Depth mode characteristics of light propagation in real turbid media and media with two-dimensional scattering

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A study of the depth mode of light propagation in turbid media with slowly decreasing scattering phase function $\chi(\gamma)$ (when the scattering probability decreases more slowly than γ^{-4} as the single scattering angle γ is enhanced) is performed outside the framework of the Fokker–Planck approximation. We propose and realize a regular procedure for optimum determination of the parameters of a postulated approximate angular spectrum in the depth mode. The dispersion in the depth mode and the depth damping coefficient are found analytically. The obtained results are in good agreement with the results of numerical calculations and transform, in the limit, into known results of the diffusion approximation. The calculations were also performed in a two-dimensional scattering medium, for which the transport equation written in the quasi-diffusion approximation was recently solved exactly for the particular case of the Henyey–Greenstein phase function. In this case very good coincidence of our approximate results with the exact solution is shown.
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1. INTRODUCTION

We study analytically the problem of the propagation of stationary light fluxes in strongly absorbing media, $\kappa \gg D$ (κ is the true absorption coefficient and D is the angular diffusion coefficient) when scattering on individual centers exhibits strongly anisotropic features: $\langle 1 - \cos\gamma\rangle \ll 1$ ($\langle\cos\gamma\rangle$ is the average cosine of the single scattering angle γ). This situation appears when the sizes of the scattering centers are significantly larger than the wavelength of light ($a \gg \lambda$), while the relative refractive index of the centers is $n_{\mathrm{rel}} \sim 1$. Scattering on such centers occurs mainly at small angles, $\gamma_{\mathrm{ef}} \sim \lambda/a \ll 1$, so $\langle\gamma^2\rangle \ll 1$. In such media the following double inequality is valid:

$$\sigma \gg \kappa \gg D, \qquad \bar{l} \ll l_a \ll l_{\mathrm{tr}}\,. \tag{1}$$

Here σ is the scattering coefficient; $\bar{l} = 1/\sigma$ is the photon elastic scattering length; $l_a = 1/\kappa$ is the absorption length; and $l_{\mathrm{tr}} = 1/2D$ is the transport scattering length, i.e., the average path along which the initially monodirected flux of photons undergoing purely elastic scattering becomes isotropic. Investigation of the light-propagation problem under the conditions above is of considerable practical and theoretical interest for many real problems relevant to the physics of the atmosphere, to hydro-optics, and also to various artificial media.[1–4]

Below we shall be interested in depth ranges $z \geq l_a$, where the scattering of light is of an essentially multiple nature: $N_{\mathrm{col}} \geq l_a/\bar{l} \gg 1$. However, owing to the strong absorption ($l_a \ll l_{\mathrm{tr}}$), the original flux does not become isotropic in practice; i.e., even at relatively large depths ($z \geq \sqrt{l_a l_{\mathrm{tr}}}$) we have only small-angle scattering: $\langle\theta^2\rangle_z \ll 1$ ($\langle\theta^2\rangle_z$ is the mean-square multiple scattering angle at the depth z). Physically, this effect is caused by the fact that, because of the strong absorption, large depths are reached only by the photons proceeding along weakly curvilinear trajectories. Thus scattering on large angles appears to be strongly suppressed because of the absorption. For this reason the small-angle approximation (SAA) may be applied to describe the propagation of photons in such media.[3–10]

Until now, the problem being considered had, in the main, been solved within the framework of the small-angle diffusion approximation (SADA), in which the elastic collision integral in the transport equation is written in the differential form.[11–16] In the Fokker–Planck approximation the scattering properties of the medium are characterized only by the numerical value of the photon angular diffusion coefficient: $D = (\sigma/2)\langle 1 - \cos\gamma\rangle$, i.e.,

$$D = \pi\sigma \int_0^{\pi} (1 - \cos\gamma)\chi(\gamma)\sin\gamma\,\mathrm{d}\gamma\,. \tag{2}$$

Here $\chi(\gamma)$ is the scattering phase function that determines the photon-scattering law on an individual center. It is normalized by the condition

$$2\pi \int_0^{\pi} \chi(\gamma)\sin\gamma\,\mathrm{d}\gamma \approx 2\pi \int_0^{\infty} \gamma\chi(\gamma)\mathrm{d}\gamma = 1\,. \tag{3}$$

One can see immediately from relation (3) that, if the phase function has at relatively large scattering angles

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($\gamma \gg \gamma_{\mathrm{ef}}$) a power form $\gamma^{-2(1+\nu)}$, then the condition $\nu > 0$ is necessary for convergence of the integral in relation (3) at the upper limit. This means that the phase function must decrease faster then γ^{-2} as the single-scattering angle γ is enhanced; otherwise the SAA practically has no domain of application.

We restrict ourselves to the case when a broad stationary light flux of intensity I_0 is normally incident upon the surface of a medium occupying the region $z > 0$ (the z axis in the medium is directed inward). Then the expression for the light intensity $I(z, \theta)$, derived within the SADA framework with the fluctuations of photon paths owing to elastic scattering taken into account, has a Gaussian form[4,6,9,11,12,15]:

$$I^D(z, \theta) = E^D(z)\Phi^D(z, \theta), \tag{4a}$$

where

$$E^D(z) = I_0 \frac{\exp(-\kappa z)}{\cosh(z/\sqrt{l_a l_{\mathrm{tr}}})},$$

$$\Phi^D(z, \theta) = \frac{1}{\pi\langle\theta^2\rangle_z^D} \exp\left(-\frac{\theta^2}{\langle\theta^2\rangle_z^D}\right), \tag{4b}$$

$$\langle\theta^2\rangle_z^D = 2\sqrt{2D/\kappa}\,\tanh(z\sqrt{2D\kappa}) = 2\sqrt{l_a/l_{\mathrm{tr}}}\,\tanh(z/\sqrt{l_a l_{\mathrm{tr}}}). \tag{4c}$$

