

# Coherent backscattering of light by a layer of discrete random medium

Victor P. Tishkovets<sup>a</sup>, Michael I. Mishchenko<sup>b,\*</sup>

<sup>a</sup>*Astronomical Institute of Kharkov National University, Sumskaya St. 35, Kharkov 61022, Ukraine*

<sup>b</sup>*NASA Goddard Institute for Space Studies, 2880 Broadway, New York, NY 10025, USA*

Received 18 February 2003; received in revised form 17 July 2003; accepted 23 July 2003

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## Abstract

We consider the problem of backscattering of light by a layer of discrete random medium illuminated by an obliquely incident plane electromagnetic wave. The multiply scattered reflected radiation is assumed to consist of incoherent and coherent parts, the coherent part being caused by the interference of multiply scattered waves. Formulas describing the characteristics of the reflected radiation are derived assuming that the scattering particles are spherical. The formula for the incoherent contribution reproduces the standard vector radiative transfer equation. The interference contribution is expressed in terms of a system of Fredholm integral equations with kernels containing Bessel functions. The special case of the backscattering direction is considered in detail. It is shown that the angular width of the backscattering interference peak depends on the polar angle of the incident wave and on the azimuth angle of the reflection direction.

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*Keywords:* Coherent backscattering; Multiple scattering; Polarization

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## 1. Introduction

The phenomenon of electromagnetic scattering is widely used in remote sensing and laboratory characterization of various objects [1–3]. Calculations of various characteristics of radiation scattered by different discrete random media is important for atmospheric optics, astrophysics, biophysics, and

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\* Corresponding author. Tel.: +1-212-678-5590; fax: +1-212-678-5622.

E-mail addresses: [tishkovets@astron.kharkov.ua](mailto:tishkovets@astron.kharkov.ua) (V.P. Tishkovets), [mmishchenko@giss.nasa.gov](mailto:mmishchenko@giss.nasa.gov) (M.I. Mishchenko).

many other areas of science and engineering. More often than not, multiple-scattering effects on the characteristics of the measured detector response must be taken into account. Multiple scattering of light by closely packed medium is a complicated problem due to potentially significant near-field effects. Indeed, the scattered electromagnetic wave in the close vicinity of the scatterer is strongly inhomogeneous, which means that in the spherical coordinate system associated with the particle, the radial components of the electric and magnetic field vectors can be comparable to, if not exceeding the respective tangential components [4,5]. The analysis of scattering of such an inhomogeneous wave by an adjacent particle requires more sophisticated techniques than those used to address the problem of scattering of a plane electromagnetic wave.

The problem becomes simpler for sparse media in which waves propagating from one scatterer to another are spherical, thereby making applicable the so-called vector radiative transfer equation [1,2,6]. However, this equation does not describe explicitly interference effects such as the effect of coherent backscattering enhancement, which manifests itself as a sharp peak of intensity centered at exactly the backscattering direction [7]. This effect is also known as weak photon localization or the coherent opposition effect. In the case of unpolarized incident light, it can be accompanied by the so-called opposition polarization effect in the form of a region of negative linear polarization within a narrow range of backscattering angles [8]. Both effects are the result of constructive interference of multiply scattered waves propagating inside the medium along certain direct and reverse trajectories [7].

The coherent backscattering effect was first predicted theoretically in studies of backscattering of electromagnetic waves by turbulent plasmas [9]. Then it has been analyzed in numerous experimental and theoretical studies (see, e.g., [7,10,11] and references therein). A strong dependence of the angular width of the interference peak on the particle number density has been demonstrated both experimentally and theoretically [12–14]. However, only recently rigorous formulas describing the opposition effects have been derived in the particular case of *normal* incidence of light on a plane-parallel layer of discrete random medium. Specifically, a complete analytical solution for a semi-infinite medium filled with nonabsorbing, randomly positioned Rayleigh scatterers has been obtained in [15–17]. The rigorous approach was later extended to plane-parallel layers composed of randomly positioned and randomly oriented particles with arbitrary sizes, shapes, and refractive indices [18]. Numerical results for semi-infinite scattering media obtained in the double-scattering approximation showed a considerable dependence of the characteristics of the opposition effects on the particle microphysical properties [11,19,20].

In this paper, we generalize the method developed in [18] in order to derive equations describing the reflection of light by a layer of discrete random medium in the general case of *oblique* illumination. As in [7], the radiation reflected by the medium is decomposed into incoherent and coherent contributions. We show that the equation describing the incoherent contribution exactly reproduces the vector radiative transfer equation. The equation for the coherent contribution describes the interference of multiply scattered waves and has a more complex form than the radiative transfer equation. A qualitative analysis of this equation shows that the interference effects are apparent in a narrow region of scattering angles in the vicinity of the exact backscattering direction. The amplitude and width of the backscattering interference peak depend on the polar angle of the incidence direction. Specifically, the amplitude decreases and the width increases with increasing polar angle. Furthermore, there is significant dependence on the azimuth angle of the reflection direction resulting in an asymmetry of the opposition peak.

## 2. Basic equations

In this section, we introduce the necessary definitions and notation. Consider a discrete random medium in the form of a homogeneous and isotropic layer consisting of randomly positioned spherical particles and denote by  $Z_0$  its geometrical thickness. The scattering geometry is specified using the coordinate system shown in Fig. 1. An incident plane wave propagates along the  $z_{in}$ -axis of a coordinate system  $\hat{\mathbf{k}}_0$ . Throughout the paper, bold letters with carets  $\hat{\mathbf{n}}_i$  denote right-handed coordinate systems  $(x_i, y_i, z_i)$  with the  $z_i$ -axis along the vectors  $\mathbf{n}_i$ , whereas symbols  $-\hat{\mathbf{n}}_i$  denote coordinate systems with axis  $(x_i, -y_i, -z_i)$ . Coordinates of scatterers are determined in the laboratory coordinate system  $\hat{\mathbf{n}}_0$  whose  $x_0y_0$  plane coincides with the upper boundary of the medium. The scattered wave propagates along the  $z_{sc}$ -axis of the coordinate system  $\hat{\mathbf{k}}_{sc}$ . The rotation from  $\hat{\mathbf{n}}_0$  to  $\hat{\mathbf{k}}_0$  is determined by the Euler angles  $\varphi_0, \vartheta_0, \gamma_0$ . The Euler angles  $\varphi, \vartheta, \gamma$  specify the rotation from  $\hat{\mathbf{n}}_0$  to  $\hat{\mathbf{k}}_{sc}$ . Finally, the rotation from  $\hat{\mathbf{k}}_0$  to  $\hat{\mathbf{k}}_{sc}$  is specified by the Euler angles  $\varphi_{sc}, \vartheta_{sc}, \gamma_{sc}$  (Fig. 1).

It is convenient to describe wave scattering by using the circular-polarization basis (the so-called CP-representation), in which the incident wave can be written as [21]

$$\mathbf{E}^0 = \mathbf{e}_n(\hat{\mathbf{k}}_0) \exp(i\mathbf{k}_0\mathbf{r}), \tag{1}$$

where  $n = \pm 1$ ,  $\mathbf{k}_0$  is the wave vector ( $k_0 = 2\pi/\lambda, \lambda$  is the wavelength), and  $\mathbf{e}_n(\hat{\mathbf{k}}_0)$  is a covariant spherical basis vector [22] in the coordinate system  $\hat{\mathbf{k}}_0$ . When  $n = 1$ , the direction of rotation of the electric vector of wave (1) corresponds to the clockwise direction when looking in the direction of the vector  $\mathbf{k}_0$ .

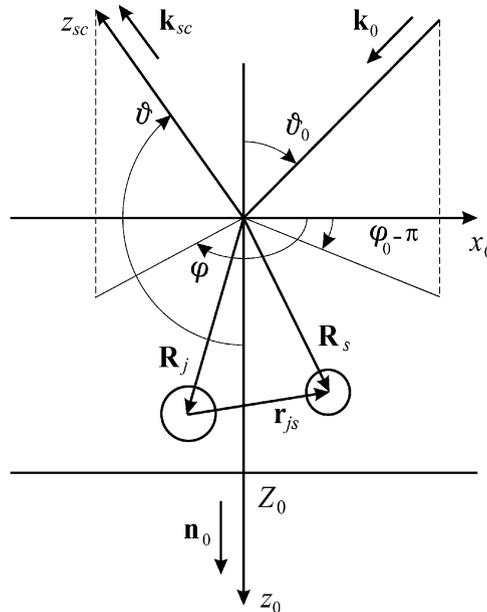


Fig. 1. Scattering geometry.

