2)(1 + |a_0|^2)^{-1} \quad (5b)$$

where $B_1(z)$ is the first baroclinic mode solution of the eigenvalue problem

$$\frac{\partial}{\partial z} \left(N^{-2} \frac{\partial B_1}{\partial z} \right) + (r_d f)^{-2} B_1(z) = 0 \quad (5c)$$

with the boundary conditions $\partial_z B_1 = 0$, at $z = -H, 0$ and $B_1(0) = 1$. The physical interpretation of (5b) is as follows. B_1 represents the *baroclinic contribution* while the a_0 terms represent the *barotropic one*. The partitioning of the eddy kinetic energy between the two modes has been studied by Wunsch (1997) who, among others things, concluded that the first baroclinic mode $B_1(z)$ dominates the surface kinetic energy. In the numerical estimates presented below we use $N^2(z) = N_0^2 e^{-2z}$, $N_0 = 7 \times 10^{-3} \text{s}^{-1}$ (z in km) first suggested by Garrett and Munk (1972) and used recently by Zang and Wunsch (2001, their Appendix 1). For maps of $N(z)$, see Emery et al. (1984).

The barotropic contribution to $K(z)$ is represented by the variable a_0 the form of which is given by

$$|a_0|^2 (1 + |a_0|^2)^{-1} K_t = \frac{1}{2} (1 + 2c_1) \mathbf{I}_1^2 + \mathbf{I}_1 \cdot \mathbf{I}_2 + \text{trace}(\mathcal{A}_{ij} + \mathcal{Q}_{ij}) \quad (5d)$$

where the vectors $\mathbf{I}_{1,2}$ and the tensors \mathcal{A} and \mathcal{Q} are given by

$$\mathbf{I}_1 = c_2 \overline{\psi_*} \quad \mathbf{I}_2 = -c_2 \overline{\Gamma^{-1/2} \psi} \quad \psi_* \equiv \psi \operatorname{sign} B_1 \quad (5e)$$

$$A_{ij} = c_3 \overline{\Pi_i \Pi_j \Gamma^{-1/2} \operatorname{sign} B_1} \quad \Omega_{ij} = c_4 \overline{\Pi_i \Pi_j} \quad \mathbf{\Pi} = f \boldsymbol{\psi} - f \partial_z \mathbf{L} + \boldsymbol{\beta} \quad (5f)$$

The constants c 's and the overbar are defined as

$$c_1 = \overline{\Gamma^{1/2}(z)}^{-1} \overline{\operatorname{sign} B_1}, \quad c_2 = 1.7 s^{1/2} \sigma_t^{-1/2} f r_d^2, \quad c_3 = \sigma_t^4 (1 + \sigma_t)^{-2} r_d^4, \quad c_4 = -c_1 c_3$$

$$\overline{X} \equiv H^{-1} \int_{-H}^0 X(z) dz \quad (5g)$$

Using Eqs. (5), we have compared the model predicted profile $K(z)/K_t$ with the measurements for the vertical section along 55°W in the interval 15°N – 45°N (Richardson, 1983a). The maximum of the eddy kinetic energy occurs at 38°N (Gulf Stream). To compute r_d and ψ , we needed the profiles of $\overline{\mathbf{u}}(z)$, $N(z)$ and $\nabla \overline{h}$ which however are not given in Richardson (1983a). For this reason, we obtained them from (Richardson, 1983b; Owens, 1984; Antonov et al., 1998; Boyer et al., 1998). The behavior predicted by the model presented in Fig. 2 is in good agreement with the measured profile by Richardson (1983a). Fig. 2 is also in agreement with the numerical simulation of Böning and Budich (1992), their Fig. 6a, b and with Fig. 14 of Wunsch (1997). An additional prediction of the present model is that the profile of $K(z)/K_t N(z)$ has a minimum at $z \sim -1$ km, in agreement with measured values (Schmitz, 1994).

5.7. Surface eddy kinetic energy K_t

For the surface value of the eddy kinetic energy, the model yields the following result (see OM1):

$$K_t = 3\eta r_d \varphi_3^{-1} \Phi \quad (6a)$$

where (p is the mean pressure):

$$\Phi = -1.7 s^{1/2} f r_d \rho_0^{-1} \int_{-H}^0 dz \Gamma^{1/2}(z) \boldsymbol{\psi} \cdot \nabla_{\text{HP}}, \quad \varphi_3 = \int_{-H}^0 dz |B_1(z)|^3 \quad (6b)$$

The parameter η will be discussed below. While these expressions are fully calculable once the mesoscale model is used with an ocean OGCM to provide the mean fields, like the mean pressure in (6b), it is possible to present a more physical expression by recalling that the rate of production of total eddy energy is given by Eq. (2b)

$$P_{\text{T}} (\text{m}^3 \text{s}^{-3}) = \int_{-H}^0 dz \mathbf{F}_{\text{H}} \cdot \mathbf{L} = \rho_0^{-1} \int_{-H}^0 dz \mathbf{u}^+ \cdot \nabla_{\text{HP}} \quad (6c)$$

In the last step, we used the first of (1d), integrated by parts and used the hydrostatic equilibrium equation. Using the expression for the eddy-induced velocity, one can relate (6c) to the function Φ . The result is

$$P_{\text{T}} = (3\eta r_d \varphi_3^{-1})^{1/2} \Phi^{3/2} \quad (6d)$$

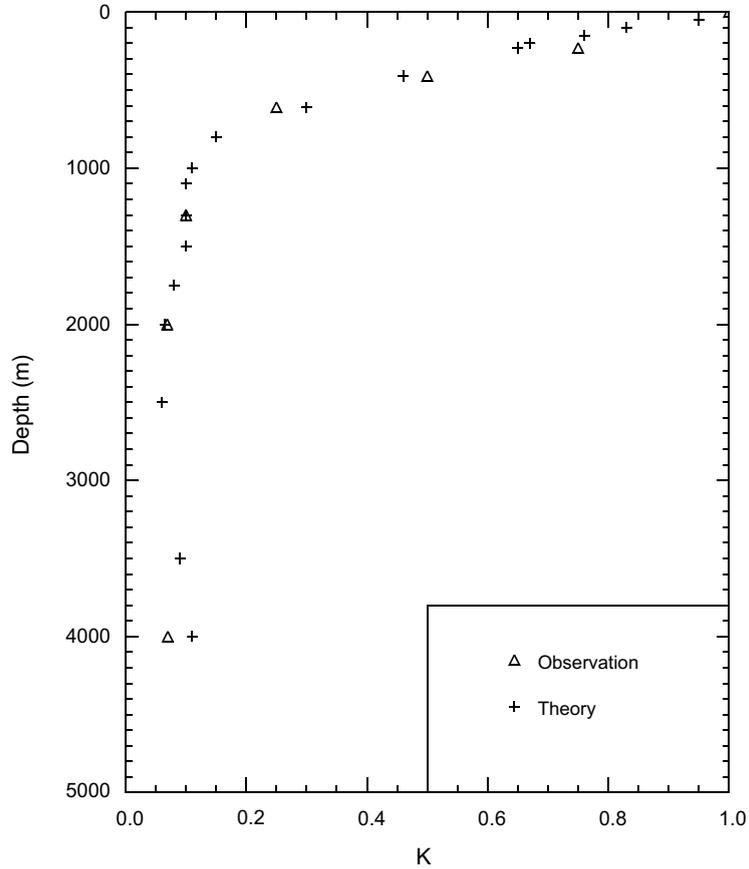


Fig. 2. The predicted profile of the eddy kinetic energy K in units of its surface value, K_t . The data are from Richardson (1983a).

and thus from (6a) we have

$$K_t = (3\eta r_d \varphi_3^{-1})^{2/3} P_T^{2/3} \tag{6e}$$

which is physically more transparent since it relates the eddy surface kinetic energy to the power available to the mesoscale field. Using computed values $r_d = 25$ km, and $\varphi_3 = 325$ m we obtain

$$K_t = 37.5(\eta P_T^*)^{2/3} \text{ cm}^2 \text{ s}^{-2} \quad P_T^* = P_T / (10^{-6} \text{ m}^3 \text{ s}^{-3}) \tag{6f}$$

In Fig. 6 of Böning and Budich (1992) the value of K_t for the subpolar region is $35 \text{ cm}^2 \text{ s}^{-2}$. We should recall that the nominal value of 25 km for the Rossby radius used in this paper is typical of latitudes about 40°N (Emery et al., 1984, Fig. 5a,b). For the meridional dependence (at 30°W) of the surface eddy kinetic energy, see Fig. 1 of Beckmann et al. (1994). Data on the eddy kinetic energy from Topex–Poseidon are discussed by Stammer and Dietrich (1999).

5.8. Ocean energy and dissipation data

Given the still incomplete understanding of the energy budget in the ocean (Wunsch and Ferrari, 2004; Huang, 2004), it is difficult to make unequivocal comparisons with ocean energy data. For example, using the GM model $\mathbf{F}_H = k_m N^2 \mathbf{L}$, Eq. (6c) gives

$$P_T(\text{GM}) = \int k_m N^2 \mathbf{L}^2 dz = \int k_m \bar{b}_z^{-1} (\nabla_H \bar{b})^2 dz \quad (6g)$$

which is the expression employed by previous authors (Gent et al., 1995; Treguier et al., 1997; Ferrari and McWilliams, 2004) who arrived at (6g) in a way different from ours. In fact, these authors did not consider $D(\text{TEE})/Dt$ as we do, but rather the total potential energy TPE given by Eq. (2c). However, since they assumed $\Sigma = 0$, the rhs of (2c) coincides with the rhs of Eq. (2b) with an obvious sign difference. Treguier et al. (1997) compared (6g) with FRAM data while Ferrari and McWilliams (2004) compared (6g) with the ~ 1 TW of energy input by the wind into the ocean (Wunsch, 1998). The present model predicts a well defined value for P_T once Eq. (6c) is computed. For the time being, we use $f = 10^{-4} \text{ s}^{-1}$, $u^+ \sim 10^{-3} \text{ m s}^{-1}$, in which case Eq. (6c) gives (A is the ocean's surface area)

$$\rho A P_T \approx \frac{1}{3} \text{ TW} \quad (6h)$$

to be compared with ~ 1 TW which represents the work by the wind (Wunsch, 1998).

5.9. Mesoscale diffusivity: Time and length scales

Consider (4b). Since the eddy kinetic energy depends on z and since the Rossby radius depends on the geographic location, the predicted mesoscale diffusivity is neither horizontally nor vertically uniform, in agreement with what several authors have concluded (e.g., Stammer, 1998). Furthermore, if one writes the mesoscale diffusivity as

$$k_m \sim l_{\text{eddy}}^2 t_{\text{eddy}}^{-1} \quad (6i)$$

where l_{eddy} is an eddy's mixing length and t_{eddy} is an internal time scale, we conclude that

$$l_{\text{eddy}} \sim r_d, \quad t_{\text{eddy}} \sim r_d K^{-1/2} \quad (6j)$$

In the past, the nature of both l_{eddy} and t_{eddy} was discussed by several authors who suggested different forms. Stone (1972) suggested the first of (6j) while Held and Larichev (1996) suggested the larger Rhines scale

$$l_{\text{eddy}} \sim (U/\beta)^{1/2} \quad (6k)$$

where U is a characteristic mean flow velocity and $\beta = \partial f / \partial y$. Using the Topex–Poseidon data, Stammer (1997, 1998) concluded that l_{eddy} is given by the first of (6j) rather than (6k). Using typical values of $K_t \sim (0.5–1) \times 10^{-2} \text{ m}^2 \text{ s}^{-2}$ (Stammer, 1998), and $r_d \approx 30 \text{ km}$, Eq. (4b) gives

$$k_m(\text{surface}) \approx 3 \text{ s}^{1/2} 10^3 \text{ m}^2 \text{ s}^{-1} \quad (6l)$$

Since the filling factor $s < 1$ (see below), Eq. (6l) is close to the heuristic values used in the GM model (Gent and McWilliams, 1990; Gent et al., 1995). Karsten and Marshall (2002) used the models of Holloway (1986) and Keffer and Holloway (2002) and the TOPEX Poseidon data for the sea surface height. Their result, shown in their Fig. 2, is

$$10^3 \leq k_m(\text{surface}) < 3 \times 10^3 \text{ m}^2 \text{ s}^{-1} \quad (6m)$$

in agreement with (6l). It may be of physical interest to relate the mesoscale diffusivity to the power P_T . Using (6e) we obtain

$$k_m(\text{surface}) = Cr_d^{4/3} \varphi_3^{-1/3} P_T^{1/3} = 10^3 CP_T^{*1/3} \text{ m}^2 \text{ s}^{-1} \quad (6n)$$

where $C = 2.45s^{1/2}\eta^{1/3}$. In the second relation we have used typical value of $\varphi_3 \approx H = 1$ km, $r_d = 30$ km as well as the dimensionless variable P_T^* defined in (6f). The last relation is in agreement with (6m).