Here $E^D(z)$ is the light flux at depth z and $\Phi^D(z, \theta)$ is the angular distribution density normalized by the condition

$$2\pi \int_0^\infty \theta\Phi^D(z, \theta)\mathrm{d}\theta = 1. \tag{4d}$$

It follows *a posteori* directly from Eq. (4c) that in strongly absorbing media [inequalities (1)] even in the depth mode ($z\sqrt{2D\kappa} \gg 1$) the value of $\langle\theta^2\rangle_\infty$ is small: $\langle\theta^2\rangle_\infty = 2\sqrt{l_a/l_{\mathrm{tr}}} = 2\sqrt{2D/\kappa} \ll 1$. (The fact that the angular spectrum dispersion is small at a large depth was revealed also, without recourse to the Fokker–Planck approximation, by means of numerical calculations[4] and, for example, in the case of the Henyey–Greenstein phase function.[5])

However, the possibility of applying the SADA is related to a very rigorous restriction imposed on the form of the scattering phase function $\chi(\gamma)$: it must fall with the rise of γ more rapidly than γ^{-4}, i.e., faster than a phase function of the Rutherford form.[8,17–20] In most real turbid media, however, quite the opposite occurs: $\chi(\gamma \gg \gamma_{\mathrm{ef}}) \sim \gamma^{-2(1+\nu)}$ and $0 < \nu < 1$. This means that there are no arguments in favor of considering the Gaussian scattering law [Eqs. (4b)] to be correct.

Therefore the problem of calculating the light intensity $I(z, \theta)$ in real turbid media (even within the framework of the SAA) becomes significantly more complicated, because now to compute $I(z, \theta)$ one must solve the transport equation with the elastic collision integral $\hat{B}_{\mathrm{el}}I$ of the general form. In other words, light propagation in such media is described adequately not by the differential Fokker–Planck equation but by the integrodifferential Boltzmann equation[1,2,4]

$$\cos\theta \frac{\partial I}{\partial z} + \kappa I = \hat{B}_{\mathrm{el}}I, \qquad I(z = 0, \mu \geq 0) = \frac{I_0}{2\pi}\delta(1 - \mu). \tag{5}$$

The elastic collision integral in the case of the flat geometry has the following form:

$$\hat{B}_{\mathrm{el}}I = \sigma \int_0^{2\pi} \mathrm{d}\varphi' \int_{-1}^{1} \mathrm{d}\mu' \chi(\cos\gamma)[I(z; \mu') - I(z; \mu)], \tag{6}$$

where $\mu = \cos\theta$ and $\mu' = \cos\theta'$. The cosine of a single-scattering angle γ from the stage (μ', φ') to the stage (μ, φ) is defined by the well-known spherical trigonometry formula

$$\cos\gamma = \mu\mu' + \sqrt{1 - \mu^2}\sqrt{1 - \mu'^2}\cos(\varphi' - \varphi). \tag{7}$$

(We note that, in the framework of the SAA, backscattering of photons can be considered an additional absorption[21] that practically leads just to some renormalization of the absorption coefficient κ and will not affect our consideration.)

2. THEORY

The light intensity at a depth z may always be represented in a form analogous to Eq. (4a):

$$I(z, \theta) = E(z)\Phi(z, \theta). \tag{8}$$

Here $\Phi(z, \theta)$ is the angular function normalized by the condition

$$2\pi \int_0^{\pi/2} \sin\theta\Phi(z, \theta)\mathrm{d}\theta \approx 2\pi \int_0^\infty \theta\Phi(z, \theta)\mathrm{d}\theta = 1 \tag{9}$$

[thus the quantity $2\pi\Phi(z, \theta)\theta\mathrm{d}\theta$ is the probability for the photons at a depth z to travel at angles within the range from θ to $\theta + \mathrm{d}\theta$), and $E(z)$ is the light flux at depth z:

$$E(z) = 2\pi \int_0^{\pi/2} \cos\theta I(z, \theta)\sin\theta\mathrm{d}\theta \approx 2\pi \int_0^\infty \theta I(z, \theta)\mathrm{d}\theta. \tag{10}$$

Inasmuch as the trajectories of photons in matter are curvilinear, the degree of their absorption depends on their mean path $\langle s\rangle_z$ covered by photons in a layer of matter of thickness z. Therefore we shall represent the light flux $E(z)$ in the form[6]

$$E(z) = I_0 \exp(-\kappa\langle s\rangle_z) = I_0 \exp\left[-\kappa\left(z + \frac{1}{2}\int_0^z \langle\theta^2\rangle_{z'}\mathrm{d}z'\right)\right]. \tag{11}$$

Here we have used the fact that in the case of small-angle scattering ($\langle\theta^2\rangle_z \ll 1$) the average path $\langle s\rangle_z$ is determined by the expression

$$\langle s\rangle_z = \int_0^z \mathrm{d}z'\langle\sec\theta\rangle_{z'} \simeq z + \frac{1}{2}\int_0^z \langle\theta^2\rangle_{z'}\mathrm{d}z', \tag{12}$$

where, in accordance with the definition of the mean,

$$\langle\theta^2\rangle_z = \frac{2\pi}{E(z)}\int_0^\infty \theta^3 I(z, \theta)\mathrm{d}\theta = 2\pi \int_0^\infty \theta^3\Phi(z, \theta)\mathrm{d}\theta. \tag{13}$$

The mutual influence of the absorption and scattering processes on the form of the photon angular spectrum

starts to manifest itself at such depths, when $(\langle s\rangle_z - z) \geq l_a$.[4,6,12]

Now, substituting Eq. (8) into Eqs. (5) and (6), taking into account relation (11) and assuming that $\mu = \cos\theta \approx 1 - \theta^2/2$, we obtain the following equation for the angular function $\Phi(z, \theta)$ in the SAA[22]:

$$\frac{\partial\Phi}{\partial z} + \frac{\kappa}{2}(\theta^2 - \langle\theta^2\rangle_z)\Phi(z, \theta) = \hat{B}_{\mathrm{el}}\Phi\,,$$

$$\Phi(z = 0, \theta) = \frac{1}{2\pi}\frac{\delta(\theta)}{\theta}\,. \tag{14}$$

Here

$$\hat{B}_{\mathrm{el}}\Phi = -\sigma\int_0^{2\pi}\mathrm{d}\varphi'\int_0^{\infty}\theta'\mathrm{d}\theta'\chi(\gamma)[\Phi(z, \theta) - \Phi(z, \theta')]\,, \tag{15a}$$

$$\gamma^2 \simeq \theta^2 + \theta'^2 - 2\theta\theta'\cos(\varphi' - \varphi)\,. \tag{15b}$$

Equation (15a) represents the elastic collision integral [Eq. (6)] written in the SAA. Expression (15b) is a small-angle variant of Eq. (7).