The linearity of the Maxwell equations allows one to define the amplitude scattering matrix of the entire layer  $T_{pn}$  as

$$\begin{pmatrix} E_1 \\ E_{-1} \end{pmatrix} = \frac{\exp(ik_0 r)}{-ik_0 r} \begin{pmatrix} T_{11} & T_{1-1} \\ T_{-11} & T_{-1-1} \end{pmatrix} \begin{pmatrix} E_1^0 \\ E_{-1}^0 \end{pmatrix}, \quad (2)$$

where  $r$  is the distance from the origin of the coordinate system  $\hat{\mathbf{n}}_0$  to the observation point, and then express it in the form

$$T_{pn} = \sum_j t_{pn}^{(j)}. \quad (3)$$

Here  $t_{pn}^{(j)}$  is the  $2 \times 2$  amplitude scattering matrix of the  $j$ th scatter. The  $4 \times 4$  scattering matrix  $S_{pn\mu\nu}$ , which transforms the Stokes parameters of the incident radiation into those of the scattered radiation, is defined by the following expression:

$$S_{pn\mu\nu} = \left\langle \sum_j t_{pn}^{(j)} t_{\mu\nu}^{*(j)} \right\rangle + \left\langle \sum_{j,\sigma \neq j} t_{pn}^{(j)} t_{\mu\nu}^{*(\sigma)} \right\rangle, \quad (4)$$

where the angular brackets denote ensemble averaging, the indices take on the values  $n, p, \mu, \nu = \pm 1$ , and the asterisk denotes complex conjugation. The first term on the right-hand side of this equation describes the incoherent part of the scattered radiation and the fraction of the coherent part caused by the interference of waves propagating along looped trajectories [18]. The second term describes the remainder of the coherent part.

We use the standard theory of light scattering by a system of spherical particles [1] to derive the requisite equations. In this case, the field scattered by the  $j$ th particle can be expressed in the form [1,4,18]

$$\mathbf{E}^{(j)} = \frac{\exp(ik_0 r_j)}{-ik_0 r_j} \sum_{LMp} \frac{2L+1}{2} A_{LM}^{(jpn)} D_{Mp}^{*L}(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_{sc}) \mathbf{e}_p(\hat{\mathbf{k}}_{sc}). \quad (5)$$

Here  $r_j$  is the distance from particle  $j$  to the observation point,  $\hat{\mathbf{k}}_{sc}$  is the wave vector in the scattering direction ( $k_{sc} = 2\pi/\lambda$ ),  $\mathbf{e}_p(\hat{\mathbf{k}}_{sc})$  is a covariant spherical basis vector in the coordinate system  $\hat{\mathbf{k}}_{sc}$ , and the  $D_{Mp}^L(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_{sc}) = D_{Mp}^L(\varphi, \vartheta, \gamma) = \exp(-iM\varphi) d_{Mp}^L(\vartheta) \exp(-iP\gamma)$  are Wigner  $D$  functions [22]. It is assumed here that the scattering directions are identical for all particles of the medium.

In contrast to the traditional representation of the scattered field using the helicity unit vector basis [22] defined with respect to the scattering plane, we express field (5) using the spherical unit vector basis defined with respect to the coordinate system  $\hat{\mathbf{k}}_{sc}$ . This approach is more convenient when one has to take into account the transformation of the electric field vector components upon rotations of a reference frame.

The coefficients  $A_{LM}^{(jpn)}$  are determined by the system of equations [1,4,18]

$$A_{LM}^{(jpn)} = a_L^{(jpn)} \exp(i\mathbf{k}_0 \mathbf{R}_j) D_{Mn}^L(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_0) + \sum_q a_L^{(jpn)} \sum_{s \neq j} \sum_{lm} A_{lm}^{(sqn)} H_{LMlm}^{(q)}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}), \quad (6)$$

where  $a_L^{(jpn)} = a_L^{(j)} + pnb_L^{(j)}$ ,  $a_L^{(j)}$  and  $b_L^{(j)}$  are the Lorenz–Mie coefficients [1–3],  $q = \pm 1$ ,  $\mathbf{R}_j$  is the radius-vector of particle  $j$  (Fig. 1), the  $H_{LMlm}^{(q)}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj})$  are coefficients of the addition theorem for the vector Helmholtz harmonics [23,24], and  $\hat{\mathbf{r}}_{sj}$  is the coordinate system with the  $z_{sj}$  axis along the vector  $\mathbf{r}_{sj}$ . When the distances between particles are sufficiently large, the wave propagating from one particle to another is spherical, and these coefficients can be written in the form [18]

$$H_{LMlm}^{(q)}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}) = \frac{2l + 1}{2} \frac{\exp(ik_0r_{js})}{-ik_0r_{js}} D_{Mq}^L(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}) D_{mq}^{*l}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}). \quad (7)$$

Eqs. (5)–(7) are applicable when the number of particles  $\tilde{N}$  is finite. The calculation of the matrix  $S_{p\mu\nu}$  in the limit  $\tilde{N} \rightarrow \infty$  is greatly simplified when the discrete random medium is characterized by a complex effective refractive index  $\varepsilon = \text{Re}(\varepsilon) + i \text{Im}(\varepsilon)$ . Following [1,3,21], the latter is given by  $\varepsilon = 1 + i2\pi v \langle t_{pn}^{(j)}(\theta_{sc}=0) \rangle \delta_{pn}/k_0^3$ , where  $v$  is the particle number density and the angular brackets denote averaging over particle microphysical characteristics. This expression for  $\varepsilon$  can be obtained directly from system (6) upon taking the limit  $\tilde{N} \rightarrow \infty$  or by using the general approach outlined in [6]. The effective extinction coefficient of the medium is given by  $2 \text{Im}(\varepsilon)k_0 = v \langle C_{\text{ext}}^{(j)} \rangle$ , where  $C_{\text{ext}}^{(j)}$  is the extinction cross section of particle  $j$  [1–3]. A related quantity widely used in the multiple-scattering theory is the so-called photon transport mean free path  $l_{tr}$  [14]. This quantity takes into account the potential asymmetry of the single-scattering phase function and allows one to reconcile the results of theoretical calculations of the interference effects performed for Rayleigh scatterers with experimental data obtained for wavelength-sized particles [17]. In this paper the value of  $\varepsilon$  is assumed to be known. Then the exponential factor in Eq. (7) depends on  $k_0\varepsilon$ , and the exponents in Eqs. (5) and (6) must be replaced according to

$$\begin{aligned} k_0r_j &\Rightarrow k_0r - \mathbf{K}\mathbf{R}_j, \\ \mathbf{k}_0\mathbf{R}_j &\Rightarrow \mathbf{K}_0\mathbf{R}_j, \end{aligned} \quad (8)$$

where

$$\begin{aligned} \mathbf{K} &= \mathbf{k}_{sc} + k_0\mathbf{n}_0 \frac{\varepsilon - 1}{\cos \vartheta}, \\ \mathbf{K}_0 &= \mathbf{k}_0 + k_0\mathbf{n}_0 \frac{\varepsilon - 1}{\cos \vartheta_0} \end{aligned} \quad (9)$$

and  $|\mathbf{n}_0| = 1$ .

In this paper, we use the description of radiation adopted in the radiative transfer theory. Specifically, the characteristics of the incident wave are defined with respect to the meridional plane through the vectors  $\mathbf{k}_0$  and  $\mathbf{n}_0$ , whereas the characteristics of the scattered wave are defined with respect to the meridional plane through the vectors  $\mathbf{k}_{sc}$  and  $\mathbf{n}_0$ .

To determine the matrix  $t_{pn}^{(j)}$ , let us introduce the basis vectors  $\mathbf{e}_\perp$  and  $\mathbf{e}_\parallel$  with respect to the plane through the vectors  $\mathbf{n}_0$  and  $\mathbf{k}_0$  and analogous vectors with respect to the plane through the vectors  $\mathbf{n}_0$  and  $\mathbf{k}_{sc}$ . The vectors  $\mathbf{e}_\perp$  are perpendicular to their respective reference planes, whereas the vectors  $\mathbf{e}_\parallel$  are parallel to them. Transforming these vectors into spherical basis vectors yields the

contravariant spherical basis vectors [22]  $\mathbf{e}^n(\hat{\mathbf{k}}_0)$  and  $\mathbf{e}^p(\hat{\mathbf{k}}_{sc})$ , which are rotated with respect to the vectors  $\mathbf{e}_n(\hat{\mathbf{k}}_0)$  and  $\mathbf{e}_p(\hat{\mathbf{k}}_{sc})$  through angles  $-\gamma_0$  and  $-\gamma$ , respectively. We, therefore, obtain from Eqs. (2) and (5)

$$t_{pn}^{(j)} = \exp(-i\mathbf{K}\mathbf{R}_j + in\gamma_0 - ip\gamma) \sum_{LM} \frac{2L+1}{2} A_{LM}^{(jpn)} D_{Mp}^{*L}(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_{sc}). \quad (10)$$

The solution of the system of equations (6) can be obtained by iteration. This representation of the solution corresponds to the expansion of the coefficients  $A_{LM}^{(jpn)}$  in a multiple-scattering series. The first two terms of this series are

$$A_{LM}^{(jpn)} = a_L^{(jpn)} D_{Mn}^L(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_0) \exp(i\mathbf{K}_0\mathbf{R}_j) + \sum_q a_L^{(j pq)} \sum_{lms \neq j} a_l^{(sqn)} H_{LMlm}^{(q)}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}) \exp(i\mathbf{K}_0\mathbf{R}_s) D_{mn}^l(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_0) + \dots, \quad (11)$$

where the first term corresponds to the single scattering by particle  $j$ , the second one corresponds to the second-order scattering first by particle  $s$  and then by particle  $j$ , and so on.