5.10. Mesoscale diffusivity: z -dependence

The z -dependence of the mesoscale diffusivity has been the subject of considerable interest in the past. Since our model (4b) predicts a diffusivity proportional to $K^{1/2}(z)$, in the present model the predicted behavior of $k_m(z)$ has a form similar to that depicted in Fig. 2, that is, $k_m(z)$ is the largest at the surface, followed by a rapid decrease, ultimately reaching a value constant with depth. This behavior is in accordance with the heuristic expressions suggested by Visbeck et al. (1997) and Karsten and Marshall (2002). Using an eddy-resolving code, Bryan et al. (1999) found that an overall consistent ocean model was obtained with

$$k_m(z) = 0.13r_d^2 f Ri^{-1/2} \quad (6o)$$

Since Ri is the smallest near the surface and then increases, (6o) predicts a behavior similar to that of our model. More recently, Ferreira and Marshall (submitted, Fig. 13) have numerically studied the problem and their result for $k_m(z)$ is very similar to ours. Olbers and Visbeck (in press, Fig. 6) have also concluded that they “require” a mesoscale diffusivity of the form obtained in this model.

5.11. Eddy potential energy EPE

The eddy potential energy EPE is given by (m^2s^{-2}):

$$\text{EPE} \equiv \frac{1}{2} N^2 \overline{z^2} = \sigma_t \left(\frac{fr_d}{N} \right)^2 K_t \left(\frac{\partial B_1}{\partial z} \right)^2 \quad (7a)$$

where the z -dependence is due to the eigenfunction $B_1(z)$, Eq. (5c). The EPE given by Eq. (7a) is plotted in Fig. 3a in units of $\xi^{-1}(1 - \xi)K_t$ where the Prandtl number σ_t is expressed in terms of ξ in Eq. (8d) below. As Böning and Budich (1992) have noticed “while the EKE (Fig. 1) decreases with increasing depth through the main thermocline, maximum values of EPE are found at the subsurface”. The profile of EPE predicted by this model is in agreement with the results of Böning and Budich (1992), their Fig. 6. In particular, using $\xi = 1/5$ (see 8f), in Fig. 3b we plot EPE/K_t only to a depth of 1400 m to facilitate the comparison with Böning and Budich’s results (their Fig. 6)

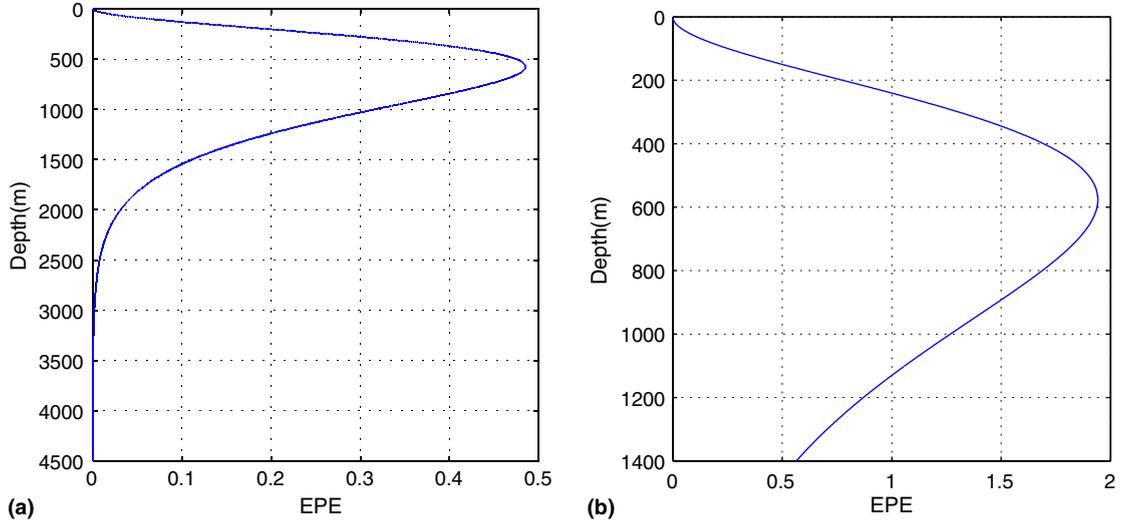


Fig. 3. (a) The predicted profile of the eddy potential energy EPE, Eq. (7a). The units are $\xi^{-1}(1 - \xi)K_t$ where ξ is defined in Eqs. (2g) and (8e,f). (b) Same as in Fig. 3a but for $\xi = 1/5$ and only up to depth of 1400 m to facilitate the comparison with the eddy-resolving data of (Böning and Budich, 1992). The shapes of the profiles are quite similar.

where the maximum values of EPE/K_t vary between 2.7 and 3.5. Our maximum value is close to those values. However, the most important feature is that the predicted profile of EPE is quite similar to the one obtained with a high resolution OGCM.

5.12. EPE

Integrating Eq. (7a) over z and expressing the result in terms of P_T , we obtain

$$\int_{-H}^0 dz EPE(z) = \xi^{-1}(1 - \xi)B_*(\eta P_T^*)^{2/3}, \quad B_* = 3.75 \times 10^{-3} \int_{-H}^0 dz B_1^2(z) \quad (7b)$$

We computed $B_* = 1.81$ m. Multiplying (7b) by ρA where $A = 3.6 \times 10^{14}$ m² is the ocean's surface area, we obtain (1ExJ = 10^{18} J):

$$EPE = 0.65 \xi^{-1}(1 - \xi)(\eta P_T^*)^{2/3} = 2.6(\eta P_T^*)^{2/3} \text{ExJ} \quad (7c)$$

which relates EPE to the rate of production of TEE and to ξ . In the last step we used $\xi = 1/5$, see Eq. (8f). Since $\eta > 1$ and $P_T^* = 3$ (see below), a value of $\eta = 5$ would yield 16ExJ, in agreement with the values cited by WF4.

5.13. Residual buoyancy flux

For the residual diapycnal buoyancy flux Σ defined in Eq. (1e), the model gives the following results:

$$\Sigma(z) = -k_m^r(z)N^2, \quad k_m^r(z) = \Gamma_m^r(z) \frac{\varepsilon_m}{N^2} \quad (7d)$$

where

$$\Gamma_m^r(z) = (1 + \sigma_t^{-1})^{-1} \left(\frac{r_{df}}{N} \right)^2 \varphi_3 \mathfrak{M}(z) \quad (7e)$$

$$\varepsilon_m = P_T \varphi_3^{-1} \quad (7f)$$

$$\mathfrak{M}(z) = \left(\frac{\partial B_1}{\partial z} \right)^2 \left(\int_{-H}^0 B_1^2(z) dz \right)^{-1} \quad (7g)$$

The form of $\mathfrak{M}(z)$ assures that the boundary condition (1g) is satisfied since as z approaches $-H$ and 0, $B_1(z)$ tends to a constant value. Here, $k_m^r(z)$ is the diapycnal residual diffusivity, $\Gamma_m^r(z)$ is a depth-dependent mixing efficiency and ε_m is the mesoscale dissipation. Several considerations are in order. The fact that the model yields a negative Σ is quite reassuring since such a behavior is indeed expected from general physical arguments about the physical role of Σ , as discussed in Section 4. The vertical diffusivity is directly proportional to the power P_T which serves to link the strength of the residual flux to the power available to the mesoscale field as drawn from the MPE. Finally, in Eq. (1c) the two types of mesoscale terms are of equal importance if

$$k_m^r \approx \mathbf{L}^2 k_m \quad (7h)$$

which represents an interesting relation between the two mesoscale diffusivities. In Fig. 4 we plot $\Sigma(z)$ normalized to $(1 - \xi)P_T^*$. The most important implication is that the bulk of the transfer from MPE to EPE is predicted to occur in the first 500 m, in accordance with previous studies (Gill et al., 1974; Böning and Budich, 1992; Beckmann et al., 1994).

5.14. Residual vertical diffusivity k_m^r

This diffusivity k_m^r , Eq. (7d), is plotted in Fig. 5 normalized to $(1 - \xi)P_T^*$. As expected, it vanished at the top and bottom but it reaches a value of almost $1 \text{ cm}^2 \text{ s}^{-1}$ at a depth of about 1 km.

5.15. Residual mesoscale mixing efficiency

This variable defined in Eq. (7e) is plotted in Fig. 6a. Contrary to the Osborn–Cox mixing efficiency for the thermocline which is usually taken to be a constant, the one corresponding to the mesoscales residual flux depends on depth.

5.16. Vertical flux

Eq. (2a) for EKE shows that the source of eddy kinetic energy is the vertical flux $F_v = \overline{w''b''}$. Using the first of (1d), together with (1e) and (7d) and $F_v(-H, 0) = 0$, we derive

$$F_v = \overline{w''b''} = k_m^v N^2 > 0 \quad k_m^v(z) = -k_m^r(z) - \mathbf{L} \cdot \int_{-H}^z \mathbf{u}^+(z') dz' \quad (8a)$$

where k_m^v is the vertical diffusivity.

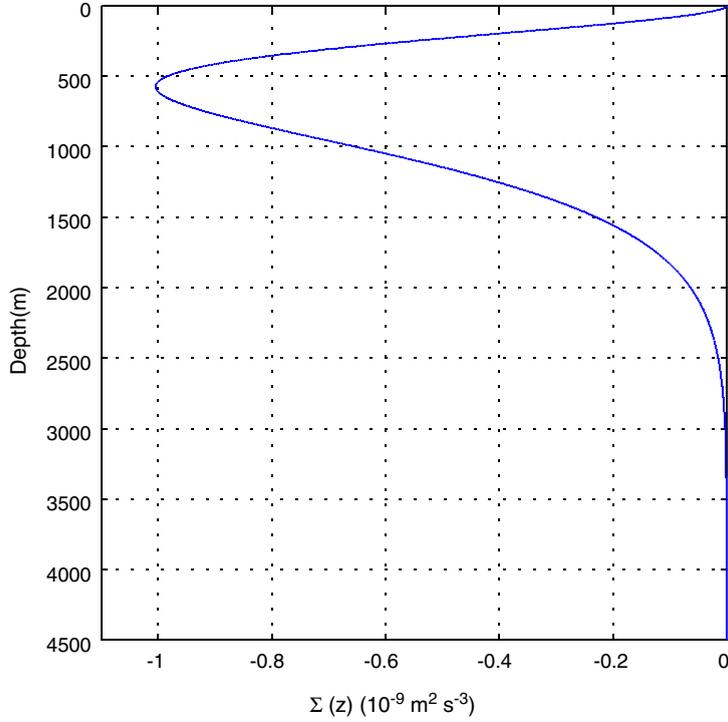


Fig. 4. Profile of the Σ term defined in Eq. (1e) and given by Eqs. (7d–g). As expected, it is negative, see discussion in Section 4. The normalization is $(1 - \xi)P_T^*$.

5.17. Diapycnal diffusivity induced by mesoscales

The two diffusivities k_m^r and k_m^v combine in a particularly interesting way since the rhs of (2b) can be written as

$$\mathbf{F}_H \cdot \mathbf{L} = F_v - \Sigma = k_m^d N^2 \quad (8b)$$

where the *mesoscale-induced diapycnal diffusivity* is given by

$$k_m^d = k_m^v + k_m^r = -\mathbf{L} \cdot \int_{-H}^z \mathbf{u}^+(z') dz' \quad (8c)$$

which is entirely expressed in terms of the eddy-induced velocity.

5.18. Efficiency ξ

The efficiency ξ is defined in Eq. (2g). Integrating over z both numerator and denominator and using Eqs. (7d–g), (6c) and (5c), we obtain

$$\xi = (1 + \sigma_t)^{-1} \quad (8d)$$

which is used to express the turbulent Prandtl number σ_t in terms of ξ which we determine next.

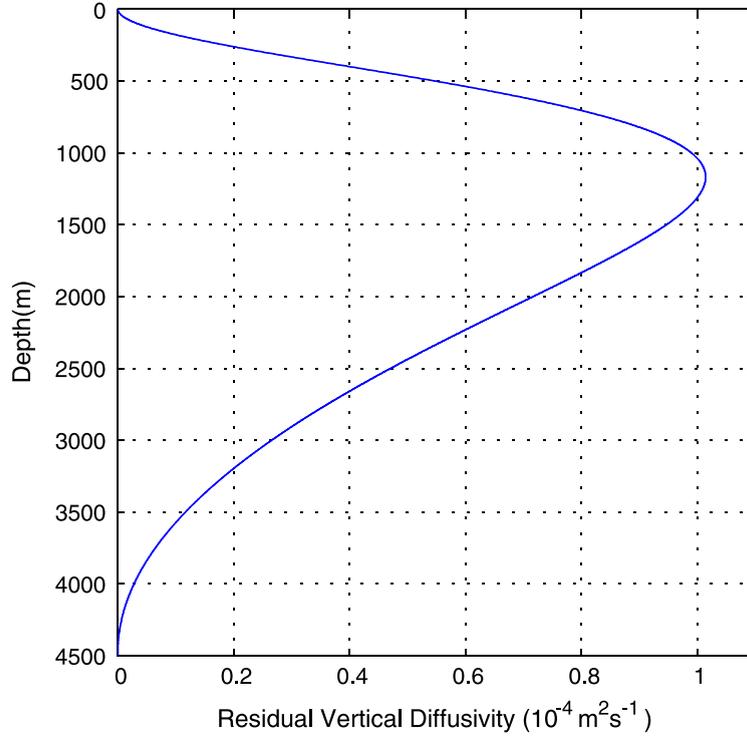


Fig. 5. Profile of the residual vertical diffusivity defined in Eq. (7d) normalized to $(1 - \xi)P_T^*$. As one can observe, values up to $1 \text{ cm}^2 \text{ s}^{-1}$ can be obtained.