Actually, the set of Eq. (13) and (14) determines two unknown functions: $\Phi(z, \theta)$ and $\langle\theta^2\rangle_z$. If these quantities are found, then one can calculate the total light flux $E(z)$ by Eq. (11) and then determine, by Eq. (8), the light intensity $I(z, \theta)$ at depth z.

Note that by using the sum rule for the Bessel functions it is possible to write the elastic collision integral [Eq. (15a)] in the form[20]

$$\hat{B}_{\mathrm{el}}\Phi = -\frac{\sigma}{2\pi}\int_0^{\infty}\omega\mathrm{d}\omega J_0(\omega\theta)\Phi_\omega(z)[1 - \chi(\omega)]\,. \tag{16}$$

Here $\Phi_\omega(z)$ and $\chi(\omega)$ are the Bessel transforms of $\Phi(z, \theta)$ and $\chi(\gamma)$, respectively:

$$\Phi_\omega(z) = 2\pi\int_0^{\infty}J_0(\omega\theta)\Phi(z, \theta)\theta\mathrm{d}\theta, \qquad \Phi_{\omega=0}(z) \equiv 1\,, \tag{17}$$

$$\chi(\omega) = 2\pi\int_0^{\infty}J_0(\omega\gamma)\chi(\gamma)\gamma\mathrm{d}\gamma\,, \qquad \chi(\omega = 0) = 1\,. \tag{18}$$

At the same time we note that, if Eq. (14) is written in the diffusion approximation, $\hat{B}_{\mathrm{el}}\Phi \simeq D(1/\theta)(\partial/\partial\theta)[\theta(\partial/\partial\theta)\Phi]$, then by solving Eq. (13) and (14) we naturally obtain the SADA result, Eqs. (4).

Here we consider only the case of relatively large depths $(\langle s\rangle_z - z) > l_a$. The main feature of the depth mode is the factorization of the emission spectrum:

$$I_{\mathrm{as}}(z, \theta) = I_0 a\exp(-kz)\Phi_{\mathrm{as}}(\theta)\,. \tag{19}$$

When normalization condition (9) is taken into account, this actually means that in the depth mode the angular function is independent of the depth: $\Phi_{\mathrm{as}}(\theta) = \Phi(z \to \infty, \theta)$. The quantity k is the depth damping index; the quantity $I_0 a\exp(-kz)$ is the light flux in the depth mode. Now, instead of Eqs. (13) and (14) we obtain the following system:

$$(\kappa/2)(\theta^2 - \langle\theta^2\rangle_\infty)\Phi_{\mathrm{as}}(\theta) = \hat{B}_{\mathrm{el}}\Phi_{\mathrm{as}}(\theta)\,, \tag{20}$$

$$\langle\theta^2\rangle_\infty = 2\pi\int_0^{\infty}\theta^3\Phi_{\mathrm{as}}(\theta)\mathrm{d}\theta\,. \tag{21}$$

Here $\langle\theta^2\rangle_\infty$ is the dispersion of the photon angular distribution in the depth mode. Using Eq. (11), we find that

$$k = \kappa(1 + {}^1\!/\!_2\langle\theta^2\rangle_\infty)\,, \tag{22}$$

$$a = \exp\left[(\kappa/2)\int_0^{\infty}(\langle\theta^2\rangle_\infty - \langle\theta^2\rangle_z)\mathrm{d}z\right]. \tag{23a}$$

Thus to calculate the damping index k it suffices to know the value of $\langle\theta^2\rangle_\infty$. To calculate the coefficient a in Eq. (19) it is necessary to know the value of $\langle\theta^2\rangle_z$ over the whole range of depths $0 \leq z \leq \infty$.

However, the coefficient a that appears in Eq. (19) can be calculated in a completely different way without the use of Eq. (23a). Indeed, at any depth z the light intensity $I(z, \theta)$ can be expanded by the complete orthonormal system of functions [see, e.g., Eq. (14) in Ref. 23]. Inasmuch as the intensity of light in the depth mode [Eq. (19)] is determined only by the term corresponding to the least eigenvalue, one can use orthonormality of the angular functions and boundary condition (14) to obtain that

$$a = \frac{\Phi_{\mathrm{as}}(\theta = 0)}{2\pi\int_0^{\infty}\Phi_{\mathrm{as}}(\theta)^2\theta\mathrm{d}\theta}\,. \tag{23b}$$

From Eqs. (4b), (4c), and (23b) it follows that, in the depth mode in the SADA,

$$\Phi_{\mathrm{as}}{}^D(\theta) = \frac{1}{\pi\langle\theta^2\rangle_\infty{}^D}\exp\left(-\frac{\theta^2}{\langle\theta^2\rangle_\infty{}^D}\right), \qquad a^D = 2\,,$$
$$\langle\theta^2\rangle_\infty{}^D = 2\sqrt{(l_a/l_{\mathrm{tr}})} = 2\sqrt{(2D/\kappa)}\,, \qquad k^D = \kappa + \sqrt{2D\kappa}\,. \tag{24}$$

Analysis of experimental data reveals that the scattering phase function of light on large-scale scattering centers in clouds, fog, aerosols, seawater, etc. exhibits, in the range of angles $1 > \gamma \gg \gamma_{\mathrm{ef}}$, a power (or close to such) form: $\chi(\gamma) \sim \gamma^{-2(1+\nu)}$, and the parameter ν usually lies in the interval $0.25 \leq \nu \leq 0.75$. Therefore we shall further take advantage, for $\chi(\gamma)$, of a two-parameter expression of the following form:

$$\chi_\nu(\gamma) = \frac{\nu\gamma_{\mathrm{ef}}{}^{2\nu}}{\pi[\gamma_{\mathrm{ef}}{}^2 + \gamma^2]^{1+\nu}}\,. \tag{25}$$