Matrix (10) has a very important property. Specifically, let us substitute the second term on the right-hand side of Eq. (11) into Eq. (10) and denote the result by  $t_{pn}^{(j)}(\hat{\mathbf{k}}_0, \hat{\mathbf{k}}_{sc}, -\mathbf{K}, \mathbf{k})$ . The latter can be transformed by using the well-known properties of the Wigner  $D$  function (see Appendix A) as follows:

$$\begin{aligned} t_{pn}^{(j)}(\hat{\mathbf{k}}_0, \hat{\mathbf{k}}_{sc}, -\mathbf{K}, \mathbf{K}_0) &= \exp(-i\mathbf{K}\mathbf{R}_j + in\gamma_0 - ip\gamma) \sum_{LMq lm} \frac{(2L+1)}{2} a_L^{(j pq)} a_l^{(sqn)} \\ &\quad \times D_{Mp}^{*L}(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_{sc}) \exp(i\mathbf{K}_0\mathbf{R}_\sigma) H_{LMlm}^{(q)}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{\sigma j}) D_{mn}^l(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_0) \\ &= \exp[i\mathbf{K}_0\mathbf{R}_\sigma - in(\pi - \gamma_0) + ip(\pi - \gamma)] \sum_{LMq lm} \frac{(2L+1)}{2} a_l^{(sqn)} a_L^{(j pq)} \\ &\quad \times D_{mn}^{*l}(\hat{\mathbf{n}}_0, -\hat{\mathbf{k}}_0) \exp(-i\mathbf{K}\mathbf{R}_j) H_{lmLM}^{(q)}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{j\sigma}) D_{Mp}^L(\hat{\mathbf{n}}_0, -\hat{\mathbf{k}}_{sc}) \\ &= t_{np}^{(\sigma)}(-\hat{\mathbf{k}}_{sc}, -\hat{\mathbf{k}}_0, \mathbf{K}_0, -\mathbf{K}). \end{aligned} \quad (12)$$

It is now seen that the amplitude scattering matrix corresponding to the wave incident along the vector  $\mathbf{k}_0$  and scattered first by particle  $\sigma$  and then by particle  $j$  in the direction  $\mathbf{k}_{sc}$  is equal to the transpose of the matrix corresponding to the wave incident in the direction  $-\mathbf{k}_{sc}$  and scattered first by particle  $j$  and then by particle  $\sigma$  in the direction  $-\mathbf{k}_0$ . Analogous equalities can be derived for any order of scattering. They represent one of the various forms of the fundamental reciprocity principle [21,25].

### 3. Incoherent scattering

It is well known [7] that the incoherent part of the reflected radiation corresponds to the sum of the ladder diagrams in the diagrammatic representation of the Bethe–Salpeter equation. The summation

of these diagrams in the case of a sparse medium leads to the vector radiative transfer equation [6,7]. To derive this equation we calculate the products

$$t_{pn}^{(j)} t_{\mu\nu}^{*(j)} = \exp[i\mathbf{R}_j(\mathbf{K}^* - \mathbf{K})] \exp[-i\gamma_0(v - n) + i\gamma(\mu - p)] \times \sum_{LMlm} \frac{(2L+1)(2l+1)}{4} A_{LM}^{(jpn)} A_{lm}^{*(j\mu\nu)} D_{Mp}^{*L}(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_{sc}) D_{m\mu}^l(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_{sc}) \quad (13)$$

and retain only the combinations  $A_{LM}^{(jpn)} A_{lm}^{*(j\mu\nu)}$  corresponding to incoherent scattering. To do that, we write series (11) for the incident wave with polarization  $n$  and the scattered wave with polarization  $p$ . We then write the same series for the incident wave with polarization  $\nu$  and the scattered wave with polarization  $\mu$ . Calculating the complex conjugate of the second series, we multiply both series term by term for every order of scattering. Since the particles are randomly positioned, a major contribution to these products give the scattering paths for which both waves are first scattered by a certain scatterer, propagate along the same trajectory, and then are scattered by particle  $j$ . This scenario corresponds to incoherent scattering. The sum of these products is equivalent to the solution of the following system of equations:

$$A_{LM}^{(jpn)} A_{L_1 M_1}^{*(j\mu\nu)} = a_L^{(jpn)} a_{L_1}^{*(j\mu\nu)} D_{Mn}^L(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_0) D_{M_1 \nu}^{*L_1}(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_0) \exp[i\mathbf{R}_j(\mathbf{K}_0 - \mathbf{K}_0^*)] + \sum_{qq_1} a_L^{(jpn)} a_{L_1}^{*(j\mu\nu)} \sum_{s \neq j} \sum_{lm_1 m_1} A_{lm}^{(sqn)} A_{l_1 m_1}^{*(sq_1 \nu)} H_{LMlm}^{(q)}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}) H_{L_1 M_1 l_1 m_1}^{*(q_1)}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}). \quad (14)$$

We then perform ensemble averaging as described in [18]. Let us denote

$$\chi_{L_1}^{(j)(pn)(\mu\nu)} = \sum_{Ll} \frac{(2L+1)(2l+1)}{4} a_L^{(jpn)} a_l^{*(j\mu\nu)} C_{L-nl\nu}^{L_1 M_0} C_{L-pl\mu}^{L_1 N_0},$$

$$\chi_L^{(pn)(\mu\nu)} = \langle \chi_L^{(j)(pn)(\mu\nu)} \rangle, \quad (15)$$

where the  $C$ s are Clebsch–Gordan coefficients [22],  $M_0 = \nu - n$ ,  $N_0 = \mu - p$ , and the angular brackets denote averaging over the particle microphysical characteristics. The coefficients  $\chi_L^{(j)(pn)(\mu\nu)}$  are coefficients of the expansion of the scattering matrix components for isolated particle  $j$  in the Wigner  $D$  functions:

$$\tilde{t}_{pn}^{(j)} \tilde{t}_{\mu\nu}^{(j)} = \sum_L \chi_L^{(j)(pn)(\mu\nu)} d_{M_0 N_0}^L(\vartheta). \quad (16)$$

Here  $\tilde{t}_{pq_1}^{(j)}$  is the amplitude scattering matrix of particle  $j$ . Expanding the products of the Wigner  $D$  functions in Eqs. (13) and (14) in Clebsch–Gordan series [22] (see Appendix A) and averaging over the ensemble leads to the following equations for the incoherent part of the reflected radiation:

$$S_{pn\mu\nu}^{(nc)} = \frac{\nu}{k_0} \sum_{LM} D_{MN_0}^L(\varphi - \varphi_0, \vartheta, 0) \int_0^{k_0 z_0} \alpha_{LM}^{(z)(pn)(\mu\nu)} \exp\left(\frac{\tau_z}{\cos \vartheta}\right) dz, \quad (17)$$

where the matrix  $S_{pn\mu\nu}^{(nc)}$  is defined per unit area of the upper boundary of the medium. The coefficients  $\alpha_{LM}^{(z)(pn)(\mu\nu)}$  are determined from the system of equations

$$\alpha_{LM}^{(z)(pn)(\mu\nu)} = \chi_L^{(pn)(\mu\nu)} d_{MM_0}^L(\vartheta_0) \exp\left(-\frac{\tau_z}{\cos \vartheta_0}\right) + \frac{2\pi\nu}{k_0^3} \sum_{qq_1} \chi_L^{(pq)(\mu q_1)} \int \sum_l \alpha_{lM}^{(y)(qn)(q_1\nu)} \exp(-\tau_\rho) d_{MN}^L(\omega) d_{MN}^l(\omega) \sin \omega \, d\omega \, d\rho, \quad (18)$$

where  $\rho, \omega$  are the polar coordinates of the integration point with respect to the point  $z$ , the angle  $\omega$  ( $0 \leq \omega \leq \pi$ ) is measured from the direction  $-\mathbf{n}_0$  (see Fig. 1),  $\tau_x = 2 \operatorname{Im}(\varepsilon)x$ ,  $M_0 = \nu - n$ ,  $N_0 = \mu - p$ ,  $N = q_1 - q$ , and  $y = z - \rho \cos \omega$ . The upper integration limit over  $\rho$  is equal to  $z/\cos \omega$  for  $\omega < \pi/2$  and to  $(z - Z_0 k_0)/\cos \omega$  for  $\omega > \pi/2$ .