5.19. Eddy kinetic and potential energy

Using Eqs. (5a) and (5b) with only the baroclinic part, as well as (7a), we obtain that the ratio of the z-integrated EKE and EPE is given by

$$\frac{\text{EKE}}{\text{EPE}} = \xi(1 - \xi)^{-1} \tag{8e}$$

which can be used to determine the value of ξ once the ratio on the lhs is known. Wunsch (1999) has recently studied this problem and his results are presented in his Fig. 2a. Böning and Budich (1992) carried out two simulations with coarse ($1/3^\circ \times 0.4^\circ$) and fine ($1/6^\circ \times 0.2^\circ$) resolutions. In the latter case, the ratio (8e) was “strikingly stable” around 1/4 (their Table 2), which implies:

$$\xi \cong 1/5 \tag{8f}$$

This confirms our first result discussed after (2g). The implications of (8f) have already been discussed after Eq. (2g). All previous analyses assumed $\xi = 1$ which corresponds to a complete transfer of MPE to EKE.

5.20. Filling factor

In Eq. (4b), the variable “s” represents the fraction of the flow’s area occupied by the meso-scales, a filling factor. For example, in Denmark Strait, the observed eddies have a diameter of

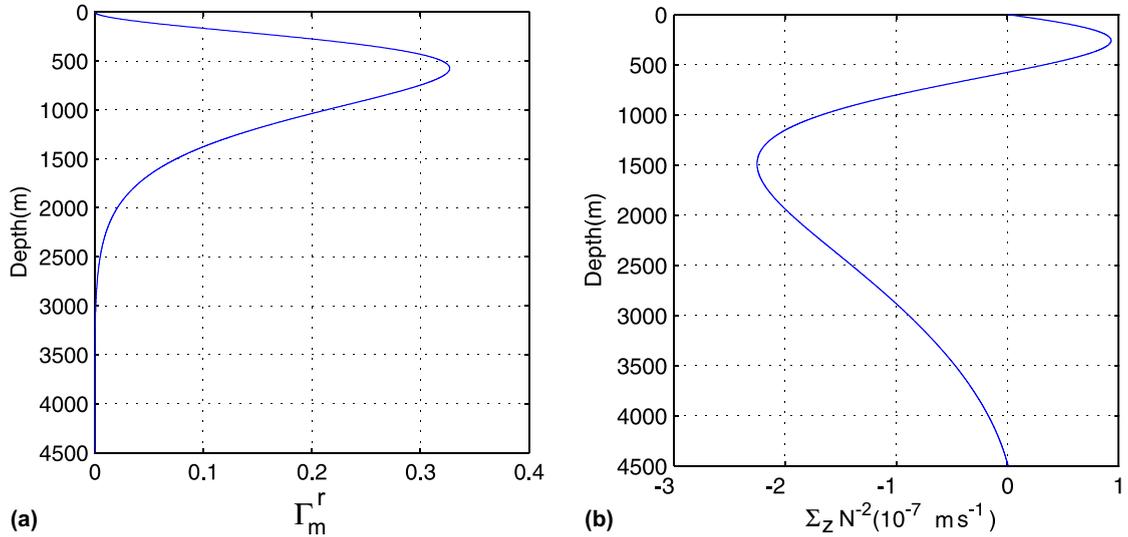


Fig. 6. (a) Profile of the mesoscale-induced mixing efficiency defined in Eq. (7e). Contrary to the Osborn–Cox mixing efficiency in the thermocline which is constant, in this case it is z -dependent. (b) From Eq. (9d) the term $N^{-2}\Sigma_z$ corresponds to an additional, mesoscale-induced diapycnal velocity to be added to the diabatic velocity in the Munk–Wunsch model for the MOC. See Fig. 7a.

about 30 km (the deformation radius is about 10 km) and a separation between eddy centers of about 70 km (Cooper, 1995; Bruce, 1995). Thus, $s \approx 0.15$. Theoretically, s can be obtained using the CD turbulence model to the whole spectra of eddies. However, at the present stage of the mesoscale modeling, we apply the turbulence model only in the vicinity of the maxima of mesoscale spectra. In this approximation, s cannot be computed.

5.21. Diabatic and adiabatic processes: stirring and mixing

In OM1 it was shown that the parameter η entering (6a) is defined as

$$\eta = \tilde{\eta}^{-1} - 1, \quad \tilde{\eta} = \varepsilon_d / \varepsilon_p \quad (8g)$$

where $\tilde{\eta}$ represents the ratio of the rate of the eddy energy dissipation (by small scales) ε_d to the eddy potential energy flux ε_p . It is important to stress that if the mesoscale model were treated with an *adiabatic approximation at all scales*, there would be no dissipation ε_d which means $\eta \rightarrow \infty$ corresponding to an unphysical piling-up of EKE, see Eq. (6a). The problem of the “ultimate fate of the eddy potential energy” (Tandon and Garrett, 1996) cannot be resolved within a purely adiabatic mesoscale model since to include dissipative processes, one must include diabatic processes. Though this topic was discussed in OM1, we recall here some important aspects of the EPE/EKE cycle that drains eddy potential energy from mean potential energy at the large scales and transfers it to smaller scales via an ordinary cascade process (see Fig. 1 of OM1). EPE flows from large to small scales until the turbulence dynamical time scale becomes equal to the Coriolis time scale and allows the release of EPE. At that scale, both the Rossby and the Richardson num-

bers become of order unity thus enabling shear fluctuations to generate vertical turbulence. At those scales, a fraction of the EPE gets dissipated (ε_d) while the rest goes into generating EKE which flows upscale to be returned partly to the EPE and partly to the mean velocity field. Extant models only treat the stirring phase of the entire mesoscale energy processes (Muller and Garrett, 2002). In the present model, this is no longer so since we have

$$k_m \sim \eta^{1/2} \sim \varepsilon_d^{-1/2} (\varepsilon_p - \varepsilon_d)^{1/2} \quad (8h)$$

Thus, mixing by the small dissipative scales (diabatic) affect the mesoscale diffusivity that represents stirring (adiabatic), as indeed expected when kinetic energy flows from small to large scales (Garrett, 2001). The model yields the following expression for η

$$\eta = 10^2 C f^{-1} \int_{-H}^0 |B_1^3(z)| dz \left[\int_{-H}^0 N(z) |B_1^3(z)| dz \right]^{-1} = 1.82 C \quad (8i)$$

where C is a universal constant of order unity. In the last step in (8i) we have used the $N(z)$ described after Eq. (6e).

5.22. Temporal–residual mean approach (McDougall, 2001)

In this approach, a distinction is made between the averaging density at a given height and averaging the height of a given density surface, in which case the buoyancy is denoted by \tilde{b} the dynamic equation of which entails the stream function Ψ_{TRM} (McDougall, 2001, Eq. (7)) which in our model is given by

$$N^2 \Psi_{\text{TRM}} = \mathbf{F}_H + \sigma_t (fr_d N^{-1})^2 K_t \left(\frac{\partial B_1}{\partial z} \right)^2 \partial_z \bar{\mathbf{u}} \quad (8j)$$

Clearly we have $\Psi_{\text{TRM}}(-H, 0) = 0$.

5.23. The advection–diffusion problem

Munk (1966) and Munk and Wunsch (1988) have proposed the well-known advection–diffusion model for the meridional overturning circulation. They employed a diapycnal diffusivity (uniform or not) that was assumed to be caused by diabatic processes represented by a flux F_d . If we include the diabatic contribution, the mean buoyancy equation (1c) becomes

$$\bar{b}_t + (\bar{\mathbf{u}} + \mathbf{u}^+) \cdot \nabla_H \bar{b} + (\bar{w} + w^+) N^2 = - \frac{\partial \Sigma}{\partial z} - \frac{\partial F_d}{\partial z} \quad (9a)$$

If, as customary, the diabatic flux is represented in the form

$$F_d = -k_d N^2 \quad (9b)$$

a value of the diabatic diffusivity k_d of $1 \text{ cm}^2 \text{ s}^{-1}$ was required by Munk and Wunsch to explain their model of the MOC. Since the measured value of the so-called “pelagic” diapycnal diffusivity is about ten times smaller, the problem has been amply discussed in the literature. Here, we want to discuss the mesoscale contribution to the problem. Eq. (9a) can be rewritten as

$$\bar{b}_t + \bar{\mathbf{u}} \cdot \nabla_H \bar{b} + \bar{w} N^2 = w^* N^2 \quad (9c)$$

where w^* is the velocity across mean isopycnals

$$-w^* = \underbrace{w^+ - \mathbf{u}^+ \cdot \mathbf{L}}_{\text{Mesoscale adiabatic}} + N^{-2} \frac{\partial \Sigma}{\partial z} + \underbrace{N^{-2} \frac{\partial F_d}{\partial z}}_{\text{MW-diabatic}} \quad (9d)$$

The last term corresponds to the MW-diabatic process while the first three terms correspond to the adiabatic, mesoscale-induced processes. In Fig. 6b we plot the third term in (9d).

Using an eddy-resolving code, Radko and Marshall (2004a,b, cited as RM) have studied w^* . Lack of a mesoscale model forced them to employ a heuristic expression for $w^* = \varepsilon \nabla^2 h$ whereas the present model gives an expression for all of them. The last term in (9d) is modeled in Canuto et al. (2001, 2002, 2004). RM concluded that “largest part of the diapycnal flux is due to the action of eddies rather than to small-scale vertical mixing”. The results of our model are as follows. Using the first of (7d) and (9b), Eq. (9d) can also be written as ($w^r = \partial k_m^r / \partial z$, $w^d = \partial k_d / \partial z$)

$$-w^* = \underbrace{w^+ - \mathbf{u}^+ \cdot \mathbf{L}}_{\text{Mesoscale adiabatic}} + \underbrace{w^r + k_m^r N^{-2} N_z^2}_{\text{mesoscale-adiabatic due to } \Sigma} + \underbrace{w^d + k_d N^{-2} N_z^2}_{\text{MW-diabatic}} \quad (9e)$$

While the first two terms on the rhs of (9e) can be evaluated once the mesoscale model is solved, the third and fourth terms can be estimated and so are the last two terms using a turbulence model (Canuto et al., 2001, 2002, 2004). The third and fourth terms are shown in Fig. 7a while the last two terms are shown in Fig. 7b.

At $z = 1$ km one can see that the mesoscale adiabatic terms are some 10 times larger than the diabatic mixing terms, in accordance with the conclusions of Radko and Marshall (2004a,b). Since the “flattening of isopycnals” is mainly due to the vertical advection process (Treguier et al., 1997; Treguier, 1999), the presence of the third and fourth terms in (9e) may significantly affect the role of mesoscales. From Fig. 7a (in Munk’s paper of, 1966, the upward vertical motion was estimated to be $1.43 \times 10^{-7} \text{ m s}^{-1}$) we further observe that in the first 500 m, the mesoscale contribution corresponds to *down-welling* while beneath 500 m it contributes to *up-welling*. The “cancellation” in the first 500 m may help reduce the “excessive up-welling in the Pacific” found in z -coordinate OGCMs (without the Σ term) discussed by Hirst and McDougall (1998, HMD, Fig. 3) which is not supported by observations (cited by HMD) and which the GM model does not help resolve. Since an analysis of the same region with an isopycnal OGCM does not exhibit such excessive upwelling (Sun and Bleck, 2001, Fig. 5b), it is conceivable that the missing Σ term in z -OGCMs may be the cause of the difference. The “cancellation” may weaken the upwelling in the upper parts of the water column and yield results more in line with both measurements and isopycnal OGCMs.