Equation (25) satisfies normalization condition (3). Typical values are $\gamma_{\mathrm{ef}} \leq 5°$ in seawater and $\gamma_{\mathrm{ef}} \sim 6°–15°$ in clouds. The parameter $\nu = 1$ corresponds to Rutherford scattering, whereas in the case of $\nu = 1/2$ one obtains the Henyey–Greenstein phase function. Equation (25) represents the two-parameter phase function

$$\chi_\nu(\cos\gamma) = \frac{\nu}{\pi}\frac{\gamma_{\mathrm{ef}}{}^{2\nu}(4 + \gamma_{\mathrm{ef}}{}^2)^\nu}{[(4 + \gamma_{\mathrm{ef}}{}^2)^\nu - \gamma_{\mathrm{ef}}{}^{2\nu}]} \times \frac{1}{[\gamma_{\mathrm{ef}}{}^2 + 2(1 - \cos\gamma)]^{1+\nu}} \tag{26}$$

written in the SAA. Phase functions of the form of Eqs. (26) and (25) were used in Refs. 17, 20, and 24–26.

3. CALCULATION OF THE DEPTH MODE ANGULAR SPECTRUM

If photon path-length fluctuations owing to scattering are not taken into account, i.e., if the standard small-angle approximation (SSAA) is used [when the term $\sim\kappa$ on the left-hand side of Eq. (14) is neglected], one can obtain the following expression[20]:

$$\Phi^{\mathrm{SSAA}}(z,\,\theta) = \frac{1}{2\pi}\int_0^\infty \omega\,\mathrm{d}\omega J_0(\omega\theta)\exp\{-3z[1-\chi(\omega)]\}\,. \tag{27}$$

It is clear from Eq. (27) that in the framework of the SSAA it is impossible to describe the depth mode, because even in the formal limit $\sigma z \to \infty$ factorization (19) of the spectrum does not occur. However, even in the framework of this simplest approximation, the angular functions $\Phi^{\mathrm{SSAA}}(z,\,\theta)$ show principally different behavior, as the scattering angle θ is increasing, for different values of the parameter ν in Eqs. (25) and (26). (A comprehensive analysis of light fields in the framework of the SSAA for a great variety of phase functions was given in a review paper[20] and in Ref. 25.) When $\nu > 0$, $\Phi^{\mathrm{SSAA}}(z,\,\theta) \sim \exp(-\theta^2/4Dz)$, i.e., it has a Gaussian form. On the other hand, in the case of interest, when $\nu < 0$, in the range of relatively large depths $\sigma z \gg 1$ (while, of course, $z \ll l_{\mathrm{tr}}$) we have $\Phi^{\mathrm{SSAA}} \sim \theta^{-2(1+\nu)}$, i.e., the light intensity decreases with an increase in the angle according to the power law with the same power index as the phase function [see Eq. (47) of Ref. 20]. For example, for the Henyey–Greenstein phase function ($\nu = 1/2$), when $\chi(\gamma) \sim \gamma^{-3}$ one has[20]

$$\Phi^{\mathrm{SSAA}}(z\sigma \gg 1,\,\theta) \approx \frac{1}{2\pi}\,\frac{\gamma_{\mathrm{ef}} z\sigma}{[(\gamma_{\mathrm{ef}} z\sigma)^2 + \theta^2]^{3/2}}\,.$$

However, in media with strong absorption, when inequalities (1) are satisfied the situation is essentially different, because scattering on large angles is heavily suppressed because of the absorption of photons moving along highly curvilinear trajectories. Therefore asymptotic behavior of angular functions in the depth mode differs dramatically[8] from that in the SSAA: $\Phi_{\mathrm{as}} \sim \theta^{-2(2+\nu)}$, whereas $\Phi_{\mathrm{as}}(\theta)$ again falls according to a power law but much faster than the phase function.

What we have said about the power law of intensity decreasing with an increase in the scattering angle θ suggests that we look for an approximate solution to Eq. (20) for the class of functions of the form of Eq. (25) but that fall, as θ is enhanced, faster than θ^{-4}. The latter obvious requirement is necessary for convergence of integral Eq. (21) at the upper limit. Basing on the above reasoning, the authors suggest the following *approximate* expression for angular function in the depth mode[22]:

$$\Phi_{\mathrm{as}}(\theta) \simeq \frac{(1+\alpha)(\alpha\langle\theta^2\rangle_\infty)^{1+\alpha}}{\pi[\alpha\langle\theta^2\rangle_\infty + \theta^2]^{2+\alpha}} \qquad (\alpha > 0)\,. \tag{28}$$

Expression (28) satisfies normalization condition (9) and Eq. (21) for $\langle\theta^2\rangle_\infty$. The quantities α and $\langle\theta^2\rangle_\infty$ are free parameters. To determine them it is necessary to formulate an appropriate set of two equations.

We immediately obtain the first of these equations by setting $\theta^2 = \langle\theta^2\rangle_\infty$ in Eq. (20):

$$\hat{B}_{\mathrm{el}}\Phi_{\mathrm{as}}|_{\theta^2=\langle\theta^2\rangle_\infty} = 0\,. \tag{29}$$

Thus in the depth mode the elastic collision integral in the transport equation is zero at a deviation angle $\theta = (\langle\theta^2\rangle_\infty)^{1/2}$. Now, using Eqs. (16)–(18) and taking into account Eq. (25) and formula (28), we obtain

$$\int_0^\infty \omega^{2+\alpha} J_0(\omega) K_{1+\alpha}(\omega\sqrt{\alpha})\left[1 - \chi_\nu\left(\omega\,\frac{\gamma_{\mathrm{ef}}}{\sqrt{\langle\theta^2\rangle_\infty}}\right)\right]\mathrm{d}\omega = 0\,, \tag{30}$$

where

$$\chi_\nu(\omega) = \frac{2^{1-\nu}}{\Gamma(\nu)}\,\omega^\nu K_\nu(\omega)\,. \tag{31}$$

Here $\Gamma(\nu)$ is the gamma function; $K_\nu(\omega)$ is the Macdonald function. If $\nu = 1/2$, we have $\chi_{1/2}(\omega) = \exp(-\omega)$. It is essential that the quantities γ_{ef} and $\langle\theta^2\rangle_\infty$ occur in Eq. (30) only in the form of the ratio $\gamma_{\mathrm{ef}}/(\langle\theta^2\rangle_\infty)^{1/2}$. Because it is obvious, beforehand, that $(\langle\theta^2\rangle_\infty)^{1/2} \gg \gamma_{\mathrm{ef}}$, then throughout the entire region of integration over ω ($\omega \le 1$) the value of $\omega\gamma_{\mathrm{ef}}/(\langle\theta^2\rangle_\infty)^{1/2} \ll 1$.