Eqs. (17) and (18) are equivalent to the vector radiative transfer equation in the CP-representation. In the case of scalar waves, an analogous equation has been obtained in [26].

Introducing the notation

$$\begin{aligned} \tau_0 &= 2k_0 Z_0 \operatorname{Im}(\varepsilon), \\ \xi_0 &= \frac{\pi\nu}{k_0^3 \operatorname{Im}(\varepsilon)}, \\ X_M^{(pn)(\mu\nu)}(\vartheta, \vartheta_0) &= \sum_{L=|M|}^{\infty} \chi_L^{(pn)(\mu\nu)} d_{M\mu-p}^L(\vartheta) d_{M\nu-n}^L(\vartheta_0), \end{aligned} \quad (19)$$

we can rewrite Eqs. (17) and (18) in the following form:

$$S_{pn\mu\nu}^{(nc)} = \frac{k_0^2 \xi_0}{2\pi} \sum_M \exp[-iM(\varphi - \varphi_0)] \int_0^{\tau_0} Y_M^{(\tau)(pn)(\mu\nu)}(\vartheta) \exp\left(\frac{\tau}{\cos \vartheta}\right) \, d\tau, \quad (20)$$

where

$$\begin{aligned} Y_M^{(\tau)(pn)(\mu\nu)}(\vartheta) &= X_M^{(pn)(\mu\nu)}(\vartheta, \vartheta_0) \exp\left(-\frac{\tau}{\cos \vartheta_0}\right) + \xi_0 \sum_{qq_1} \int X_M^{(pq)(\mu q_1)}(\vartheta, \omega) \\ &\quad \times Y_M^{(\tau-x)(qn)(q_1\nu)}(\omega) \exp\left(-\frac{x}{\cos \omega}\right) \tan \omega \, d\omega \, dx. \end{aligned} \quad (21)$$

Here the upper integration limit over  $x$  is  $\tau$  for  $\omega < \pi/2$  and  $\tau - \tau_0$  for  $\omega > \pi/2$ .

Eq. (21) has a simpler form than Eq. (18), but depends on  $\vartheta$ . It should be noted that the equations derived can also be applied to light transmitted by the medium provided that the exponent  $z$  in Eq. (17) is replaced by  $z - k_0 Z_0$  and  $\tau$  in Eq. (20) is replaced by  $\tau - \tau_0$ . These equations can be easily generalized to the case of randomly oriented nonspherical particles and vertically inhomogeneous media. The coefficients  $\chi_L^{(pn)(\mu\nu)}$  determine the dependence of the matrix  $S_{pn\mu\nu}^{(nc)}$  on particle properties. If we replace these coefficients by the corresponding coefficients for randomly oriented nonspherical particles, Eqs. (18) and (20) remain valid. In the case of vertically inhomogeneous media the parameters  $\nu$  and  $\operatorname{Im}(\varepsilon)$  (and possibly the coefficients  $\chi_L^{(pn)(\mu\nu)}$ ) depend on  $z$  and should appear under the integration sign.

Since  $\text{Im}(\varepsilon) \sim v$ , matrix (20) depends on the product  $Z_0v$  rather than on  $v$  and  $Z_0$  separately. In other words, matrix (20) remains the same for both geometrically thin and geometrically thick layers provided that the products  $Z_0v$  and the particle properties remain the same.

#### 4. Coherent backscattering

The coherent part of the radiation reflected by the medium is caused by the constructive interference of multiply scattered waves propagating along direct and reverse paths and corresponds to the sum of the cyclical diagrams in the diagrammatic representation of the Bethe–Salpeter equation [7].

To derive an equation describing the coherent part, we write series (11) for the incident wave with initial polarization  $n$  that is scattered first by particle  $\sigma$ , propagates along a certain trajectory, is scattered by particle  $j$ , and has final polarization  $p$ . This series is then substituted in Eq. (10). Next we rewrite the series for the incident wave with initial polarization  $\mu$  that is scattered first by particle  $j$ , propagates along the same trajectory but in the reverse direction, is then scattered by particle  $\sigma$ , and has final polarization  $v$ . Applying the reciprocity principle (12) to the second series, we multiply both series term by term for every order of scattering and sum up these products over all  $\sigma$ . Note that the value of  $\sigma$  can be equal to  $j$  starting from the third order of scattering, which corresponds to the propagation of waves along looped trajectories. The series of such products thus obtained can be interpreted as an iterative solution of a system of equations. In other words, the reciprocity principle (12) allows one to transform the cyclical diagrams into ladder diagrams and, using the approach outlined in the previous section, to derive an equation describing the coherent part of the reflected light. The result is as follows:

$$\sum_{\sigma} t_{pn}^{(j)} t_{v\mu}^{*(\sigma)} + t_{pn}^{(0j)} t_{v\mu}^{*(0j)} = \exp[i\gamma(v - p) - i\gamma_0(\mu - n)] \exp[-i\mathbf{R}_j(\mathbf{K} + \mathbf{K}_0^*)] \times \sum_{LMlm} \frac{(2L + 1)(2l + 1)}{4} A_{LM}^{(jpn)} A_{lm}^{*(j\mu v)} D_{Mp}^{*L}(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_{sc}) D_{m\mu}^L(\hat{\mathbf{n}}_0, -\hat{\mathbf{k}}_0) \quad (22)$$

and the coefficients  $A_{LM}^{(jpn)} A_{L_1M_1}^{*(j\mu v)}$  are determined from the system of equations

$$A_{LM}^{(jpn)} A_{L_1M_1}^{*(j\mu v)} = a_L^{(jpn)} a_{L_1}^{*(j\mu v)} D_{Mn}^L(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_0) D_{M_1v}^{*L_1}(\hat{\mathbf{n}}_0, -\hat{\mathbf{k}}_{sc}) \exp[i\mathbf{R}_j(\mathbf{K}_0 + \mathbf{K}^*)] + \sum_{q_1} a_L^{(jpn)} a_{L_1}^{*(j\mu q_1)} \sum_{s \neq j} \sum_{lm_1 m_1} A_{lm}^{(sqn)} A_{l_1 m_1}^{*(sq_1 v)} H_{LMlm}^{(q)}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}) H_{L_1 M_1 l_1 m_1}^{*(q_1)}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}). \quad (23)$$

The matrix  $t_{pn}^{(0j)} t_{v\mu}^{*(0j)}$  describes the single scattering by particle  $j$  assuming that the latter is imbedded in the effective medium [cf. the first term on the right-hand side of Eq. (23)].

Eqs. (22) and (23) can be also derived using a simpler approach. Specifically, Eq. (12) for an arbitrary fixed couple of particles  $j$  and  $\sigma$  can be rewritten in the form valid for any order of scattering between the particles. Then summing up over all possible  $\sigma$  and over all orders of scattering and taking into account that  $\sigma$  can become equal to  $j$  starting from the third order of

scattering, we obtain

$$\sum_{\sigma} t_{\nu\mu}^{(\sigma)}(\hat{\mathbf{k}}_0, \hat{\mathbf{k}}_{sc}, -\mathbf{K}, \mathbf{K}_0) + t_{\nu\mu}^{(0j)} = t_{\mu\nu}^{(j)}(-\hat{\mathbf{k}}_{sc}, -\hat{\mathbf{k}}_0, \mathbf{K}_0, -\mathbf{K}). \quad (24)$$

Using this equality, we obtain from Eqs. (10) and (6)

$$\sum_{\sigma} t_{\nu\mu}^{(\sigma)} + t_{\nu\mu}^{(0j)} = \exp[i\mathbf{K}_0\mathbf{R}_j + i\nu(\pi - \gamma) - i\mu(\pi - \gamma_0)] \sum_{LM} \frac{2L+1}{2} \hat{A}_{LM}^{(j\mu\nu)} D_{M\mu}^{*L}(\hat{\mathbf{n}}_0, -\hat{\mathbf{k}}_0), \quad (25)$$

where the coefficients  $\hat{A}_{LM}^{(j\mu\nu)}$  are determined from the system of equations

$$\hat{A}_{LM}^{(j\mu\nu)} = a_L^{(j\mu\nu)} \exp(-i\mathbf{K}\mathbf{R}_j) D_{M\nu}^L(\hat{\mathbf{n}}_0, -\hat{\mathbf{k}}_{sc}) + \sum_q a_L^{(j\mu q)} \sum_{s \neq j} \sum_{lm} \hat{A}_{lm}^{(sq\nu)} H_{LMlm}^{(q)}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}). \quad (26)$$

Further straightforward manipulations lead to Eqs. (22) and (23).