Abyssal circulation. In the classical Stommel–Arons model (for a detailed discussion, see Pedlosky, 1998, Chapter 7), the dynamics of the abyssal waters (>1 km) is set in motion by a diapycnal velocity w_∞ at the base of the thermocline. Its value is usually taken to be

$$w_\infty = k_d / D \quad (9f)$$

where D is the scale height of the heat flux. While in the original Stommel–Arons theory w_∞ was contributed by diabatic processes only, and as such, it is just a simplified form of the last term in (9e), the presence of mesoscales means that its form is actually given by

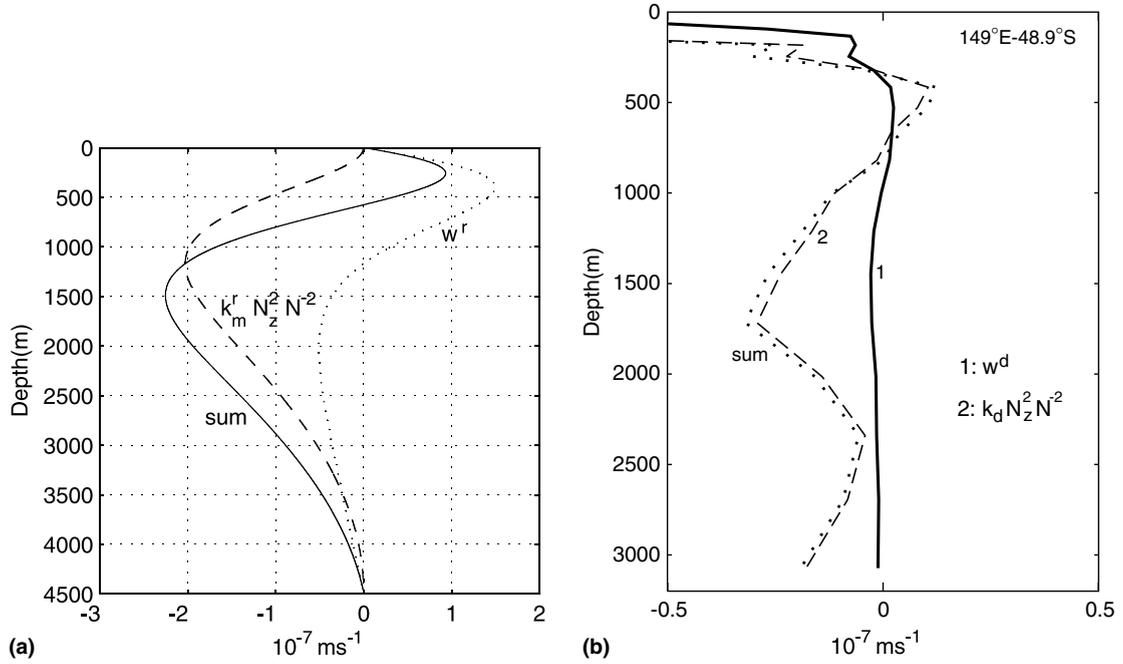


Fig. 7. (a) The third and fourth terms in Eq. (9e) as well as their sum in units of 10^{-7} m s^{-1} . The normalization is $(1 - \xi)P_T^*$. We recall that the original “upward vertical velocity” in Munk (1966) was $1.43 \times 10^{-7} \text{ m s}^{-1}$. We must also recall that the numerical values are contingent upon the use of the stratification law discussed after Eq. (5c). (b) The fifth and sixth terms in Eq. (9e) as well as their sum in units of 10^{-7} m s^{-1} as from the GISS mixing model (Canuto et al., 2001, 2002). As one can observe, at the base of the thermocline $\sim 1 \text{ km}$ depth, the (diabatic) diapycnal velocity is much smaller than the one induced by the mesoscales exhibited in Fig. 7a (see however the last remark in the caption of Fig. 7a).

$$w_\infty = w^* \tag{9g}$$

As discussed by Pedlosky (1998, Chapter 7), to obtain a flux of about 20 Sv from the abyss to the thermocline, a value $w_\infty = 0.7 \times 10^{-7} \text{ m s}^{-1}$ is required. Without the contribution of mesoscales, diabatic processes characterized by a global $0.1 \text{ cm}^2 \text{ s}^{-1}$ diffusivity and with a $D = 1 \text{ km}$, fall short of what is required. On the other hand, Fig. 7a shows that the addition of mesoscales would significantly help in fulfilling the required value of w_∞ .

5.24. Vertical diffusivity vs. wind stresses

The well-known relation $OV \sim k_d^{2/3}$ between the strength of the meridional overturning (OV) and the diapycnal diffusivity (Bryan, 1987), has recently been generalized by Gnanadesikan (1999) to include the effect of wind stresses. In a recent study, Klinger et al. (2003) conclude that “the globally averaged vertical diffusivity may need to be on the low side of the observational estimates ($0.1 \text{ cm}^2 \text{ s}^{-1}$) for the OV to be dominated by southern winds”, as suggested by Toggweiler and Samuels (1998). Since mesoscales contribute an additional diapycnal diffusivity with a larger

patchiness than the ubiquitous breaking of internal gravity waves responsible for $0.1 \text{ cm}^2\text{s}^{-1}$, it would be interesting to revisit Klinger et al.'s conclusion in the presence of mesoscales.

6. Mean momentum equations: Reynolds stresses

6.1. Eulerian mean (EM) formalism

Neglecting diapycnal terms due to diabatic process which are modeled elsewhere (Canuto et al., 2001, 2002, 2004), the dynamic equations for the two-dimensional mean velocity field in the EM formalism are given by (Gent and McWilliams, 1996)

$$D_t \bar{\mathbf{u}} + f \mathbf{e}_z \times \bar{\mathbf{u}} = -\nabla_H \bar{p} - \nabla \cdot \mathbf{F}_m \quad (10a)$$

where $\rho_0 = 1$, $D_t = \partial_t + \bar{\mathbf{U}} \cdot \nabla$ is the advective derivative and $\bar{\mathbf{U}} = \bar{\mathbf{u}} + \overline{w} \mathbf{e}_z$. Mesoscales contribute the term $\nabla \cdot \mathbf{F}_m$ given by

$$\nabla \cdot \mathbf{F}_m \equiv \overline{\mathbf{U}'' \cdot \nabla \mathbf{u}''} = \nabla \cdot \overline{\mathbf{U}'' \mathbf{u}''} = \nabla \cdot \mathbf{R} \quad (10b)$$

where \mathbf{R} are the Reynolds stresses

$$\mathbf{R} = \overline{\mathbf{U}'' \mathbf{u}''} = \mathbf{R}_H + \mathbf{R}_\perp = \overline{\mathbf{u}'' \mathbf{u}''} + \mathbf{e}_z \overline{w'' \mathbf{u}''} \quad (10c)$$

It is important to stress that the vertical velocity w'' in (10c) does not include the diapycnal vertical velocity corresponding to diabatic processes leading to dissipation. Eq. (10b) can alternatively be written as

$$\nabla \cdot \mathbf{R} \equiv \overline{\mathbf{u}'' \cdot \nabla_H \mathbf{u}''} + \overline{w'' \partial_z \mathbf{u}''} \quad (10d)$$

which has the following advantages. Use of the identity

$$\overline{\mathbf{u}'' \cdot \nabla_H \mathbf{u}''} = \mathbf{e}_z \times \mathbf{F}_\zeta + \nabla_H K \quad (10e)$$

helps exhibit the mesoscale kinetic energy and the flux of the relative vorticity ζ''

$$K = 1/2 \overline{\mathbf{u}'' \mathbf{u}''} \quad \mathbf{F}_\zeta = \overline{\zeta'' \mathbf{u}''} \quad \zeta'' = \nabla_H \times \mathbf{u}'' \cdot \mathbf{e}_z \quad (10f)$$

Irrespective of any mesoscale modeling, Eq. (10e) shows that mesoscale models ought to exhibit the *horizontal gradients of K* which data show to be large (Wyrki et al., 1976, Fig. 4; Schmitz, 1976; Wunsch, 1981). Traditionally, the most widely used model in the mean momentum has been a down-gradient mixing of momentum (Rosati and Miyakoda, 1988; Griffies and Hallberg, 2000) which is not a model for mesoscales but a device to assure numerical stability.

6.2. Transformed Eulerian mean (TEM)

In this case, the mean momentum equations acquire the form (Andrews et al., 1987; Gent and McWilliams, 1996; Lee and Leach, 1996; Greatbatch, 2001)

$$\bar{\mathbf{u}}_t + \mathbf{u}^\# \cdot \nabla \bar{\mathbf{u}} + f \mathbf{e}_z \times \mathbf{u}^\# = -\nabla_H \bar{p} + \nabla \cdot \mathbf{E} \quad (10g)$$

where the 3D velocity field $\mathbf{u}^\#$ is called the *residual mean circulation*

$$\mathbf{u}^\# \equiv \bar{\mathbf{U}} + \mathbf{u}_M \quad (10h)$$

and \mathbf{u}_M is the eddy-induced velocity defined earlier in Eq. (1d). The second term on the rhs of (10g) is the divergence of the Eliassen–Palm fluxes

$$\nabla \cdot \mathbf{E} = f \mathbf{e}_z \times \mathbf{u}_M + \nabla \cdot (-\mathbf{R} + \mathbf{u}_M \bar{\mathbf{u}}) \quad (10i)$$

Below we present the model results for the Reynolds stresses.

7. Mean momentum equations: Model results

7.1. Eulerian mean formalism

The model result for the Reynolds stresses defined in Eq. (10c) is

$$\mathbf{R}_H = \overline{\mathbf{u}''\mathbf{u}''} = K\delta_H + \boldsymbol{\tau}, \quad \mathbf{R}_\perp = \mathbf{e}_z \mathbf{S} \quad (11a)$$

where δ_H is the 2D Kroneker tensor in a horizontal plane and $\boldsymbol{\tau}$ is a barotropic, traceless tensor given by

$$\boldsymbol{\tau} = \mathbf{T} - \frac{1}{2} \delta_H \text{Trace}(\mathbf{T}), \quad \mathbf{T} = \frac{1}{2} (1 + 2c_1) \mathbf{I}_1 \mathbf{I}_1 + \frac{1}{2} (\mathbf{I}_1 \mathbf{I}_2 + \mathbf{I}_2 \mathbf{I}_1) + \boldsymbol{\Lambda} + \boldsymbol{\Omega} \quad (11b,c)$$

where $\mathbf{I}_{1,2}$, $A_{i,j}$ and Ω_{ij} are given by Eqs. (5e–g). The vector \mathbf{S} in (11a) is given by

$$\mathbf{S} = - \int_{-H}^z [(1 + \sigma_t)^{-1} f^{-1} E \boldsymbol{\Pi} - \mathbf{L} \cdot \partial_z \mathbf{R}_H - K f^{-1} \beta \mathbf{e}_y] dz \quad (12a)$$

$\boldsymbol{\Pi}$ is defined in Eq. (5f) and E is the total eddy energy ($E = \text{EKE} + \text{EPE}$) given by

$$E = \left[B_1^2 + \sigma_t (r_d f)^2 N^{-2} \left(\frac{\partial B_1}{\partial z} \right)^2 \right] K_t \quad (12b)$$

Finally, the divergence of the Reynolds stress is computed to be

$$\nabla \cdot \mathbf{R} = (\nabla_H + \mathbf{L} \partial_z) \cdot \mathbf{R}_H - (1 + \sigma_t)^{-1} f^{-1} E \boldsymbol{\Pi} + K f^{-1} \beta \mathbf{e}_y \quad (12c)$$

As shown in Section 10, the first term in (12c) is contributed only by the geostrophic component of the mesoscale velocity while the remaining terms are contributed by the product of geostrophic and a-geostrophic components. The largest contribution to (12c) comes from the first term since the remaining terms are $O(10^{-1})$ smaller.

7.2. Shear contribution

The above results do not account for shear in the mean velocity field $\bar{\mathbf{u}}$. If one accounts for such terms, one obtains the additional term (Appendix E)

$$\nabla \cdot \mathbf{F}_m(\text{shear}) = v_m(\text{model}) (\nabla_H + \mathbf{L} \partial_z)^2 \bar{\mathbf{u}} \quad (12d)$$

where $v_m \sim k_m \sim 10^3 \text{ m}^2 \text{ s}^{-1}$. As expected, (12d) represents an upscale gradient and it acts like a source for the mean velocity. In conclusion, the full mesoscale contribution to the mean momentum equations (10a) is given by

$$\nabla \cdot \mathbf{F}_m \equiv \nabla \cdot \mathbf{R} + \nabla \cdot \mathbf{F}_m(\text{shear}) \quad (12e)$$

With (12c,d) and the typical values $|\bar{\mathbf{u}}| \sim 10^{-2} \text{ m s}^{-1}$, $k \sim 10^{-2} \text{ m}^2 \text{ s}^{-2}$, $\nabla \sim 10^{-6} \text{ m}^{-1}$, we obtain

$$\nabla \cdot \mathbf{F}_m = \underbrace{(\nabla_H + \mathbf{L}\partial_z) \cdot \mathbf{R}_H}_{10^{-8}} - \underbrace{(1 + \sigma_t)^{-1} f^{-1} E\Pi}_{10^{-9}} + \underbrace{Kf^{-1}\beta\mathbf{e}_y}_{10^{-10}} + \underbrace{\nabla \cdot \mathbf{F}_m(\text{shear})}_{10^{-11}} \quad (12f)$$

Therefore, *the Reynolds stresses are the largest contribution*. At the same time, they are also larger than the non-linear terms contained in the lhs of Eq. (10a) since

$$D\bar{\mathbf{u}}/Dt = \partial\bar{\mathbf{u}}/\partial t + \underbrace{\bar{\mathbf{u}} \cdot \nabla_H \bar{\mathbf{u}}}_{10^{-10}} \quad (12g)$$

Thus, since in OGCMs the non-linear, advective terms (12g) are accounted for, there is no justification in neglecting the Reynolds stresses which are considerably larger.