This circumstance permits essential simplification of Eq. (30). To see that, let us use the following approximate relation:

$$\sigma[1 - \chi_\nu(\omega\gamma_{\mathrm{ef}} \ll 1)] \approx \begin{cases} D_\nu\omega^2 & \text{if } \nu > 1 \\ D_\nu{}^{\mathrm{QD}}\omega^{2\nu} & \text{if } \nu < 1 \end{cases}\,. \tag{32}$$

The coefficient at ω^2 in formula (32) is the usual angular diffusion coefficient that appears in the Fokker–Planck equation, which is valid only when $\nu > 1$. In the case of interest, when $\nu < 1$, it seems natural to give the name quasi-diffusion coefficient to the coefficient at $\omega^{2\nu}$, because approximation (32) is called the quasi-diffusion approximation[20] (QDA) when $\nu < 1$. It is clear that the angular diffusion coefficient in the case of phase functions of the form of Eq. (26) can be calculated for any value of the parameter ν, both $\nu > 1$ and $\nu < 1$ (Ref. 20):

$$D_\nu = \frac{\sigma\langle 1 - \cos\gamma\rangle}{2} \approx \begin{cases} \dfrac{\sigma}{\nu - 1}\,\gamma_{\mathrm{ef}}{}^2 & \text{if } \nu > 1 \\ \sigma\,\dfrac{\nu}{\nu - 1}\left(\dfrac{\gamma_{\mathrm{ef}}}{2}\right)^{2\nu} & \text{if } \nu < 1 \end{cases}\,. \tag{33}$$

Inasmuch as $D_\nu{}^{\mathrm{QD}} = \sigma[\Gamma(1-\nu)/\Gamma(1+\nu)](\gamma_{\mathrm{ef}}/2)^{2\nu}$, it is easy to see that in a three-dimensional medium D_ν and $D_\nu{}^{\mathrm{QD}}$ obey the following simple relation:

$$D_\nu{}^{\mathrm{QD}} = \frac{1-\nu}{\nu}\,\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\,D_\nu\,. \tag{34}$$

[Thus, for example, in the case of the Henyey–Greenstein phase function ($\nu = 1/2$) $D_{1/2}{}^{\mathrm{QD}} = 2D_{1/2} = 4\gamma_{\mathrm{ef}}$.] On this basis, from now on we shall express all quantities sought (e.g., $\langle\theta^2\rangle_\infty$) not in terms of $D_\nu{}^{\mathrm{QD}}$ but in terms of the more conventional quantity D_ν according to Eq. (34).

Now, taking into account that in the quasi-diffusion approximation the expression in brackets in Eq. (30) is proportional to $\omega^{2\nu}$, after calculation of the integral over ω we obtain

$$ {}_2F_1\left(2 + \nu + \alpha,\, 1 + \nu;\, 1;\, -\frac{1}{\alpha}\right) = 0\,. \tag{35} $$

Here ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function. Thus in the first approximation the value of the unknown parameter α is independent of the quantity $\langle\theta^2\rangle_\infty$. Moreover, the value of α is determined uniquely by the sole parameter ν: $\alpha = \alpha(\nu)$. Thus the speed with which the angular function $\Phi_{\mathrm{as}}(\theta)$ decreases in the depth mode depends only on the single scattering law and is independent of both the scattering coefficient σ and the true absorption coefficient κ.

A numerical solution of Eq. (35) was carried out for the entire range $0 < \nu < 1$. The dependence $\alpha(\nu)$ is plotted in Fig. 1. It can be seen that $\alpha(\nu)$ increases monotonically from 0 at $\nu = 0$ toward ∞ as $\nu \to 1$. In other words, the faster the phase function $\chi_\nu(\gamma)$ falls with enhancement of the single scattering angle γ, the faster the photon angular spectrum in the depth mode decreases, as the scattering angle θ increases.

In certain cases it turns out to be possible to obtain an analytical solution of Eq. (35) or to reduce it to a simpler one. Thus, if $\nu \ll 1$, one can obtain that $\alpha(\nu) \approx \nu$. On the contrary, when $\nu \sim 1$, i.e., $1 - \nu \ll 1$, the parameter $\alpha(\nu)$ is determined by the following formula: $\alpha(\nu) \approx 5.14/(1 - \nu)$. When $\alpha = 1$, Eq. (35) reduces to a simpler equation for determination of the corresponding value of ν:

$$ \tan\left(\frac{\pi\nu}{2}\right) = \frac{2(1+\nu)}{(2+\nu)\nu^2}\left[\frac{\Gamma\left(\frac{1+\nu}{2}\right)}{\Gamma(\nu/2)}\right]^2 . \tag{36} $$

From Eq. (36) we find that $\nu \approx 1/3$. This coincides with the solution to the general equation (35). Now, having found $\alpha(\nu)$, it is necessary to formulate the second equation to determine the quantity $\langle\theta^2\rangle_\infty$. That equation is readily obtained from the correct normalization condition for the angular function of the second approximation when Eq. (20) is iterated and the value [formula (28)] for $\Phi_{\mathrm{as}}(\theta)$ is used as the first approximation:

$$ 2\pi \int_0^\infty \theta\, \mathrm{d}\theta \left(\frac{2}{\kappa}\, \frac{\hat{B}_{\mathrm{el}}\Phi_{\mathrm{as}}}{\theta^2 - \langle\theta^2\rangle_\infty}\right) = 1\,. \tag{37} $$

On performing simple transformations, similar to the ones described above, for Eq. (37) we obtain the following expression for the angular dispersion in the depth mode:

$$ \langle\theta^2\rangle_\infty = \left(8 d_\nu \frac{D_\nu}{\kappa}\right)^{1/1+\nu} . \tag{38} $$

Here D_ν is the coefficient of angular diffusion [formula (33)].