Substituting Eq. (23) into Eq. (22), decomposing the Wigner function products into the Clebsch–Gordan series [22] (see Appendix A), and denoting

$$\begin{aligned} \eta_{LM}^{(j)(pn)(\mu\nu)} &= \exp(i\nu\gamma + i n\gamma_0) \sum_{M_1} \zeta_{LM_1}^{(j)(pn)(\mu\nu)} D_{MM_1}^{*L}(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_0), \\ \zeta_{L_1M_1}^{(j)(pn)(\mu\nu)} &= \sum_{Ll} \frac{(2L+1)(2l+1)}{4} a_L^{(jpn)} a_l^{*(j\mu\nu)} (-1)^{l+m} C_{L-nl-m}^{L_1M_1} C_{L-p\mu}^{L_1\mu-p} D_{mv}^l(\hat{\mathbf{k}}_0, \hat{\mathbf{k}}_{sc}), \end{aligned} \quad (27)$$

where  $m = -M_1 - n$ , yield

$$\begin{aligned} \sum_{\sigma} t_{pn}^{(j)} t_{\nu\mu}^{*(\sigma)} &= \exp(-i\mathbf{R}_j\mathbf{K}_1) \sum_{LMqq_1} (-1)^L \eta_{LM}^{*(j)(q_1\mu)(qp)} \sum_{slm} W_{lm}^{(s)(qn)(q_1\nu)} \\ &\times \frac{\exp[i\mathbf{r}_{js}(\mathbf{k}_0 + \mathbf{k}_{sc}) - 2 \operatorname{Im}(\varepsilon)k_0r_{js}]}{(k_0r_{js})^2} D_{MN}^{*L}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}) D_{mN}^l(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}). \end{aligned} \quad (28)$$

Here  $N = q_1 - q$ ,  $\mathbf{K}_1 = k_0\mathbf{n}_0[(\varepsilon - 1)/\cos\vartheta + (\varepsilon^* - 1)/\cos\vartheta_0]$ , and the coefficients  $W_{LM}^{(j)(pn)(\mu\nu)}$  follow from the system of equations

$$\begin{aligned} W_{LM}^{(j)(pn)(\mu\nu)} &= \exp(i\mathbf{R}_j\mathbf{K}_1^*) \eta_{LM}^{(j)(pn)(\mu\nu)} + \sum_{qq_1} \chi_L^{(j)(pq)(\mu q_1)} \sum_{slm} W_{lm}^{(s)(qn)(q_1\nu)} \\ &\times \frac{\exp[i\mathbf{r}_{js}(\mathbf{k}_0 + \mathbf{k}_{sc}) - 2 \operatorname{Im}(\varepsilon)k_0r_{js}]}{(k_0r_{js})^2} D_{MN}^{*L}(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}) D_{mN}^l(\hat{\mathbf{n}}_0, \hat{\mathbf{r}}_{sj}). \end{aligned} \quad (29)$$

Taking into account that

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \exp[iM\varphi + i\mathbf{r}(\mathbf{k}_0 + \mathbf{k}_{sc})] d\varphi \\ &= i^{-M} \exp\left(iM\varphi_1 - 2ik_0r \cos \frac{\vartheta_{sc}}{2} \cos\vartheta_1 \cos\omega\right) J_M\left(2k_0r \cos \frac{\vartheta_{sc}}{2} \sin\vartheta_1 \sin\omega\right), \end{aligned} \quad (30)$$

where  $J_M(x)$  is the Bessel function,  $\vartheta_1$  and  $\varphi_1$  are the spherical coordinates of the vector  $\mathbf{k}_1 = \mathbf{k}_0 + \mathbf{k}_{sc}$  in the coordinate system  $\hat{\mathbf{n}}_0$ , and the angle  $\omega$  is the same as in Eq. (18), and averaging over the ensemble gives

$$S_{pnv\mu}^{(co)} = \frac{2\pi v^2}{k_0^4} \sum_{qq_1LM} (-1)^L \exp(iM\varphi_1) \eta_{LM}^{*(q_1\mu)(qp)} \int_0^{k_0 Z_0} \beta_{LM}^{(z)(qn)(q_1v)} \exp(-\varepsilon_1 z) dz. \quad (31)$$

Here the matrix  $S_{pnv\mu}^{(co)}$  is defined per unit surface area of the upper boundary of the medium,

$$\varepsilon_1 = i \left( \frac{\varepsilon - 1}{\cos \vartheta} + \frac{\varepsilon^* - 1}{\cos \vartheta_0} + 2 \cos \frac{\vartheta_{sc}}{2} \cos \vartheta_1 \right),$$

$$\eta_{LM}^{(pn)(\mu v)} = \langle \eta_{LM}^{(j)(pn)(\mu v)} \rangle, \quad (32)$$

and the coefficients  $\beta_{LM}^{(z)(pn)(\mu v)}$  are determined from the system

$$\begin{aligned} \beta_{LM}^{(z)(pn)(\mu v)} &= \exp(-\varepsilon_1^* z) B_{LM}^{(z)(pn)(\mu v)} \\ &+ \frac{2\pi v}{k_0^3} \sum_{qq_1lm} \chi_l^{(pq)(\mu q_1)} i^{M-m} \int \beta_{lm}^{(y)(qn)(q_1v)} \exp(-\tau_\rho) \\ &\times d_{MN_0}^L(\omega) d_{mN_0}^l(\omega) J_{m-M} \left( 2\rho \sin \vartheta_1 \cos \frac{\vartheta_{sc}}{2} \sin \omega \right) d\rho \sin \omega d\omega. \end{aligned} \quad (33)$$

The integral in Eq. (33) has the same meaning as that in Eq. (18),  $\tau_\rho = 2 \operatorname{Im}(\varepsilon)\rho$ ,  $N_0 = \mu - p$ ,  $y = z - \rho \cos \omega$ , and the coefficients  $B_{LM}^{(z)(pn)(\mu v)}$  are given by

$$\begin{aligned} B_{LM}^{(z)(pn)(\mu v)} &= \sum_{lm} \exp(-im\varphi_1) \eta_{lm}^{(pn)(\mu v)} i^{M-m} \int \exp(-\tilde{\tau}_\rho) \\ &\times J_{m-M} \left( 2\rho \sin \vartheta_1 \cos \frac{\vartheta_{sc}}{2} \sin \omega \right) d_{MN_0}^L(\omega) d_{mN_0}^l(\omega) d\rho \sin \omega d\omega, \end{aligned} \quad (34)$$

where

$$\tilde{\tau}_\rho = \rho [2 \operatorname{Im}(\varepsilon) - \varepsilon_1^* \cos \omega]. \quad (35)$$

The meaning of the coefficients  $\zeta_{LM}^{(pn)(\mu v)}$  is as follows. Let the first wave propagate along the vector  $\mathbf{k}_0$  and be scattered by a particle in the direction  $\mathbf{r}$ . The second wave propagates along the vector  $-\mathbf{k}_{sc}$  and is scattered by the same particle in the same direction  $\mathbf{r}$ . The corresponding amplitude matrices are

$$\begin{aligned} \tilde{i}_{pn}^{(j)} &= \exp(-iv\psi_1 - ip\gamma_1) \sum_L \frac{2L+1}{2} a_L^{(jpn)} D_{np}^{*L}(\hat{\mathbf{k}}_0, \mathbf{r}), \\ \tilde{i}_{\mu v}^{(j)} &= \exp(-iv\psi_2 - i\mu\gamma_2) \sum_L \frac{2L+1}{2} a_L^{(j\mu v)} D_{v\mu}^{*L}(-\hat{\mathbf{k}}_{sc}, \mathbf{r}), \end{aligned} \quad (36)$$

where  $\psi_1$  and  $\psi_2$  are the azimuth angles of the vector  $\mathbf{r}$  in the coordinate systems  $\hat{\mathbf{k}}_0$  and  $-\hat{\mathbf{k}}_{sc}$ , respectively. Expanding the products  $\hat{t}_{pn}^{(j)} \hat{t}_{\mu\nu}^{*(j)}$  in functions  $D_{MN}^L(\hat{\mathbf{k}}_0, \hat{\mathbf{r}})$  gives

$$\begin{aligned} \hat{t}_{pn}^{(j)} \hat{t}_{\mu\nu}^{*(j)} &= \exp(i\nu\psi_2 + i\mu\gamma_2 - in\psi_1 - ip\gamma_1) \sum_{L_1 lm} \frac{(2L_1 + 1)(2l + 1)}{4} a_{L_1}^{(jpn)} a_l^{*(j\mu\nu)} \\ &\quad \times D_{np}^{*L_1}(\hat{\mathbf{k}}_0, \hat{\mathbf{r}}) D_{mv}^{*l}(\hat{\mathbf{k}}_0, -\hat{\mathbf{k}}_{sc}) D_{m\mu}^l(\hat{\mathbf{k}}_0, \hat{\mathbf{r}}) \\ &= -\exp(i\nu\psi_2 + i\mu\gamma_2 - in\psi_1 - ip\gamma_1) \sum_{LM} \zeta_{LM}^{(j)(pn)(\mu\nu)} D_{M\mu-p}^L(\hat{\mathbf{k}}_0, \hat{\mathbf{r}}). \end{aligned} \quad (37)$$