8. Generalized Eliassen–Palm theorem

8.1. Transformed Eulerian formalism

In this representation, the divergence of the Eliassen–Palm tensor is given by (10i) which we now transform as follow. The potential vorticity (PV) $q'' = \zeta'' + f\partial_z(\rho''/\bar{\rho}_z)$, the PV-flux $\mathbf{F}_q \equiv \overline{\mathbf{u}''q''}$, the relative vorticity flux (RV) $\mathbf{F}_\zeta \equiv \overline{\mathbf{u}''\zeta''}$ and the eddy-induced velocity are related by

$$\mathbf{F}_q = \mathbf{F}_\zeta - (f + \bar{\zeta})\mathbf{u}^+ \quad (13a)$$

The geostrophic and a-geostrophic components of \mathbf{F}_ζ have been computed to be

$$\mathbf{F}_\zeta^g = -\mathbf{e}_z \times \nabla_H \cdot \boldsymbol{\tau}, \quad \mathbf{F}_\zeta^{\text{ag}} = (1 + \sigma_t)^{-1} f^{-1} K \mathbf{e}_z \times \boldsymbol{\Pi} \quad (13b)$$

where the tensor $\boldsymbol{\tau}$ is given by (11b,c). The two components of \mathbf{F}_ζ are of the same magnitude. Eq. (10i) thus becomes

$$\nabla \cdot \mathbf{E} = -\mathbf{e} \times \mathbf{F}_q - \nabla \cdot \mathbf{R} + \mathbf{e} \times \mathbf{F}_\zeta + \nabla \cdot \mathbf{u}_M \bar{\mathbf{u}} \quad (13c)$$

which we consider a generalized Eliassen–Palm theorem. Using the typical values $|\bar{\mathbf{u}}| \sim 10^{-2} \text{ m s}^{-1}$, $u^+ \sim 10^{-3} \text{ m s}^{-1}$, $K \sim 10^{-2} \text{ m}^2 \text{ s}^{-2}$, $\nabla \sim 10^{-6} \text{ m}^{-1}$, $f \sim 10^{-4} \text{ s}^{-1}$, $r_d \sim 3 \times 10^4 \text{ m}$, a scale analysis of the terms in the rhs of (13c) reveals the sequence

$$\nabla \cdot \mathbf{E} = \underbrace{-\mathbf{e} \times \mathbf{F}_q}_{10^{-7}} - \underbrace{\nabla \cdot \mathbf{R}}_{10^{-8}} + \underbrace{\mathbf{e} \times \mathbf{F}_\zeta}_{10^{-9}} + \underbrace{\nabla \cdot \mathbf{u}_M \bar{\mathbf{u}}}_{10^{-11}} \quad (13d)$$

On this basis, one may conclude that it is permissible to retain only the first term in (13d). If so, the TEM momentum equation (10g) becomes

$$\bar{\mathbf{u}}_t + \mathbf{u}^\# \cdot \nabla \bar{\mathbf{u}} + f \mathbf{e}_z \times \mathbf{u}^\# \cong -\rho_0^{-1} \nabla_H \bar{p} - \mathbf{e}_z \times \mathbf{F}_q \quad (13e)$$

which is the form used in many oceanic studies (e.g., Lee and Leach, 1996; Wardle and Marshall, 2000). However, if one carries out the same scale analysis in the mesoscale terms on the lhs of (13e), with (10h), one obtains

$$\bar{\mathbf{u}}_t + \underbrace{(\bar{\mathbf{u}} + \mathbf{u}^+) \cdot \nabla \bar{\mathbf{u}}}_{10^{-11}} + \underbrace{f \mathbf{e}_z \times (\mathbf{u}^+ + \bar{\mathbf{u}})}_{10^{-7}} \cong -\nabla_H \bar{p} - \underbrace{\mathbf{e}_z \times \mathbf{F}_q}_{10^{-7}} \quad (13f)$$

If one keeps only the largest term on the lhs, the latter just cancels the last term on the rhs of (13f), leaving no mesoscale terms, a conclusion that cannot be physically correct. The reason is that given the zeroth-order cancellation, one must keep the second term in (13c). In that case, Eq. (13e) changes to

$$\bar{\mathbf{u}}_t + \mathbf{u}^\# \cdot \nabla \bar{\mathbf{u}} + f \mathbf{e}_z \times \mathbf{u}^\# \cong -\nabla_H \bar{p} - \mathbf{e}_z \times \mathbf{F}_q - \nabla \cdot \mathbf{R} \quad (13g)$$

Since, as before, the largest term in the lhs of (13g) cancels the second term on the rhs, this leaves as the first non-zero term the gradient of the Reynolds stresses. This is the same conclusion one arrives at in the EM formalism, Eq. (12f).

The above analysis shows that Eq. (13e) is not the correct form of the TEM momentum equations for it keeps only zeroth-order terms which cancel each other out leaving no trace of mesoscale influence. The correct procedure requires that one keeps the next term and that brings the EM and the ETM formalism in full agreement. In conclusion, use of either formalism requires the knowledge of the Reynolds stresses.

8.2. Sverdrup relation

Since the Sverdrup relation (Pedlosky, 1998) has played a major role in the study of ocean dynamics, it seems natural to inquire about the effects of mesoscales since the original derivation neglects the non-linear terms in the momentum equations. Taking the curl of (10a) and including the mesoscale-induced Reynolds stresses, (10b) and (12c), we obtain a new Sverdrup relation. For example, keeping only the first term in (12c) gives

$$\underbrace{\beta v = f \frac{\partial w}{\partial z}}_{O(1)} + \underbrace{M(z)}_{O(0.1)} \quad (13h)$$

where the mesoscale term $M(z)$ is given by

$$\mathbf{M}(z) = K_z (\mathbf{e}_z \times \mathbf{L}) \cdot \nabla_H \Phi_z, \quad \Phi = \ln(\rho/K) \quad (13i)$$

In (13i), $a_z = \partial a / \partial z$, $\beta = 2\Omega R^{-1} \cos \theta$, $f = 2\Omega \sin \theta$, θ is the latitude and R is the earth's radius. Two considerations are of interest. First, using typical values, $M(z)$ is estimated to be of order $O(0.1)$ of the standard Sverdrup terms; second, integration of (13h) over z gives

$$\bar{v} = A \tan \theta + B \beta^{-1} \quad (13j)$$

where

$$\bar{v} = \int v dz, \quad A = R \int dz \frac{\partial w}{\partial z}, \quad B = \int dz M(z) \quad (13k)$$

While the proper choice of the lower limit of the z -integration is discussed in Wunsch and Roemich (1985), it is clear from (13j) that as one approaches the equator, the first term becomes increasing small while the mesoscale term remains finite. As discussed in Pedlosky (1998), without mesoscales there is no interior flow crossing the equator which is then relegated to the western boundaries where Sverdrup's relation no longer holds. In the presence of mesoscales, a non-zero meridional crossing ($\bar{v} \neq 0$) at the equator is allowed.

9. Derivation of the model results: Mean density equation

As discussed in the Introduction, transforming from isopycnal coordinates is rather complex process due to the random nature of the density field. However, the main contributions can be sorted out thanks to the existence of the small parameter h'/\bar{h} where $h = \partial z/\partial \rho$ is the layer thickness in isopycnal coordinates. In fact, we have

$$\frac{h'}{\bar{h}} = \frac{\partial z'}{\partial z} \sim z'H^{-1} \quad (14a)$$

where z' represents the characteristic variation of a level $z(\rho)$ which is $\sim 10^2$ m (Richardson, 1993), whereas the typical scale height H is $\geq 10^3$ m. Thus, the ratio (14a) is ~ 0.1 . In reality, it is even smaller if one considers the filling factor of the eddies within a given volume which is typically a few tens of a percent. Then, $\overline{z'^2} \sim 10^3$ m² and the ratio (14a) becomes $\sim 3 \times 10^{-2}$. Thus, the coordinate transformation problem can be treated perturbatively in powers of h'/\bar{h} . Such an approach is developed in Appendix A (see also McDougall, 1998 and McDougall and McIntosh, 2001) where we derive the transformation formulae for the eddy fields. Using such formulae, one can express second-order moments in level coordinates in terms of the corresponding moments in isopycnal coordinates developed in OM1.

Eddy-induced velocity. We begin by considering the eddy-induced velocity. To this end, it is sufficiently to take into account the transformation formulae of Appendix A in the lowest order of h'/\bar{h} (see also McDougall, 1998)

$$\bar{\rho}_z = \bar{h}^{-1}, \quad \rho'' = -z'\bar{\rho}_z, \quad \mathbf{u}'' = \mathbf{u}', \quad \bar{\mathbf{u}} = \bar{\mathbf{u}} \quad (14b)$$

The first of Eq. (1d) then becomes

$$\bar{h}\mathbf{u}^+ = (\overline{z'\mathbf{u}'})_\rho \quad (14c)$$

Differentiating the product $(z'\mathbf{u}')$ yields $\mathbf{u}^+ = \mathbf{u}_* + \mathbf{u}_{**}$ where \mathbf{u}_* is the bolus velocity defined in OM1 (Eq. (D.2)) whereas $\mathbf{u}_{**} = h^{-1}\overline{z'\mathbf{u}'}_\rho$. Using the geostrophic relation

$$\mathbf{u}'_\rho = (g/f\rho_0)[\mathbf{e}_z \times \nabla z'] \quad (14d)$$

we obtain

$$\mathbf{u}_{**} = -(N^2/f) \left[\mathbf{e}_z \times \nabla \left(\frac{1}{2} \overline{z'^2} \right) \right] \quad (14e)$$

With $f \sim 10^{-4}$ s⁻¹, $\nabla \sim 10^{-6}$ m⁻¹, $N^2 \sim 10^{-5}$ s⁻², we obtain $\mathbf{u}_{**} \sim 10^{-4}$ m s⁻¹ which is an order of magnitude smaller than $\mathbf{u}_* \sim 3 \times 10^{-3}$ m s⁻¹. The estimate of \mathbf{u}_{**} is confirmed by computing (14e) in the framework of the present model

$$\mathbf{u}_{**} = -(N^2/f)[\mathbf{e}_z \times \nabla(K/N^2)] \quad (14f)$$

which yields the same value of \mathbf{u}_{**} . Thus, we can take $\mathbf{u}^+ = \mathbf{u}_*$. What remains to be done is express the mesoscale eddy-induced velocity in terms of level coordinates. Using Eqs. (17a–c) of OM1, we obtain Eqs. (4) cited above.

The Σ -term. We begin by considering the dynamic equation for the variance of the buoyancy fluctuations in the adiabatic approximation (McDougall and McIntosh, 1996).

$$D_t \frac{1}{2} \overline{b''^2} + \nabla \cdot \frac{1}{2} \overline{\mathbf{U}'' b''^2} = -N^2 \Sigma \quad (14g)$$

where $D_t = \partial/\partial t + \overline{\mathbf{U}} \cdot \nabla$. It was argued by McDougall and McIntosh (1996) that the triple correlation term is negligible. Furthermore, it was also suggested by Treguier et al. (1997) to consider the stationary limit in which case (14g) becomes

$$N^2 \Sigma = -\frac{1}{2} \overline{\mathbf{U}} \cdot \nabla \overline{b''^2} \quad (14h)$$

To evaluate (14h), we use the relation between the eddy potential energy EPE and the buoyancy variance

$$\text{EPE} = \frac{1}{2} N^{-2} \overline{b''^2} \quad (14i)$$

Then, Eq. (14h) gives

$$N^2 \Sigma = -\overline{\mathbf{U}} \cdot \nabla (N^2 \text{EPE}) \quad (14j)$$

Since the vertical and horizontal contributions to the scalar production are of the same order, we may evaluate Σ using only the horizontal one. Using the characteristic values

$$|\overline{\mathbf{u}}| \sim 10^{-2} \text{ m s}^{-1}, \quad \nabla \sim 10^{-6} \text{ m}^{-1}, \quad \text{EPE} \sim K \sim 10^{-2} \text{ m}^2 \text{ s}^{-2} \quad (14k)$$

we obtain

$$\Sigma \sim 10^{-10} \text{ m}^2 \text{ s}^{-3}, \quad \Sigma_z \sim 10^{-13} \text{ m s}^{-3} \quad (14l)$$

which must be compared with the terms containing \mathbf{u}^+ , w^+ in Eq. (1c)

$$\mathbf{u}^+ \cdot \nabla_H \overline{b} \sim w^+ N^2 \sim u^+ H L^{-1} N^2 \sim 10^{-11} \text{ m s}^{-3} \quad (14m)$$

where we have used $N^2 \sim 10^{-5} \text{ s}^{-2}$ and $u^+ \sim 10^{-3} \text{ m s}^{-1}$. Thus, one is led to conclude that the terms (14m) exceed Σ_z by two orders of magnitude and that the latter can be neglected. This conclusion is however incorrect since it follows from (14h) which is the stationary limit of Eq. (14g). *Such a limit is impossible in the adiabatic limit underlying (14g).* In fact, in general the complete $\partial_t \overline{b''^2}$ equation consists of *production* and *dissipation* terms of the density variance and in the stationary case the two contributions balance out. However, in the adiabatic approximation dissipation is absent and such balance cannot exist. We must substitute the left hand side of (14g) with the adiabatic growth rate of the density variance which, because of relation (14i), is proportional to the adiabatic growth rate of the eddy potential energy R_W . Thus, when Σ is larger than the diffusion term in (14g), we have

$$R_W = -\Sigma \quad (15a)$$

To evaluate Σ , we adopt $R_W \sim \varepsilon_d$ which in OM1 was estimated to be of the order $10^{-8} \text{ m}^2 \text{ s}^{-3}$, a value consistent with the data. Therefore, from (15a) we obtain

$$\Sigma \sim 10^{-8} \text{ m}^{-2} \text{ s}^{-3} \quad \Sigma_z \sim 10^{-11} \text{ m s}^{-3} \quad (15b)$$

The latter is of the order of (14m) and thus the Σ -term in (1d) cannot be neglected. To express Σ in terms of the large scale fields, we use the fact that the characteristic time of eddy energy $L/\bar{u} \sim 10^8 \text{ s}$ is much longer than the time to achieve the virial relation (for simplicity of notation we call $W \equiv \text{EPE}$), $\tilde{W} \equiv \int W dz = \sigma_t \tilde{K}$ between eddy potential and kinetic energy per the unit area. The latter time is of order $r_d/u' \sim 10^5\text{--}10^6 \text{ s}$. This implies that the vertical profiles of R_W and W are proportional to each other. Thus, we may write

$$R_W(R_{\tilde{W}} + R_{\tilde{K}})^{-1} = W(\tilde{W} + \tilde{K})^{-1} \quad (15c)$$

where

$$\text{EPE} = \frac{1}{2} N^2 \overline{z'^2} = \frac{1}{2} \rho_0^{-2} N^{-2} \overline{\left(\frac{\partial B'}{\partial z}\right)^2} \quad (15d)$$

Next, from Appendix D we have that

$$B'^2 = 2\sigma_t \rho_0^2 f^2 r_d^2 K_t B_1^2 \quad (15e)$$

which transforms Eq. (15d), to its final form

$$\text{EPE}(z) = \sigma_t (f r_d / N)^2 K_t \left(\frac{\partial B_1}{\partial z}\right)^2 \quad (15f)$$

which is Eq. (7a). As usual, the z -dependence originates from the first baroclinic mode eigenfunction $B_1(z)$. In the adiabatic approximation, the denominator of the left hand side of Eq. (15c) equals the rate of production of total eddy energy P_T given (6c). Therefore, solving (15c) for R_W , substituting in (15a), after some manipulations and use of the boundary conditions (OM1, D.9c), we obtain Eqs. (7a)–(7c).