The function d_ν in Eq. (38) is determined by the following expression:

$$ d_\nu = \frac{4^{\nu-1}\Gamma[2 + \nu + \alpha(\nu)]\Gamma(2 - \nu)}{\nu\alpha^{1+\nu}\Gamma[1 + \alpha(\nu)]}\, b_\nu\,, \tag{39a} $$

where

$$ b_\nu = \int_0^\infty \frac{\mathrm{d}x}{1 - x}\, {}_2F_1\left[2 + \nu + \alpha(\nu),\, 1 + \nu;\, 1;\, -\frac{x}{\alpha(\nu)}\right]. \tag{39b} $$

Note that the integrand in Eq. (39b) has no singularity at $x = 1$ because by Eq. (35) the function ${}_2F_1[2 + \nu + \alpha(\nu), 1 + \nu; 1; -x/\alpha(\nu)]$ is equal to zero at that point.

Investigation of Eq. (39b) shows that, with good accuracy (not worse than 5%), the following simple formula holds:

$$ b_\nu \simeq \frac{\alpha(\nu)}{\alpha(\nu) + (\pi/4)\nu + 1.125}\,. \tag{40} $$

Thus, unlike $\alpha(\nu)$, the dispersion of the spectrum in the depth mode depends not only on ν but on the ratio (σ/κ) also, i.e., on the scattering coefficient σ and the light-absorption coefficient κ. In the limit case of $\nu \to 1$ the value of $\alpha \to \infty$ and $d_\nu \to 1$. Therefore, in this case,

$$ \langle\theta^2\rangle_\infty = 2\left(\frac{2D}{\kappa}\right)^{1/2} = \langle\theta^2\rangle_\infty{}^D\,. \tag{41} $$

Passing to the limit $\alpha \to \infty$ in formula (28), we obtain that $\Phi_{\mathrm{as}}(\theta;\, \nu \to 1)$ coincides exactly with the result [Eq. (24)] of the diffusion approximation.

The dependence of d_ν is shown in Fig. 2. Unlike $\alpha(\nu)$, the function d_ν decreases monotonically as ν increases. Here, if $\nu \to 0$, $d_\nu \sim \nu^{-1} \to \infty$; consequently $\langle\theta^2\rangle_\infty \to \infty$ also. This means that the SAA cannot be applied in the case of phase functions $\chi(\gamma) \sim \gamma^{-2}$. Within the wide range $1/3 \le \nu \le 1$ the value of $d_\nu \approx 1$. Thus in this interval of

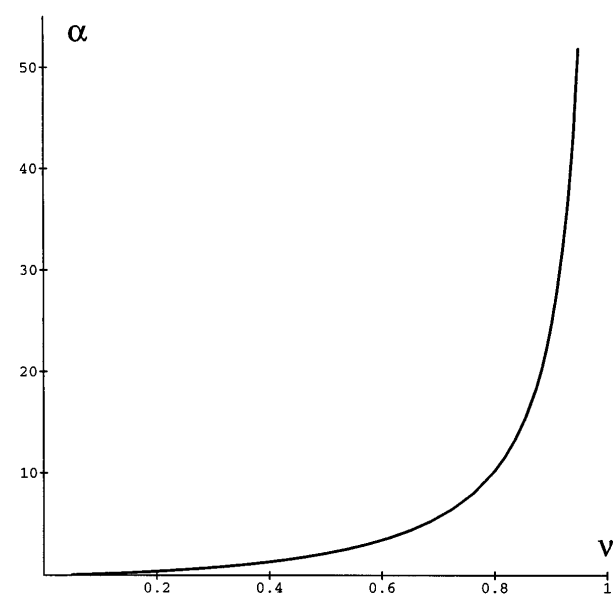

Fig. 1. Dependence of the power index α of the depth mode angular distribution [formula (28)] on the parameter ν of the phase function [Eq. (25)], obtained by numerical solution of Eq. (35) in the range $0 < \nu < 1$.

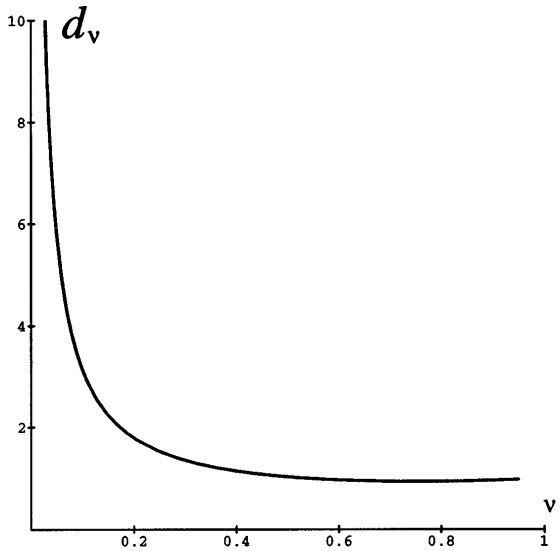

Fig. 2. Dependence [Eqs. (39)] of the coefficient d_ν, an element of Eq. (38) for the depth mode angular dispersion $\langle\theta^2\rangle_\infty$, on the phase function parameter ν.

ν values the dispersion is determined by the very simple formula:

$$\langle\theta^2\rangle_\infty \simeq \left(8\,\frac{D_\nu}{\kappa}\right)^{1/1+\nu} = \left(4\frac{\sigma}{\kappa}\right)^{1/1+\nu} \langle 1 - \cos\,\gamma\rangle^{1/1+\nu}. \quad (42)$$

The quantity D_ν here is defined by formula (33).