Decomposing Eq. (37) in functions  $D_{MN}^L(\hat{\mathbf{n}}_0, \hat{\mathbf{r}})$  yields the coefficients  $\eta_{LM}^{(j)(pn)(\mu\nu)}$ . It should be noted that the coefficients  $\eta_{LM}^{(pn)(\mu\nu)}$  do not depend on  $\gamma_0$  and  $\gamma$ . They can be rewritten in the form

$$\begin{aligned} \eta_{LM}^{(pn)(\mu\nu)} &= (-1)^{M-n} \exp(in\gamma_0 + i\nu\gamma) \sum_{L_1 M_1 l m} \frac{(2L_1 + 1)(2l + 1)}{4} \langle a_{L_1}^{(jpn)} a_l^{(j\mu\nu)} \rangle \\ &\quad \times (-1)^l C_{L_1-M_1 l-m}^{LM} C_{L_1-p l \mu}^{L\mu-p} D_{M_1 n}^{L_1}(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_0) D_{mv}^l(\hat{\mathbf{n}}_0, \hat{\mathbf{k}}_{sc}) \\ &= (-1)^{M-n} \sum_{L_1 M_1 l m} \frac{(2L_1 + 1)(2l + 1)}{4} \langle a_{L_1}^{(jpn)} a_l^{(j\mu\nu)} \rangle \\ &\quad \times (-1)^l C_{L_1-M_1 l-m}^{LM} C_{L_1-p l \mu}^{L\mu-p} D_{M_1 n}^{L_1}(\varphi_0, \vartheta_0, 0) D_{mv}^l(\varphi, \vartheta, 0), \end{aligned} \quad (38)$$

where  $M_1 + m = -M$ .

The generalization of the equations for the coherent part to the case of randomly oriented non-spherical particles and/or inhomogeneous media is similar to that for the incoherent part. Eq. (21) is a Fredholm integral equation of the second kind, and Eq. (33) is a system of Fredholm integral equations of the second kind. The kernels of these equations are oscillating functions, in contrast to those in Eq. (21). Moreover, the coefficients  $B_{LM}^{(z)(pn)(\mu\nu)}$  are also oscillating functions. The numerical solution of the system of such equations is a very complex problem. Therefore, in this paper we will consider only simplified particular cases of this system.

## 5. Results and discussion

### 5.1. The case of the exact backscattering direction

In the case ( $\mathbf{k}_{sc} = -\mathbf{k}_0$ )

$$\vartheta + \vartheta_0 = \pi, \quad \vartheta_{sc} = \pi, \quad \varphi = \varphi_0 + \pi \quad (39)$$

and the coefficients  $\eta_{LM}^{(pn)(\mu\nu)}$  become (see Eqs. (27) and (38))

$$\eta_{LM}^{(pn)(\mu\nu)} = -\chi_L^{(pn)(\mu\nu)} D_{M\nu-n}^{*L}(\varphi_0, \vartheta_0, 0). \quad (40)$$

Taking into account the symmetry property

$$\chi_L^{*(pn)(\mu\nu)} = \chi_L^{(\mu\nu)(pn)}, \tag{41}$$

we derive from Eqs. (17) and (18) and Eqs. (31) and (33)

$$S_{pn\mu\nu}^{(nc)} = S_{pn\nu\mu}^{(co)} + S_{pn\nu\mu}^1, \tag{42}$$

where the matrix  $S_{pn\nu\mu}^1$  corresponds to single scattering.

Eq. (42) allows one to calculate the amplitude of the interference peak using the vector radiative transfer equation only. This equation exactly reproduces Eq. (19) in [27]. It should be noted that in the case of the exact backscattering direction, Eq. (13) is equivalent to Eq. (22), and the system of equations (14) is equivalent to the system of equations (23). In other words, for any system of particles the relation between the incoherent part of the reflected light and the coherent one can be written as

$$t_{pn}^{(j)} t_{\mu\nu}^{*(j)} = \sum_{\sigma} t_{pn}^{(j)} t_{\nu\mu}^{*(\sigma)} + t_{pn}^{(0j)} t_{\nu\mu}^{*(0j)}. \tag{43}$$

### 5.2. The case of normal incidence

In this case ( $\mathbf{k}_0 = \mathbf{n}_0$ )

$$\vartheta_0 = 0, \quad \vartheta_1 = \frac{\vartheta}{2}, \quad \vartheta_{sc} = \vartheta, \quad \varphi_1 = \varphi \tag{44}$$

and the coefficients  $\eta_{LM}^{(pn)(\mu\nu)}$  are given by (see Eq. (38))

$$\eta_{LM}^{(pn)(\mu\nu)} = \exp[in(\varphi - \varphi_0) + iM\varphi] \zeta_{LM}^{\tilde{z}(pn)(\mu\nu)},$$

$$\zeta_{L_1 M_1}^{\tilde{z}(pn)(\mu\nu)} = \sum_{Llm} \frac{(2L+1)(2l+1)}{4} \langle a_L^{(jpn)} a_l^{*(j\mu\nu)} \rangle (-1)^{l+m} C_{L-nl-m}^{L_1 M_1} C_{L-pl\mu}^{L_1 \mu-p} d_{m\nu}^l(\vartheta), \tag{45}$$

where  $m = -M_1 - n$ . Eqs. (31)–(34) then yield

$$S_{pn\nu\mu}^{(co)} = \frac{2\pi v^2}{k_0^4} \exp[i(n - \mu)(\varphi - \varphi_0)] \sum_{qq_1 LM} (-1)^L \zeta_{LM}^{\tilde{z}(q_1\mu)(qp)}$$

$$\times \int_0^{k_0 Z_0} \beta_{LM}^{(z)(qn)(q_1\nu)} \exp(-\varepsilon_1 z) dz, \tag{46}$$

where

$$\tilde{\varepsilon}_1 = \text{Im}(\varepsilon) \left( 1 - \frac{1}{\cos \vartheta} \right) + i(1 + \cos \vartheta) \left( \frac{\text{Re}(\varepsilon) - 1}{\cos \vartheta} + 1 \right) \tag{47}$$

and the coefficients  $\beta_{LM}^{(z)(pn)(\mu\nu)}$  follow from the system of equations

$$\beta_{LM}^{(z)(pn)(\mu\nu)} = \exp(-\tilde{\varepsilon}_1^* z) \tilde{B}_{LM}^{(z)(pn)(\mu\nu)} + \frac{2\pi\nu}{k_0^3} \sum_{qq_1lm} i^{M-m} \chi_l^{(pq)(\mu q_1)} \int \beta_{lm}^{(y)(qn)(q_1\nu)} \times d_{MN_0}^L(\omega) d_{mN_0}^l(\omega) \exp(-\tau_\rho) J_{m-M}(\rho \sin \vartheta \sin \omega) d\rho \sin \omega d\omega. \quad (48)$$

Here  $\tau_\rho = 2 \operatorname{Im}(\varepsilon)\rho$ ,  $N_0 = \mu - p$ , and  $y = z - \rho \cos \omega$ . The second-order-scattering coefficients  $\tilde{B}_{LM}^{(z)(pn)(\mu\nu)}$  are

$$\tilde{B}_{LM}^{(z)(pn)(\mu\nu)} = \sum_{lm} \tilde{\zeta}_{lm}^{(pn)(\mu\nu)} i^{M-m} \int d_{MN_0}^L(\omega) d_{mN_0}^l(\omega) \exp(-\tilde{\tau}_1 \rho) J_{m-M}(\rho \sin \vartheta \sin \omega) d\rho \sin \omega d\omega, \quad (49)$$

where  $\tilde{\tau}_1 = 2 \operatorname{Im}(\varepsilon) - \tilde{\varepsilon}_1^* \cos \omega$ .

The Stokes parameters of the incident radiation are defined with respect to the plane through the vectors  $\mathbf{k}_0$  and  $\mathbf{n}_0$ , whereas the Stokes parameters of the reflected radiation are defined with respect to the plane through the vectors  $\mathbf{n}_0$  and  $\mathbf{k}_{sc}$ . Therefore, matrix (46) is proportional to  $\exp[i(n - \mu)(\varphi - \varphi_0)]$ . When the Stokes parameters are defined with respect to the scattering plane (the plane through the vectors  $\mathbf{k}_0$  and  $\mathbf{k}_{sc}$ ), this proportionality factor reduces to 1. Eq. (42) is also equitable in the case of  $\mathbf{k}_{sc} = -\mathbf{k}_0$ .

Eqs. (46)–(49) are identical to those of [18] except for a small difference in notation (cf. [20]). It should be noted that the contribution of the looped trajectories having the form of polylines were considered in [18] to be part of the incoherent radiation. But a more careful consideration shows that this contribution to the reflected radiation is coherent. Therefore, Eqs. (41) and (42) of [18] should be ignored.