Surface kinetic energy K_t . As discussed in OM1, the surface kinetic energy is obtained by considering the relations

$$P_T = \int dz \varepsilon_d = \eta^{-1} \int dz \varepsilon_k = (3\eta r_d)^{-1} K_t^{3/2} \int dz \Gamma^{3/2}(z) \quad (15g)$$

where use was made of the Kolmogorov spectrum and where the function $\Gamma(z)$ is given by Eq. (5b) of the text.

10. Derivation of the model results: Mean momentum equations

There are the two possible ways of finding the eddy contribution to the momentum equation (10a). They are either through Eqs. (10b–f) or through (10g). We have performed both computa-

tions with the same final result. We begin by separating the geostrophic and a-geostrophic contributions into (10e). We have the two exact relations

$$\overline{\mathbf{u}'' \cdot \nabla_{\mathbf{H}} \mathbf{u}''} = \mathbf{e}_z \times \mathbf{F}_\zeta + \nabla_{\mathbf{H}} K, \quad \overline{\mathbf{u}_g'' \cdot \nabla_{\mathbf{H}} \mathbf{u}_g''} = \mathbf{e}_z \times \mathbf{F}_\zeta^g + \nabla_{\mathbf{H}} K \quad (16a)$$

where we do not affix the subscript “g” to K since the latter is known to be contributed mostly by the geostrophic component. From Eqs. (10c) and (11a) and considering that the geostrophic velocity is much larger than the a-geostrophic one, one derives that $(\nabla_{\mathbf{H}} \cdot \mathbf{u}_g'' = 0)$

$$\overline{\mathbf{u}_g'' \cdot \nabla_{\mathbf{H}} \mathbf{u}_g''} = \nabla_{\mathbf{H}} \cdot \overline{\mathbf{u}_g'' \mathbf{u}_g''} = \nabla_{\mathbf{H}} \cdot \mathbf{R}_{\mathbf{H}} = \nabla_{\mathbf{H}} \cdot (K \delta_{\mathbf{H}} + \boldsymbol{\tau}) \quad (16b)$$

From the second of (16a) and (16b), we conclude that

$$\mathbf{e}_z \times \mathbf{F}_\zeta^g = \nabla_{\mathbf{H}} \cdot \boldsymbol{\tau} \quad \mathbf{F}_\zeta^g = -\mathbf{e}_z \times \nabla_{\mathbf{H}} \cdot \boldsymbol{\tau} \quad (16c)$$

which, from (16b), implies that

$$\nabla_{\mathbf{H}} \cdot \mathbf{R}_{\mathbf{H}} = \mathbf{e}_z \times \mathbf{F}_\zeta^g + \nabla_{\mathbf{H}} K \quad (16d)$$

As for the ageostrophic component \mathbf{F}_ζ^a , we compute it directly from definition in (10f). Making use of Eqs. (1f) and (21) of OM1, we obtain (Appendix B)

$$\mathbf{F}_\zeta^a = -(\sigma_t r_d^2 f)^{-1} K (\bar{\mathbf{u}} - \mathbf{u}_d + \mathbf{c}_R) \quad (16e)$$

$$\mathbf{u}_d = \langle \bar{\mathbf{u}} \rangle + (1 + \sigma_t)^{-1} \mathbf{c}_R - f r_d^2 (1 + \sigma_t^{-1})^{-1} \mathbf{e}_\rho \times \langle \partial_z \mathbf{L} \rangle \quad (16f)$$

Next, we study the second term in (10d). We begin with the field w'' given by

$$w_z'' = -\nabla_{\mathbf{H}} \cdot \mathbf{u}'' \quad (17a)$$

We then substitute (A.18) and (A.19), keep only the first terms and obtain

$$w_z'' = -\nabla_\rho \cdot \mathbf{u}' \quad (17b)$$

In Appendix C we show that, to first-order, the corrections to (17b) cancel out. Fourier transforming (17b) and using Eqs. (4g–i), (10a,b) of OM1, we obtain

$$w_z''(\mathbf{k}) = -ik_0 u_p = k_0^2 (f \rho)^{-2} [\tilde{v} + \mathbf{i} \mathbf{k} \cdot (\mathbf{u}_d - \bar{\mathbf{u}} - \mathbf{c}_R)] B'(\mathbf{k}) \quad (17c)$$

which shows explicitly that only the a-geostrophic component of \mathbf{u}' contributes to the right hand side, see Eq. (5e) of OM1. Next, we substitute Eqs. (11a–f) of OM1 into (17c) and keep terms of the zeroth- and first-order in $\Omega\tau$ taking into account that in the square brackets of (17c) the first term \tilde{v} dominates while all the other terms are of the first-order in $\Omega\tau$. Using the geostrophic relations, we transform (17c) to

$$w_z''(\mathbf{k}) = \partial_z ([\tilde{\chi} + \mathbf{i} \mathbf{k} \cdot (\bar{\mathbf{u}} - \mathbf{u}_d)] z'(\mathbf{k}) - \mathbf{i} g^{-1} \mathbf{k} \cdot \bar{\mathbf{u}}_\rho) B'(\mathbf{k}) - \chi_z z'(\mathbf{k}) \quad (17d)$$

Integrating over z and applying the geostrophic relation (B.2) and the second relation (A.19), we obtain

$$w''(\mathbf{k}) = \tilde{\chi} z'(\mathbf{k}) + \mathbf{i} \mathbf{k} \cdot (\bar{\mathbf{u}} - \mathbf{u}_d) z'(\mathbf{k}) + \mathbf{u}'(\mathbf{k}) \cdot \nabla_\rho \bar{z} - g^{-1} \int_{\rho_s}^\rho \tilde{\chi}_\rho z'(k) d\rho \quad (17e)$$

Next we express the field \mathbf{u}''_z entering relation (10d) in terms of fields in the isopycnal system. To this end, we differentiate relation (A.18). Fourier transforming, we get

$$\mathbf{u}''_z(\mathbf{k}) = \overline{\overline{\rho}}_z \mathbf{u}'_\rho(\mathbf{k}) - \overline{\overline{\mathbf{u}}}_{zz} z'(\mathbf{k}) - \overline{\overline{\mathbf{u}}}_z \overline{\overline{\rho}}_z z'_\rho(\mathbf{k}) \quad (17f)$$

The first term in the rhs is the largest. In Appendix C we show that the contribution to (10d) due to the other terms in (17f) is at least an order of magnitude smaller than the first term. Thus, we write Eq. (17f) in a different form which accounts for only the main term

$$\mathbf{u}''_z(\mathbf{k}) = \overline{\overline{\rho}}_z \mathbf{u}'_\rho(\mathbf{k}) = -iN^2 f^{-1} \mathbf{e}_z \times \mathbf{k} z'(\mathbf{k}) \quad (17g)$$

The second equality is just the geostrophic relation (B.2) in Fourier space. Thus, with Eqs. (17e, g), we have modeled the ingredients of the term $\overline{\overline{w'' \partial_z \mathbf{u}''}} \equiv \mathbf{A}$. Defining

$$\delta(\mathbf{k} - \mathbf{k}') \tilde{\mathbf{A}}(\mathbf{k}) = \text{Re} \overline{\overline{w''(\mathbf{k}) \mathbf{u}''^*(\mathbf{k}')}} \quad (18a)$$

one needs to integrate (18a) over \mathbf{k}' and $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$. In accordance with the general definition of spectra, we have

$$\mathbf{A}(k) = k \int \tilde{\mathbf{A}}(\mathbf{k}) d\mathbf{n}, \quad (18b)$$

Substituting expressions (17e,g) into (18a), we notice that the contributions of the first and last terms in the right hand side of (17e) vanish because of the factor i in front of the last expression in (17g). The other two terms of (17e) yield the following result:

$$\mathbf{A}(k) = k^2 f^{-1} (\overline{\mathbf{u}} - \mathbf{u}_d) \times \mathbf{e}_z W(k) + (\nabla_{\rho \bar{z}}) \cdot \partial_z \mathbf{R}_H(k) \quad (18c)$$

where $\mathbf{R}_H(k)$ and $W(k) \equiv \text{EPE}(k)$ are the spectra of the horizontal Reynolds stress and potential energy. As discussed in OM1, the integration (18c) over k reduces to substituting the spectra with the correspondent functions. In addition, we express \bar{z} and $\nabla_{\rho \bar{z}}$ in terms of level coordinates. The result is

$$\overline{\overline{w'' \partial_z \mathbf{u}''}} = -(\sigma_t r_d^2 f)^{-1} W \mathbf{e}_z \times (\overline{\mathbf{u}} - \mathbf{u}_d) + \mathbf{L} \cdot \partial_z \mathbf{R}_H(k) \quad (18d)$$

where W is given by (15d). Substituting Eqs. (16c,d) and (18d) into (10d), we obtain Eq. (12c).

11. Conclusions

We have presented a dynamical mesoscale model in z -coordinates. As in OM1, where we derived the dynamical model in isopycnal coordinates, we have worked out the mesoscales contributions to the mean buoyancy and mean momentum equations. In the former, we have found the expression for the eddy-induced velocity, Eqs. (4a–c), the first term of which is of the GM form. As discussed in OM1, the necessity of additional terms, not directly related to density gradients, was first suggested by Bryan et al. (1999) on the basis of an eddy-resolving ocean code. The new terms in (4a) may be the ones predicted to exist by Bryan et al. (1999). The mesoscale diffusivity which is undetermined in the GM model, is here computed in terms of the resolved scales, Eqs. (4b). The residual diapycnal flux Σ in Eqs. (1c,e) is given by Eqs. (7d–g). We show that it is of

the same order as the other mesoscale terms and thus it is not negligible. This result agrees with the conclusions of Gille and Davis (1999) and McDougall (2004) who employed eddy-resolving ocean codes to compute Σ and also concluded that it is not negligible. The boundary conditions (1f) and (1g) are satisfied by the model results.

As for the mean momentum equations, to the best of our knowledge, this is the first model that explicitly derives an expression for the divergence of the momentum fluxes (Reynolds stresses), the Eliassen–Palm and PV fluxes without using a heuristic model. The so-called default down-gradient approximation is not a model for mesoscales and represents processes at much smaller scales. The mesoscales contribution (11) and (12) exhibit the horizontal gradient of the eddy kinetic energy, the mean flow and the β -term. Since data (Schmitz, 1976; Wyrski et al., 1976, Fig. 4; Wunsch, 1981) show that there are large horizontal gradients of eddy kinetic energy, their presence in the model is a welcome feature. We have also shown that the Reynolds stresses contribute a term that is larger than the non-linear terms in the mean velocity field. Thus, accounting for the latter while neglecting the former (as done in all OGCMs) is not correct. We further show that in the TEM formalism, it is not sufficient to take into account the PV-flux *only* since, to a large extent, this term is cancelled by the mesoscale terms in the residual mean circulation. After the cancellation, the first term is the Reynolds stresses which must therefore be accounted for.

The mesoscale effect on the Sverdrup relation was also studied and shown to introduce a non-negligible modification of the order of 10%. Interestingly, as one approaches the equator, mesoscales allow for meridional currents while the standard Sverdrup relation does not.

Two major topics remain to be studied. First, since the mesoscale model shows that in addition to the eddy-induced velocities the buoyancy equation contains a residual diapycnal flux, the latter must also be present in the temperature/salinity equations but it has thus far been neglected since Σ has always been taken to be zero. The second problem to be investigated is the role of the strongly diabatic mixed layer on the mesoscale field.