Now, when the parameters $\alpha(\nu)$ and $\langle\theta^2\rangle_\infty$ of the angular function $\Phi_{\mathrm{as}}(\theta)$ in the depth mode are defined, one can find the value of the coefficient a according to the general formula, Eq. (23b):

$$a = \frac{2\alpha(\nu) + 3}{\alpha(\nu) + 1}. \quad (43)$$

In the limit case $\nu = 1$ (when $\alpha \to \infty$) one can obtain from Eq. (43) that $a(\nu = 1) = 2$. This value exactly coincides with the value of a^D calculated in the framework of the Fokker–Planck approximation, Eq. (24).

4. TWO-DIMENSIONAL CASE

Above we have dealt with the scattering of light in an ordinary three-dimensional (3D) medium. Recently, however, a number of publications raised the issues of multiple scattering of light in two-dimensional (2D) matter, which can be represented as an ensemble of disordered parallel fibers or rods (see, for example, Refs. 23, 27, and 28–33).

As in the usual 3D case, the degree of scattering anisotropy depends on the ratio of the size of the scattering center a (in two dimensions, of the transverse cross section of the fiberlike inhomogeneity) and the wavelength of the incident radiation, λ. The effective multiple scattering angle γ_{ef} can be estimated as $\gamma_{\mathrm{ef}} \sim \lambda/a$, and therefore, when conditions (1) are valid, the case in which we are interested, of sharply anisotropic small-angle scattering at any depth, occurs.

Unlike in a 3D medium, in the 2D small-angular case there are other exactly solvable models, besides the known small-angle diffusion approach to the problem of light propagation. One of the most interesting models is the scattering model that complies with the Henyey–Greenstein law and that describes the diffraction cross section averaged over oscillations.[20,23,27]

One of the advantages of these models is the possibility to estimate the accuracy of any approximation just by direct comparison of its results in the 2D case with the exact results. In this section we shall proceed according to this scheme: we shall use the exact solution[23] to check the accuracy of the approximation described in Section 3.

We consider a broad stationary light beam of intensity I_0, normally incident upon the plane boundary of a 2D medium, which occupies the semispace $z > 0$ (the z axis is normal to the surface and directed inward). Scattering occurs in the xOz plane.

As in the 3D case we assume single scattering to be sharply anisotropic ($1 - \langle\cos\,\gamma\rangle \ll 1$) and the condition of strong absorption to be satisfied:

$$l_a \ll l_{\mathrm{tr}}, \qquad \text{i.e.,}\ \kappa \gg D_\nu,$$

$$D_\nu = \frac{\sigma}{2}\langle\gamma^2\rangle = \frac{\sigma}{2}\int_{-\pi}^{\pi} \gamma^2 \chi_\nu(\gamma)\mathrm{d}\gamma.$$

The transfer equation for the radiation intensity $I(z, \theta)$ is of the following form:

$$\cos\,\theta\,\frac{\partial I(z, \theta)}{\partial z} + \kappa I(z, \theta) = \hat{B}_{\mathrm{el}} I(z, \theta),$$

$$I(z = 0, \theta) = I_0\delta(\theta), \quad (44)$$

where $\hat{B}_{\mathrm{el}} I(z, \theta)$ is the linearized Boltzmann collision integral[23]:

$$\hat{B}_{\mathrm{el}} I(z, \theta) = \sigma\int_{-\pi}^{\pi} \mathrm{d}\theta'\chi(\gamma)[I(z, \theta') - I(z, \theta)] \quad (45)$$

and θ is the angle between the direction in which the radiation propagates and the yOz plane. In the 2D case $\gamma = \theta' - \theta$ is the angle of single scattering from θ' to θ. As usual in the SAA, one may formally consider all angular variables to vary within infinite limits: $-\infty < \theta, \gamma < \infty$.

Let us single out in the intensity that we are seeking the exponential attenuation of the beam, which takes into account the absorption of light in the medium:

$$I(z, \theta) = I_0\,\exp(-\kappa z)\tilde{I}(z, \theta). \quad (46)$$

Then, substituting Eq. (46) into Eqs. (44) and (45) and expanding all the coefficients in small angle θ up to the first nonnegligible terms, we get the transport equation in the SAA and account for the influence of absorption on the photon angular distribution:

$$\frac{\partial\tilde{I}(z, \theta)}{\partial z} + \frac{\kappa}{2}\,\theta^2\tilde{I}(z, \theta) = \hat{B}_{\mathrm{el}}\tilde{I}(z, \theta),$$

$$\tilde{I}(z = 0, \theta) = \delta(\theta), \quad (47)$$

where

$$\hat{B}_{\mathrm{el}}\tilde{I}(z, \theta) = \sigma\int_{-\infty}^{\infty} \mathrm{d}\theta'\chi(\gamma)[\tilde{I}(z, \theta') - \tilde{I}(z, \theta)]. \quad (48)$$

The analog of the phase function [Eq. (25)] in the 2D case is

$$\chi_\nu(\gamma) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu)} \frac{{\gamma_{\rm ef}}^{2\nu}}{[{\gamma_{\rm ef}}^2 + \gamma^2]^{\nu+1/2}} \tag{49}$$

(we suppose that $0 < \nu < 1$), which satisfies the normalization condition

$$\int_{-\infty}^{+\infty} \chi(\gamma) \mathrm{d}\gamma = 1 . \tag{50}$$

In the particular case when $\nu = 1/2$ we obtain the 2D Henyey–Greenstein phase function

$$\chi^{\rm HG}(\gamma) \approx \frac{1}{\pi} \frac{\gamma_{\rm ef}}{{\gamma_{\rm ef}}^2 + \gamma^2} . \tag{51}$$

Because factorization of the angular spectrum of radiation occurs in the depth mode, we introduce, as before, together with the intensity $I_{\rm as}(z, \theta)$, the angular function

$$\Phi_{\rm as}(\theta) = \frac{1}{\gamma_{\rm ef}} f\left(\frac{\theta}{\gamma_{\rm ef}}\right), \tag{52}$$

normalized by the condition

$$\int_{-\infty}^{\infty} \Phi_{\rm as}(\theta) \mathrm{d}\theta = \int_{-\infty}^{\infty} f(\psi) \mathrm{d}\psi = 1 , \tag{53}$$

so it may be interpreted as the probability density of photon scattering through angles between θ and $\theta + \mathrm{d}\theta$. Then $I_{\rm as}(z, \theta)$ may be represented in the form

$$I_{\rm as}(z, \theta) = E_{\rm as}(z) \Phi_{\rm as}(\theta) , \tag{54}$$

where

$$E_{\rm as}(z) = I_0 a \exp[-(\kappa + k_0) z] \tag{55}$$

is the total radiation flux in the depth mode.