A general theoretical analysis of the dependence of matrices (31) and (46) on the parameter  $\vartheta$  is very complicated. We examine this dependence in the special case of a semi-infinite medium in the second-order scattering approximation. Consider first the case of normal incidence. Let the Stokes parameters of the incident and reflected light be specified with respect to the scattering plane (i.e., the one through the vectors  $\mathbf{k}_0$  and  $\mathbf{k}_{sc}$ ). The substitution of the first term on the right-hand side of Eq. (48) into Eq. (46) followed by integration gives [11,19,20]

$$S_{pn\nu\mu}^{(co)} = -\frac{\pi\nu^2 \cos \vartheta}{k_0^4 \operatorname{Im}(\varepsilon)(1 - \cos \vartheta)} \sum_{qq_1LMlm} (-1)^L \tilde{\zeta}_{LM}^{*(q_1\mu)(qp)} \tilde{\zeta}_{lm}^{(qn)(q_1\nu)} \times i^{-|m-M|} \int_0^\pi d_{MN}^L(\omega) d_{mN}^l(\omega) I_{|m-M|} \sin \omega d\omega, \quad (50)$$

where  $N = q_1 - q$ ,

$$I_m = \frac{c^m}{\sqrt{c^2 + f^2}(f + \sqrt{c^2 + f^2})^m},$$

$$c = \sin \vartheta \sin \omega,$$

$$f = 2 \operatorname{Im}(\varepsilon) + |\cos \omega| \operatorname{Im}(\varepsilon) \left(1 - \frac{1}{\cos \vartheta}\right) + i \cos \omega (1 + \cos \vartheta) \left(\frac{\operatorname{Re}(\varepsilon) - 1}{\cos \vartheta} + 1\right). \quad (51)$$

If the size of the scatterers is not much greater than the wavelength and the angle  $\vartheta$  is close to the exact backscattering direction then the coefficients  $\zeta_{LM}^{(pn)(\mu\nu)}$  take the form

$$\zeta_{LM}^{(pn)(\mu\nu)} \simeq -\chi_L^{(pn)(\mu\nu)} \delta_{M,\nu-n} \tag{52}$$

and matrix (50) depends on  $\vartheta$  via the coefficients  $I_m$  only. In the limit  $c \rightarrow 1$ , all the coefficients  $I_m \rightarrow 1$ , whereas in the limit  $c \rightarrow 0$  only the coefficient  $I_0$  differs significantly from 0. For sparse media,  $2 \operatorname{Im}(\varepsilon) \ll 1$  and  $\operatorname{Re}(\varepsilon) - 1 \rightarrow 0$ . Under these conditions, the coefficient  $I_0$  is dominant at angles  $\vartheta \simeq \pi$  and can be approximated by the following asymptotic formula:

$$I_0 \simeq \begin{cases} \frac{1}{2 \operatorname{Im}(\varepsilon)(1 + |\cos \omega|)} & \text{for } f \gg c, \\ \frac{1}{\sin \vartheta \sin \omega} & \text{for } f \ll c. \end{cases} \tag{53}$$

As follows from Eq. (53), the coefficient  $I_0$  as a function of the angle  $\vartheta$  has a narrow peak with a maximum at  $\vartheta = \pi$  and a rapid fall with decreasing  $\vartheta$ . The amplitude and the width of this peak are determined by the value of  $2 \operatorname{Im}(\varepsilon)$  [11,19]. The coefficient  $I_0$  causes an interference peak in the intensity of the reflected light, whereas the coefficients  $I_m$  ( $m \neq 0$ ) cause a nonzero linear polarization [28]. Note that the condition of Eq. (52) leads to a simplification of Eq. (50). For example, the reflection matrix elements  $R_{11}^{(\text{co})}$  and  $R_{21}^{(\text{co})}$ , which describe the coherent intensity and degree of linear polarization of the reflected radiation, respectively, in the linear polarization basis can be written in the form [28]

$$\begin{aligned} R_{11}^{(\text{co})} &= \frac{\pi v^2}{k^6 \operatorname{Im}(\varepsilon)(1 - \cos \vartheta)} \int_0^\pi I_0 Q_{11}(\omega) \sin \omega \, d\omega, \\ R_{21}^{(\text{co})} &= \frac{\pi v^2}{k^6 \operatorname{Im}(\varepsilon)(1 - \cos \vartheta)} \int_0^\pi I_2 Q_{21}(\omega) \sin \omega \, d\omega, \end{aligned} \tag{54}$$

where

$$\begin{aligned} Q_{11}(\omega) &= g_{11}(\omega)g_{11}(\pi - \omega) + g_{21}(\omega)g_{21}(\pi - \omega), \\ Q_{21}(\omega) &= -g_{11}(\omega)g_{21}(\pi - \omega) - g_{21}(\omega)g_{22}(\pi - \omega). \end{aligned} \tag{55}$$

Here  $g_{ij}$  are the elements of the scattering matrix for an isolated particle in the linear polarization basis averaged over the microphysical properties of the particle. Eqs. (54) and (55) allow one to predict the sign of linear polarization of the coherently scattered radiation. If the element  $g_{21}$  is positive at all scattering angles then the element  $R_{21}^{(\text{co})}$  is negative; if the element  $g_{21}$  is negative then the element  $R_{21}^{(\text{co})}$  is positive [28].

We also note another feature of the scattering pattern which follows from Eqs. (51) and (53). In the limit  $\vartheta \rightarrow \pi$ , the coefficient  $I_0$  becomes  $I_0 \simeq 1/4 \operatorname{Im}(\varepsilon)$  when  $\omega = 0$  or  $\pi$  and does not depend on  $\vartheta$ . If  $\omega = \pi/2$  then this coefficient becomes  $I_0 \simeq 1/2 \operatorname{Im}(\varepsilon)$  and decreases with decreasing  $\vartheta$ . This implies that for fixed  $\operatorname{Im}(\varepsilon)$  the angular region affected by the interference is broader for strongly forwardscattering and backscattering particles than for nearly isotropically scattering particles. In fact,

the directions  $\omega \approx 0$  and  $\omega \approx \pi$  give a major contribution to integral (50) for the former particles, whereas the range of  $\omega \approx \pi/2$  dominates this integral for the latter particles. This conclusion follows from Eqs. (54) and (55) as well. Calculations presented in [11,19,20] allow one to reach more detailed conclusions about the dependence of the interference effects on the properties of the scattering medium. They agree well with the results of the semi-qualitative analysis discussed above.

5.3. Second-order-scattering approximation for oblique illumination of a semi-infinite layer

The previous qualitative analysis can be easily extended to the case of slant illumination. Let us begin by substituting the first term on the right-hand side of Eq. (33) into Eq. (31) and integrating over  $z$ . A derivation analogous to that used to obtain Eq. (50) yields

$$S_{p\nu\mu}^{(co)} = -\frac{\pi v^2 \cos \vartheta \cos \vartheta_0}{k_0^4 \text{Im}(\varepsilon)(\cos \vartheta_0 - \cos \vartheta)} \sum_{qq_1, LMlm} (-1)^L \exp[i(M - m)\varphi_1] \times \eta_{LM}^{*(q_1\mu)(qp)} \eta_{lm}^{(qn)(q_1\nu)} i^{-|m-M|} \int_0^\pi d_{MN}^L(\omega) d_{mN}^l(\omega) \tilde{I}_{|m-M|} \sin \omega \, d\omega, \tag{56}$$

where  $N = q_1 - q$ . The coefficients  $\tilde{I}_{|m-M|}$  have the form of Eq. (51) if  $c$  and  $f$  are replaced by  $\tilde{c}$  and  $\tilde{f}$ , respectively, where

$$\begin{aligned} \tilde{c} &= 2 \sin \vartheta_1 \sin \omega \cos \frac{\vartheta_{sc}}{2}, \\ \tilde{f} &= 2 \text{Im}(\varepsilon) + |\cos \omega| \text{Im}(\varepsilon) \left( \frac{1}{\cos \vartheta_0} - \frac{1}{\cos \vartheta} \right) \\ &\quad + i \cos \omega \left( \frac{\text{Re}(\varepsilon) - 1}{\cos \vartheta} + \frac{\text{Re}(\varepsilon) - 1}{\cos \vartheta_0} + 2 \cos \frac{\vartheta_{sc}}{2} \cos \vartheta_1 \right). \end{aligned} \tag{57}$$

If the scatterers are not much larger than the wavelength and  $\vartheta_{sc} \simeq \pi$  then the coefficients  $\eta_{LM}^{(pn)(\mu\nu)}$  can be approximated by Eq. (40). As in the case of normal incidence, the interference peak is determined by the coefficient  $\tilde{I}_0$ . However, the properties of this coefficient depend on the azimuth angle of the scattering direction and on the polar angle of the incidence direction. Let us consider this dependence in two scattering angle ranges denoted in Fig. 2 as (a) and (b) assuming that the  $z$  axes of all coordinate systems and the  $x_0$ -axis lie in the same plane.