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Appendix A. Transformation from isopycnal to level coordinates

Given the two random fields $\rho(r, z)$ and $z(r, \rho)$, we split them into average and fluctuating components. Omitting for simplicity the variable \mathbf{r} , we write

$$z(\rho) = \bar{z}(\rho) + z'(\rho), \quad \rho(z) = \bar{\rho}(z) + \rho''(z) \quad (\text{A.1})$$

For arbitrary z and ρ , we have the exact relations

$$\rho = \rho[z(\rho)], \quad z = z[\rho(z)] \quad (\text{A.2})$$

Assuming z' to be small, we substitute the first relation (A.1) into the first of (A.2) and expand. Keeping only terms to the first-order in z' , we obtain

$$\rho = \rho[\bar{z}(\rho) + z''(\rho)] = \rho[\bar{z}(\rho)] + \rho_z[\bar{z}(\rho)]z' + O(z'^2)$$

Substituting here the second relation (A.1), we obtain

$$\rho = \bar{\rho}[\bar{z}(\rho)] + \rho''[\bar{z}(\rho)] + \bar{\rho}_z[\bar{z}(\rho)]z' + O_2 \quad (\text{A.3})$$

where O_2 is a second-order term in the fluctuating fields, that is

$$O_2 = \rho_z''[\bar{z}(\rho)]z' + O(z'^2)$$

If we choose a fixed value of ρ , then lhs of (A.3) does not contain fluctuating components. This implies that the fluctuating part of the rhs also equals zero. Thus, with an accuracy up to the second-order terms, we obtain

$$\rho = \bar{\rho}[\bar{z}(\rho)] \quad (\text{A.4})$$

$$\rho''[\bar{z}(\rho)] + \bar{\rho}_z[\bar{z}(\rho)]z'(\rho) = 0 \quad (\text{A.5})$$

An analogous procedure applied to the variable $z(\rho)$ leads to the following relations:

$$z = \bar{z}[\bar{\rho}(z)] \quad (\text{A.6})$$

$$z'[\bar{\rho}] + \bar{h}[\bar{\rho}(z)]\rho''(z) = 0 \quad (\text{A.7})$$

Since in (A.4, 6) ρ and z are arbitrary, one may choose

$$\rho = \bar{\rho}(z) \quad (\text{A.8})$$

Then, (A.6) yields

$$z = \bar{z}(\rho) \quad (\text{A.9})$$

Relations (A.8, 9) imply that the functions $\bar{z}(x)$ and $\bar{\rho}(x)$ are the inverse of each other, a property which is fulfilled only to the first-order in fluctuating fields. Use of (A.8, 9) in (A.5, 7) yields

$$\rho''(z) + \bar{\rho}_z(z)z' = 0 \quad (\text{A.10})$$

$$z'(\rho) + \bar{h}(\rho)\rho'' = 0 \quad (\text{A.11})$$

which proves that

$$\bar{\rho}_z(z)h(\rho) = 1 \quad (\text{A.12})$$

provided that ρ and z are related by (A.8) or (A.9). The last result becomes an identity when one neglects the random nature of $\rho(z)$ whereas when accounting for the random nature of the variables, (A.12) contains a correction of the second-order in fluctuating fields. Next, we consider the velocity field in isopycnal and level coordinates and split it into mean and fluctuating parts

$$\mathbf{u}(\rho) = \bar{\mathbf{u}}(\rho) + \mathbf{u}'(\rho), \quad \mathbf{u}(z) = \bar{\mathbf{u}}(z) + \mathbf{u}''(z) \quad (\text{A.13})$$

We begin with the exact relation

$$\mathbf{u}(z) = \mathbf{u}[\rho(z)] \tag{A.14}$$

and substitute the decomposition (A.1) for $\rho(z)$. Expanding the right hand side of (A.14) in a power series in $\rho''(z)$ and keeping terms of the first-order, we obtain

$$\mathbf{u}(z) = u[\bar{\rho}(z)] + u_\rho[\bar{\rho}(z)]\rho''(z) \tag{A.15}$$

Substituting the first of relations (A.13), separating mean and fluctuating parts and using (A.10), we get

$$\bar{u}[\bar{z}(\rho)] = \bar{u}(\rho) - \bar{h}(\rho)^{-1} \overline{z'(\rho)u'_\rho(\rho)} \tag{A.16}$$

$$\mathbf{u}''[\bar{z}(\rho)] = \mathbf{u}'(\rho) - \bar{h}(\rho)^{-1} \bar{\mathbf{u}}_\rho(\rho)z'(\rho) \tag{A.17}$$

Notice that a formal evaluation of the second term in the right hand side of (A.16) based on the mean square roots of the correlating fields, yields a result which is smaller than the first term by an order of magnitude. But in reality, due to a weak correlation, the second term is smaller than the first one by more than two orders of magnitude and may be neglected. In fact, as one can see, the discussed term equals $-\mathbf{u}_{**}$ defined in Eq. (14e). Below Eq. (14e), we have evaluated \mathbf{u}_{**} and shown that typically it is $\sim 10^{-4} \text{ m s}^{-1}$, more than two orders of magnitude smaller than $\bar{\mathbf{u}}$. Thus, the last relation (14b) is satisfied to high accuracy. On the contrary, in (A.17) to order h'/\bar{h} , the last term must be accounted for. For further use, we transform this term using $h = z_\rho$, $B_\rho = gz$, (A.12) and the last relation of (14b). Then (A.17) yields

$$\mathbf{u}'' = \mathbf{u}' - g^{-1} \bar{\mathbf{u}}_z B'_\rho \tag{A.18}$$

It is worth noticing that results of Appendix A are not new. Thus, relations (A.8, 9) are frequently used by several authors: for example, Eq. (14) of Gent et al. (1995), Eq. (4) of Treguier et al. (1997), Eqs. (B.4), (B.8), (B.12) of Smith (1999). Relations (A.10, 11) were employed, for example, by Treguier et al. (1997), Rix and Willebrand (1996) and McDougall and McIntosh (1996). Equations analogous to (A.16, 17) were obtained by McDougall (2001). Nevertheless, in our opinion, it is useful to present a concise review of transformation formulae that are essential to our derivations. For references, we also recall the well known transformation relations

$$\nabla_H = \nabla_\rho - \mathbf{L}\partial_z, \quad \mathbf{L} = -\nabla\bar{\rho}/\rho_z \tag{A.19}$$

In addition, up to the first-order of h'/\bar{h} , we may adopt

$$\mathbf{e}_\rho = \mathbf{e}_z = \mathbf{e}_v \tag{A.20}$$

Appendix B. Derivation of Eq. (16e)

To transform the flux $\mathbf{F}_{\zeta''}$ defined in (10f), we need the corresponding transformations for \mathbf{u}'' and ζ'' . For the former, the transformation is given in (A.18) which we also use to derive the transformation for ζ'' . To this end we substitute (A.18)–(A.20) into (10f). Substituting the first relation

(A.19), we notice that the ∇ -operator in the first term is of order r_d^{-1} while the second term is of order L^{-1} . Thus, the second operator of (A.19) is a small correction. Using the definition of the relative vorticity $\zeta = (\nabla_\rho \times \mathbf{u})\mathbf{e}_\rho$ for the case of fluctuating fields, we obtain

$$\zeta'' = \zeta' + (g^{-1}\bar{\mathbf{u}}_z \times B'_\rho - u'_\rho \times \nabla\bar{\rho}) \times \mathbf{e}_\rho \quad (\text{B.1})$$

where we keep the leading term (which equals to ζ') as well as the main correction to it and neglect the term, which is obtained by the action of the second operator of (A.19) on the second term of (A.18). Now we notice that, to the main order, both correction terms in (B.1) equal each other. This fact can be checked by applying the geostrophic relations

$$\mathbf{u}'_\rho = (\rho f)^{-1} \mathbf{e}_\rho \times \nabla_\rho B'_\rho \quad (\text{B.2})$$

$$\mathbf{u}_z = -g(f\rho)^{-1} \mathbf{e}_\rho \times \nabla_H \rho \quad (\text{B.3})$$

Next, we Fourier transform (B.1) using Eq. (10a) of I. We get

$$\zeta''(\mathbf{k}) = \zeta'(\mathbf{k}) - 2f\rho g^{-1} \mathbf{n} \times \mathbf{e}_\rho \cdot \bar{\mathbf{u}}_z u_s(\mathbf{k}) \quad (\text{B.4})$$

Now we consider the vorticity flux density in Fourier space in level coordinates which is given by

$$\delta(\mathbf{k} - \mathbf{k}') \mathbf{F}_{\zeta''}(\mathbf{k}) = \text{Re} \overline{\zeta''(\mathbf{k}) \mathbf{u}'^*(\mathbf{k}')} \quad (\text{B.5})$$

Substituting here relations (B.4), (A.18), Eqs. (4g–i) and (10b) of I, we derive

$$\delta(\mathbf{k} - \mathbf{k}') [\mathbf{F}_{\zeta''}(\mathbf{k}) - \mathbf{F}_{\zeta'}(\mathbf{k})] = f\rho g^{-1} [\bar{\mathbf{u}}_z - 2(\mathbf{n}' \times \mathbf{e}_V) \mathbf{n} \times \mathbf{e}_V \cdot \bar{\mathbf{u}}_z] \text{Re} \overline{u_s(\mathbf{k}) u_s^*(\mathbf{k}')} \quad (\text{B.6})$$

where in right hand side we keep only the main order of terms correcting the approximate equality $\mathbf{F}_{\zeta''}(\mathbf{k}) = \mathbf{F}_{\zeta'}(\mathbf{k})$. Further, to obtain a relation for the spectra, we need to integrate (B.6) over $\mathbf{n} = \mathbf{k}/k$. Since the eddy field $u_s(\mathbf{k})$ is axially symmetric to the main order, the integration of right hand side (B.6) yields zero. Thus, to the same order, the corrections to (16e) vanishes. This conclusion is certainly valid if the corrections due to each term in right hand side of (B.6) do not exceed (16e). To check this condition, we integrate the first term in right hand side of (B.6) over \mathbf{k}, \mathbf{k}' and use the definition of the energy spectrum $E(k)$, (A.1d) of I. The integration yields

$$f\rho g^{-1} \bar{\mathbf{u}}_z \partial_\rho K = -fN^{-2} K_z \bar{\mathbf{u}}_z \quad (\text{B.7})$$

Taking into account that $\partial_z \sim H^{-1}$ and $r_d \sim Nf^{-1}H$, we conclude that (16e) and (B.7) are of the same order. Thus, the cancellation to the main order when integrating the right hand side of (B.6), ensures the validity of (16e).

Appendix C. Correction terms to Eqs. (17b) and (18c)

Substituting the first relation (A.19) into (17a), we notice that as in the case considered in Appendix B, the second operator of (A.19) is a small correction. However, this is no longer the case when the operator (A.19) is applied to \mathbf{u}' since the geostrophic components of \mathbf{u}' yields zero. Thus, we have to account for corrections. Substituting (A.18, 19) into (17a), and keeping terms up to the first-order (i.e., neglect only the term coming from the second terms of (A.18, 19), we obtain

$$w_z'' = -\nabla_\rho \cdot \mathbf{u}' + g^{-1} \bar{\mathbf{u}}_{,z} \cdot \nabla_\rho B'_\rho - \mathbf{u}'_\rho \cdot \nabla_H \bar{\rho} \quad (\text{C.1})$$

Using the geostrophic relations (B.2, 3) one can check that the last two terms cancel each other. Thus, relation (17b) is valid. Next, we compute the contribution of the last two terms (17f) to spectra (18a–c). Substituting these terms, together with (17e) into (18a), and using geostrophic relations, we obtain

$$\delta \mathbf{A} \bar{\rho}_z = -\tilde{\chi} f^{-1} N^2 \partial_z (W/N^2) \mathbf{e} \times \nabla_H \bar{\rho} - 2\tilde{\chi} W f^{-1} \mathbf{e} \times \nabla_H \bar{\rho}_z \quad (\text{C.2})$$

Both terms are of the same order. To evaluate them, we take into account that $\tilde{\chi} r_d^{-1} K^{1/2} \sim 3 \times 10^{-6} \text{ s}^{-1}$. Then from (C.2) we get $\delta \mathbf{A} \sim 3 \times 10^{-10} \text{ m s}^{-2}$ that is smaller than the dominating term in (17f) by more than an order of magnitude. Thus, $\delta \mathbf{A}$ may be neglected.

Appendix D. Eddy potential energy in terms of large scale fields

As we discussed in I, to the lowest order in $\Omega\tau$, the field B' is proportional to the eigenfunction B_1 of (5c), i.e.,

$$B' = AB_1 \quad (\text{D.1})$$

To compute A , we express eddy kinetic energy in terms of B' using Eqs. (4g–i), (7a) and (10b) of I

$$K = \frac{1}{2} \sigma_t^{-1} (r_d \rho_0 f)^{-2} \overline{B'^2} \quad (\text{D.2})$$

Substituting (D.1), we obtain

$$K = \frac{1}{2} \sigma_t^{-1} (r_d \rho_0 f)^{-2} A^2 B_1^2 \quad (\text{D.3})$$

On the other hand, to the lowest order in $\Omega\tau$, Eqs. (24a)–(26) of I yield $K = K_s B_1^2$. Substituting (D.3), we get

$$A^2 = 2\sigma_t (r_d \rho_0 f)^2 K_t \quad (\text{D.4})$$

Using (D.1, 4), we obtain (15e).

Appendix E. Derivation of (12d)

To account for the gradients of the mean velocity gradient, we must carry out the following substitution in Eqs. (I.5)

$$\bar{\mathbf{u}} \rightarrow \bar{\mathbf{u}} + i(\partial \bar{\mathbf{u}} / \partial x_\alpha) \partial / \partial k_\alpha \quad (\text{E.1})$$

Since Eqs. (I.5) are written in \mathbf{k} -space, differentiations are understood at constant ρ . In addition, in rhs of Eq. (I.5a) one must add a gradient term equivalent to the substitution

$$\omega \rightarrow \omega + i\gamma_\alpha \gamma_\beta (\partial \bar{u}_\alpha / \partial x_\beta) \quad (\text{E.2})$$

where

$$\vec{\gamma} = \mathbf{n} \times \mathbf{e}, \quad \mathbf{n} = \mathbf{k}/k \quad (\text{E.3})$$

and \mathbf{e} , the unit vector along $-\nabla_3\rho$, is almost coincides with \mathbf{e}_z . Taking into account the identity

$$\gamma_\alpha \gamma_\beta = \delta_{\alpha\beta} - n_\alpha n_\beta \quad (\text{E.4})$$

and that the mean velocity is almost geostrophic, i.e.

$$\nabla_\rho \cdot \bar{\mathbf{u}} = 0 \quad (\text{E.5})$$

one can rewrite (E.2) as follows:

$$\omega \rightarrow \omega - \mathbf{i} n_\alpha n_\beta (\partial \bar{u}_\alpha / \partial x_\beta) \quad (\text{E.6})$$

The above modifications of Eqs. (I.5) results in corrections of the basic model equations (I.11). First, in the expressions (I.11e,f) for Ω_1 , Ω_2 , we must make the substitution (E.1) and second, in the expression for Ω_2 only, substitution (E.6) must be carried out. This results in the following substitution in Eq. (I.12d):

$$\Omega \rightarrow \Omega - \mathbf{i} \mathbf{n} \cdot \partial \bar{\mathbf{u}} / \partial x_\beta (n_\beta + k \partial / \partial k_\beta) \quad (\text{E.7})$$

Because of relation (I.10b), the geostrophic component of the eddy velocity u_s also satisfies the basic equations (I.11) whose solution we expand in powers of the mean velocity gradient

$$u_s = u^0 + u^1 + \dots \quad (\text{E.8})$$

Substituting this expansion into (I.11) and using Eqs. (I.12) and (E.7), we obtain the following equation in the first-order of Ω and of the mean velocity gradient:

$$\left(\frac{\partial^2}{\partial \rho^2} + \Lambda \right) u^1 = -\Lambda \tau \mathbf{n} \cdot \partial \bar{\mathbf{u}} / \partial x_\alpha [(1 + \sigma_t)^{-1} n_\alpha + k \partial / \partial k_\alpha] u^0 \quad (\text{E.9})$$

Neglecting the derivatives in the lhs and considering that the main contribution to u^0 is axisymmetric, we rewrite (E.9) as follows:

$$u^1 = -\tau n_\alpha n_\beta (\partial \bar{u}_\alpha / \partial x_\beta) [(1 + \sigma_t)^{-1} + k \partial / \partial k] u^0 \quad (\text{E.10})$$

since in the case of axi-symmetric functions, the operator in (E.10) satisfies the relation

$$k \partial / \partial k_\alpha = k_\alpha \partial / \partial k \quad (\text{E.11})$$

Next, we consider the large scale momentum equation in isopycnal coordinates (I.1d) and present the eddy contribution to its rhs in vector form as

$$A_j^i = \overline{u'_k \partial u'_j / \partial x_k} \quad (\text{E.12})$$

Substituting only the geostrophic component of \mathbf{u}' and using (E.8), we obtain

$$\delta \mathbf{A}_j^i = \overline{u_k^0 \partial u_j^1 / \partial x_k} \quad (\text{E.13})$$

Since the relation between the fields u^0 and u^1 (E.10) is given in \mathbf{k} -space, we can express the spectrum of $\delta \mathbf{A}(k)$ through the energy spectrum $E(k)$ which is contributed mainly by the component u^0 . Using the relation (4i) of I and (E.3), we obtain

$$\mathbf{u}^{0,1}(\mathbf{k}) = \vec{\gamma} u^{0,1}(\mathbf{k}) \tag{E.14}$$

and (E.4, 10), we obtain

$$\delta \mathbf{A}^i(k) = \frac{1}{2} \tau \left((1 + \sigma_t)^{-1} E(k) + \frac{1}{2} k^2 \frac{\partial}{\partial k} [E(k)/k] \right) \frac{\partial^2 \bar{\mathbf{u}}}{\partial x_\alpha^2} \tag{E.15}$$

In deriving this result, we made use of the following relation:

$$\langle n_i n_j n_l n_m \rangle = (1/8) (\delta_{ij} \delta_{lm} + \delta_{il} \delta_{jm} + \delta \delta_{jl}) \tag{E.16}$$

where the averaging is performed over directions of \mathbf{n} . Since (E.15) is valid near the maximum of the spectrum $E(k)$, we have

$$k^2 \frac{\partial}{\partial k} [E(k)/k] = -E(k) \tag{E.17}$$

Thus

$$\delta \mathbf{A}^i(k) = \frac{1}{4} \tau (1 + \sigma_t)^{-1} (1 - \sigma_t) E(k) \frac{\partial^2 \bar{\mathbf{u}}}{\partial x_\alpha^2} \tag{E.18}$$

Integrating over k and using (I.11d), we get

$$\delta \mathbf{A}^i = v^i \nabla_\rho^2 \bar{\mathbf{u}} \tag{E.19}$$

$$v^i = [(1 - \sigma_t)(4\sigma_t)^{-1}] \frac{K}{\chi} \tag{E.20}$$

Transforming (E.19) to z -coordinates, we obtain (12d) with $v_m = v^i$.

Appendix F. Horizontal component of the Reynolds stress, \mathbf{R}_H

In accordance with our approach, we compute the Reynolds stress \mathbf{R}_I in isopycnal coordinates which is two-dimensional, and then transform the result to z -coordinates to obtain \mathbf{R}_H . \mathbf{R}_I is contributed mostly by the geostrophic component of the eddy velocity field. Thus, we begin by computing the spectrum

$$\mathbf{R}_I(k) = k \int d\mathbf{n} \widetilde{\mathbf{R}}_I(\mathbf{k}) \tag{F.1}$$

where we account for only the geostrophic component of the eddy velocity. In accordance with Eqs. (4i), (5e) of I, we can write

$$\widetilde{\mathbf{R}}_I(\mathbf{k}) \delta(\mathbf{k} - \mathbf{k}') = (\mathbf{e} \times \mathbf{n})(\mathbf{e} \times \mathbf{n}') \overline{u(\mathbf{k}) u^*(\mathbf{k}')} \tag{F.2}$$

where \mathbf{n}, \mathbf{n}' are the unit vectors in the directions of \mathbf{k}, \mathbf{k}' , and \mathbf{e} is the unit vector in the direction of $-\text{grad } \rho$ which in this approximation coincides with \mathbf{e}_z . In the zeroth approximation in the small parameter $\Omega\tau$ in the eigenvalue equation (11a), (12c) of I, the equation has no external directions and therefore spectrum (F.2) is axisymmetric. Thus, to obtain non-axisymmetric terms of (F.2), we need to expand the field $u(\mathbf{k})$ in (F.2) in powers of $\Omega\tau$

$$u(k) = u_0(k) + u_1(k) + u_2(k) + \dots \quad (\text{F.3})$$

Compared to $u_0(\mathbf{k})$, the term $u_1(\mathbf{k})$ contains the phase factor i . Therefore the first-order correction \mathbf{R}_I^1 vanishes and thus the lowest correction to \mathbf{R}_I is quadratic in $u_1(k)$, i.e., it is \mathbf{R}_I^2 . At the same time, the term $u_2(\mathbf{k})$ has the same phase as $u_0(\mathbf{k})$ and so the product $u_0 u_2$ also contributes to \mathbf{R}_I^2 . The detailed computation of the fields $u_1(\mathbf{k}, z)$ and $u_2(\mathbf{k}, z)$ is presented in [Appendix G](#) where it is shown that the main contributions to both fields are baroclinic, i.e., ρ -independent (in z -coordinates z -independent). They are

$$(K_t^0)^{1/2} u_1(k, \rho) / u_0(k, \rho_t) = i \mathbf{e}_z \times \mathbf{I}_1 \cdot \mathbf{n} \quad (\text{F.4})$$

$$\mathbf{K}_t^0 u_2(\mathbf{k}, \rho) / u_0(\mathbf{k}, \rho_t) = (\mathbf{e}_z \times \mathbf{I}_1 \cdot \mathbf{n})(\mathbf{e}_z \times \mathbf{I}_2 \cdot \mathbf{n}) + \frac{1}{2} \mathbf{e}_z \times \mathbf{n} \cdot \mathbf{\Lambda} \cdot \mathbf{e}_z \times \mathbf{n} + \overline{\text{sign } B_1} \tau_t \text{Im } \omega_2 \quad (\text{F.5})$$

where the vectors \mathbf{I}_1 , \mathbf{I}_2 and the tensor $\mathbf{\Lambda}$ in terms of z -coordinates are given in [Eqs. \(5e,f\)](#) (it is easy to rewrite them in terms of isopycnal coordinates), ω_2 is the second-order correction to the first-order result $\omega_1 = \mathbf{k} \cdot \mathbf{u}_d$ and K_t^0 is the surface kinetic energy in the zeroth approximation. The computation of ω_2 is given in [Appendix G](#). Substituting, [\(G.2\)](#), [\(F.3\)–\(F.5\)](#) into [\(F.2\)](#), we get

$$K_t^0 \mathbf{R}_I^2(k, z) = \left[\frac{1}{4} (\mathbf{I}_1 \cdot \mathbf{I}_1 + 2 \mathbf{I}_1 \cdot \mathbf{I}_2 + 2 \text{Trace}(\mathbf{\Lambda} + \mathbf{\Omega})) \delta_\rho + \frac{1}{2} (\mathbf{I}_1 \mathbf{I}_1 + \mathbf{I}_1 \mathbf{I}_2 + \mathbf{I}_2 \mathbf{I}_1 + \mathbf{\Lambda} + \mathbf{\Omega}) \right] E_0(k, \rho_t) \quad (\text{F.6})$$

where δ_ρ is the 2-D Kroneker tensor within an isopycnal surface, $E_0(k, \rho_t)$ is the eddy energy spectrum at the surface in the zeroth approximation. In deriving [\(F.6\)](#), we have used the relation

$$(2\pi)^{-1} \int \mathbf{e} \times \mathbf{n} \mathbf{e} \times \mathbf{n} \mathbf{A} \cdot \mathbf{n} \mathbf{B} \cdot \mathbf{n} d\mathbf{n} = \frac{3}{8} \mathbf{A} \cdot \mathbf{B} \delta_\rho - \frac{1}{8} (\mathbf{A} \mathbf{B} + \mathbf{B} \mathbf{A}) \quad (\text{F.7})$$

where \mathbf{A} , \mathbf{B} are constant vectors. From [\(F.6\)](#), we obtain [\(11a–c\)](#).

Appendix G. Computing $u_{1,2}(\mathbf{k})$ in expansion [\(F.3\)](#) and ω_2 in [\(F.5\)](#)

Using the first relation (10b) of I, one can find the expansion [\(F.3\)](#) by computing the expansion $B' = b_0 + b_1 + b_2 + \dots$ in powers of $\Omega\tau$. In the spirit of the second relation of [\(14a\)](#) of I, we will search for the both $b_{1,2}$ in terms of the eigenfunctions $B_n(\rho)$ of the eigenvalue problem (6d) of I. In Section XII of I we have shown that b_1 is dominated by the barotropic component with $n = 0$. The result [\(24b, c\)](#) of I is equivalent to [\(F.4\)](#) above with account of [Eqs. \(17\), \(18\)](#) of I, [Eq. \(5e\)](#) and $\mathbf{u}^+ \approx \mathbf{u}_*$. The equation for the second-order perturbation b_2 can be obtained from [Eqs. \(11\), \(12\)](#) of I in analogy with [Eq. \(13a\)](#) of I

$$\hat{L}_0 b_2 = -i\Omega\tau b_1 + \Omega_1^2 \tau^2 b_0 - i\omega_2 \tau b_0 \quad (\text{G.1})$$

where ω_2 is the second-order correction to the first-order result $\omega_1 = \mathbf{k} \cdot \mathbf{u}_d$; Ω and ω_1 are given in [Eqs. \(11e\) and \(12d\)](#) of I. To compute ω_2 , we multiply [\(G.1\)](#) by $\bar{h} B_1(\rho)$ and integrate over ρ . The integral of the left hand side of [\(G.1\)](#) yields zero since, as pointed out in I, the expansions of the functions $b_1, b_2 \dots$ in eigenfunctions $B_n(\rho)$ do not contain terms with B_1 . Then, using [\(F.4\)](#), from the integral of the right hand side we deduce that

$$-\overline{i\Gamma(\rho)^{1/2}\omega_2(k)} = \frac{1}{4}\tau_t(1 + \sigma_t)^{-2}[(\mathbf{k} \cdot \mathbf{I}_1)^2 - 4\sigma_t^4 r_d^4 (\mathbf{e}_z \times \boldsymbol{\Pi} \cdot \mathbf{k})^2] \quad (\text{G.2})$$

where τ_t is the value of τ at the surface. We use the notation ((5e, f) and (5g) for the barotropic average, as well as relation (17a) of I and the zeroth-order term in the kinetic energy profile (5b). We solve equation (G.1) for b_2 as we solved Eq. (13a) of I. Writing b_2 as a series analogous to (14a) of I, and using methodology similar to the one presented in Sc. XII of I, we conclude that the main contribution to b_2 comes from the term $a_0 B_0$ which can be found by multiplying (G.1) by \bar{h} and then integrating over ρ . Using (10b), (12), (15a,b) of I, and $\mathbf{u}^+ \approx \mathbf{u}_*$, we arrive at (F.5).

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