Range (a): In this case  $\varphi_0 = \varphi_1 = 0$ ,  $\varphi = \pi$ ,  $\vartheta_{sc} = 2\pi - \vartheta - \vartheta_0$ , and  $\vartheta_1 = \vartheta_{sc}/2$  so that the analog of Eq. (53) has the form

$$\tilde{I}_0 \simeq \begin{cases} \frac{1}{2 \text{Im}(\varepsilon) + \text{Im}(\varepsilon) |\cos \omega| (1/\cos \vartheta_0 - 1/\cos \vartheta)} & \text{for } \tilde{f} \gg \tilde{c}, \\ \frac{1}{\sin \phi \sin \omega} & \text{for } \tilde{f} \ll \tilde{c}, \end{cases} \tag{58}$$

where  $\phi = \vartheta + \vartheta_0 - \pi$  (see Fig. 2). It follows from Eq. (58) that the amplitude and the width of the interference peak decrease with increasing angle  $\vartheta_0$ . Note that the incoherent contribution to the reflected radiation also decreases with increasing  $\vartheta_0$  so that relation (42) remains valid [29].

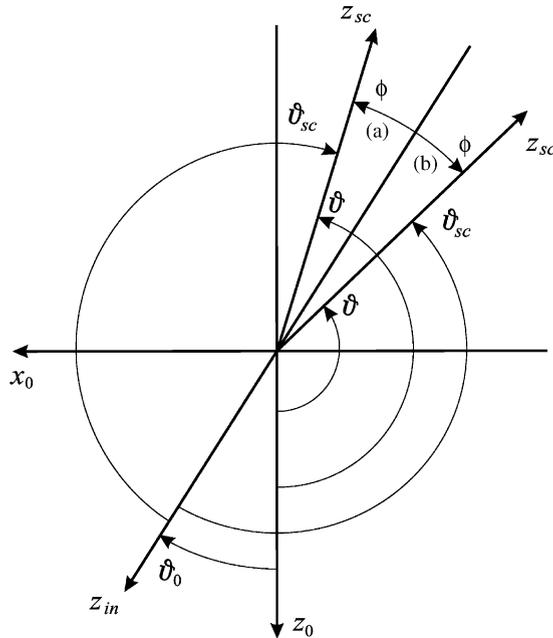


Fig. 2. Angular ranges (a) and (b) (see text).

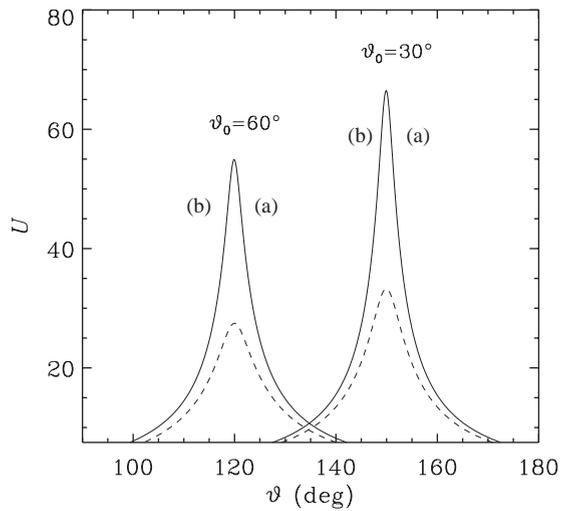


Fig. 3. Dependence of  $U$  on the scattering angle for  $\vartheta_0 = 30^\circ$  and  $\vartheta_0 = 60^\circ$ . The solid curves correspond to  $\text{Im}(\epsilon) = 0.01$  and the dashed curves correspond to  $\text{Im}(\epsilon) = 0.02$ .

*Range (b):* In this case  $\varphi_0 = 0$ ,  $\varphi = \varphi_1 = \pi$ ,  $\vartheta_{sc} = \vartheta + \vartheta_0$ , and  $\vartheta_1 = \vartheta_{sc}/2$ . As in case (a), the coefficient  $I_0$  is determined by Eq. (58) with  $\phi = \pi - \vartheta - \vartheta_0$ . However, the angle  $\vartheta$  is smaller than in range (a). Therefore, the interference peak in range (a) is broader than in range (b).

In the particular case of scalar waves,  $M = m = N = 0$  in Eq. (56). Furthermore, if scattering is isotropic, then  $L = l = 0$ , and the integral in Eq. (56) becomes simple:

$$U = \int_0^\pi \tilde{I}_0 \sin \omega \, d\omega. \quad (59)$$

The scattering-angle dependence of this integral is illustrated in Fig. 3 for  $\varepsilon = 1 + i \operatorname{Im}(\varepsilon)$ .

Finally we note that a forthcoming publication will report on numerical results obtained on the basis of the theoretical formalism outlined in this paper.

## Acknowledgements

The work of V. Tishkovets has been supported in part by the DFG Schwerpunktprogramm 1115 “Mars und die terrestrischen Planeten” (Germany). He also thanks Prof. Klaus Jockers for helpful discussions on the subjects. M. Mishchenko acknowledges funding from the NASA Radiation Sciences Program managed by Dr. Donald Anderson.

## Appendix A.

Wigner  $D$  functions [22]  $D_{Mm}^L(\hat{\mathbf{n}}_0, \hat{\mathbf{n}}_1) = D_{Mm}^L(\alpha, \beta, \gamma)$  are determined as the matrix elements of the rotation operator in the  $JM$ -representation, where the Euler angles  $\alpha, \beta, \gamma$  specify the rotation from the coordinate system  $\hat{\mathbf{n}}_0$  to the coordinate system  $\hat{\mathbf{n}}_1$ . In this paper, the following properties of the Wigner  $D$  functions and the Clebsch–Gordan coefficients [22] are used:

the unitary condition

$$\sum_{m_1} D_{Mm_1}^L(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2) D_{mm_1}^{*L}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2) = \delta_{Mm},$$

the group property (the addition theorem)

$$D_{Mm}^L(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2) = \sum_{m_1} D_{m_1 M}^{*L}(\hat{\mathbf{n}}_0, \hat{\mathbf{n}}_1) D_{m_1 m}^L(\hat{\mathbf{n}}_0, \hat{\mathbf{n}}_2)$$

and the symmetry properties

$$D_{Mm}^L(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2) = (-1)^{M-m} D_{-M-m}^{*L}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2),$$

$$D_{Mm}^L(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2) = D_{mM}^{*L}(\hat{\mathbf{n}}_2, \hat{\mathbf{n}}_1),$$

$$D_{Mm}^L(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2) = (-1)^{M-m} D_{Mm}^{*L}(-\hat{\mathbf{n}}_1, -\hat{\mathbf{n}}_2),$$

$$D_{Mm}^L(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2) = (-1)^L D_{M-m}^L(\hat{\mathbf{n}}_1, -\hat{\mathbf{n}}_2).$$

In the last two equations it is assumed that the coordinate system  $\hat{\mathbf{n}}_i$  has the axis  $(x_i, y_i, z_i)$  and the coordinate system  $-\hat{\mathbf{n}}_i$  has the axis  $(x_i, -y_i, -z_i)$ .

The product of two Wigner  $D$  functions can be expanded in the Clebsch–Gordan series

$$D_{Mm}^L(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2) D_{M_1 m_1}^{L_1}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2) = \sum_{L_2 M_2 N_2} C_{LM_1 M_1}^{L_2 M_2} C_{Lm_1 m_1}^{L_2 N_2} D_{M_2 N_2}^{L_2}(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2),$$

where  $C_{LM_1 m_1}^{L_1 M_1}$  are Clebsch–Gordan coefficients. The unitary relation for these coefficients reads

$$\sum_{Mm} C_{LM_1 m}^{L_1 M_1} C_{LM_2 m}^{L_2 M_2} = \delta_{L_1 L_2} \delta_{M_1 M_2}.$$

The Clebsch–Gordan coefficients have many symmetry properties [22] including the following:

$$\begin{aligned} C_{LM_1 m}^{L_1 M_1} &= (-1)^{L_1+L+l} C_{L-M_1 -m}^{L_1 -M_1} = (-1)^{L_1+L+l} C_{lmLM}^{L_1 M_1} \\ &= (-1)^{l+m} \sqrt{\frac{2L_1+1}{2L+1}} C_{L_1-M_1 lm}^{L-M} = (-1)^{L-M} \sqrt{\frac{2L_1+1}{2l+1}} C_{LM_1 -M_1}^{l-m}. \end{aligned}$$

The coefficients  $C_{LM_1 m}^{L_1 M_1}$  differ from 0 if  $|L-l| \leq L_1 \leq L+l$  and  $M_1 = M+m$ .

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