The exact solution[23] to the problem of Eqs. (47) and (48) with the phase function of formula (51) is rather complicated, so here we present only the depth mode results obtained in the framework of the quasi-diffusion approximation. (Actually, these are the only results that we need for our purposes.) Of course, they can be readily obtained from the exact expressions. So, we have from Ref. 23 the following expressions:

$$a = \frac{1}{\beta_0 Ai(-\beta_0)} \int_{-\beta_0}^{\infty} Ai(x) \mathrm{d}x \approx 1.47 , \tag{56}$$

$$f(\psi) = \frac{1}{Ai(-\beta_0)} P_{\beta_0}(\psi \eta^{-1/3}) ,$$

$$P_\beta(x) = \frac{1}{\pi \eta^{1/3}} \int_0^{\infty} Ai(y - \beta) \cos(xy) \mathrm{d}y , \tag{57}$$

where $\beta_0 \approx 1.019$ is the first root of the formula:

$$Ai'(-\beta) = 0 , \tag{58}$$

$Ai(x)$ is the Airy function, and

$$\eta = \frac{2\sigma}{\kappa {\gamma_{\rm ef}}^2} \tag{59}$$

is the main parameter of the problem. The dispersion of the angular distribution and the coefficient k_0 calculated by means of the formulas

$$\langle \theta^2 \rangle_\infty = {\gamma_{\rm ef}}^2 \int_0^{\infty} \psi^2 f(\psi) \mathrm{d}\psi , \qquad k_0 = \frac{\kappa}{2} \langle \theta^2 \rangle_\infty \tag{60}$$

take the following values:

$$\langle \theta^2 \rangle_\infty \approx 1.019 \eta^{2/3} {\gamma_{\rm ef}}^2 , \qquad k_0 \approx 1.019 \eta^{-1/3} \sigma . \tag{61}$$

Now we will apply the approximate method developed above for the 3D case to define the optimal parameters of the depth angular function in two dimensions [analogously to formula (28)]:

$$\Phi_{\rm as}(\theta) \simeq \frac{1}{\sqrt{\pi}} \frac{\Gamma(\alpha + 3/2)}{\Gamma(\alpha + 1)} \frac{(2\alpha \langle \theta^2 \rangle_\infty)^{\alpha+1}}{[2\alpha \langle \theta^2 \rangle_\infty + \theta^2]^{\alpha+3/2}} \qquad (\alpha > 0) , \tag{62}$$

which satisfies normalization condition (53) and Eqs. (60) for $\langle \theta^2 \rangle_\infty$. The quantities α and $\langle \theta^2 \rangle_\infty$, as before, are free parameters. Let us set in Eqs. (47)

$$\tilde{I}_{\rm as}(z, \theta) = a \exp(-k_0 z) \Phi_{\rm as}(\theta) , \tag{63}$$

where $\Phi_{\rm as}(\theta)$ is determined by formula (62) and k_0 can be expressed in terms of $\langle \theta^2 \rangle_\infty$ by Eqs. (60). Then, expressing the right-hand side of Eqs. (47), determined by Eq. (48), as a Fourier transform, one can get the equation

$$\frac{\kappa}{2} (\theta^2 - \langle \theta^2 \rangle_\infty) \Phi_{\rm as}(\theta) = -\frac{\sigma}{\pi} \int_0^{\infty} \mathrm{d}\omega \cos(\omega\theta) \Phi_{\rm as}(\omega) \times [1 - \chi_\nu(\omega)] , \tag{64}$$

where the Fourier transforms of $\Phi_{\rm as}$ and χ_ν are the following:

$$\Phi_{\rm as}(\omega) = \frac{2^{-\alpha}}{\Gamma(\alpha + 1)} [(2\alpha \langle \theta^2 \rangle_\infty)^{1/2} \omega]^{\alpha+1} K_{\alpha+1}[(2\alpha \langle \theta^2 \rangle_\infty)^{1/2} \omega] , \tag{65}$$

$$\chi_\nu(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\omega \cos(\omega\gamma) \chi_\nu(\gamma) = \frac{2^{1-\nu}}{\Gamma(\nu)} |\omega \gamma_{\rm ef}|^\nu K_\nu(|\omega \gamma_{\rm ef}|) . \tag{66}$$

Here K_μ is the Macdonald function. Now one can take the expansion of $1 - \chi_\nu(\omega)$ in the small $\omega \gamma_{\rm ef}$ up to the first nonvanishing term:

$$\sigma[1 - \chi_\nu(\omega)] \approx {D_\nu}^{\rm QD} \omega^{2\nu} , \tag{67}$$

where

$${D_\nu}^{\rm QD} = \sigma \frac{\Gamma(1 - \nu)}{\Gamma(1 + \nu)} \left(\frac{\gamma_{\rm ef}}{2}\right)^{2\nu} \tag{68}$$

is the angular quasi-diffusion coefficient in the 2D case for $\nu < 1$.

After substitution of Eqs. (65) and (66) including formula (67) into Eq. (64) and performing integration over ω, we get the following equation:

$$\frac{\kappa}{2}\,\frac{1}{\sqrt{\pi}}\,\frac{\Gamma(\alpha+3/2)}{\Gamma(\alpha+1)}\,\frac{(2\alpha\langle\theta^2\rangle_\infty)^{\alpha+1}}{(2\alpha\langle\theta^2\rangle_\infty+\theta^2)^{\alpha+3/2}}\,(\theta^2-\langle\theta^2\rangle_\infty)$$
$$=-\frac{D_\nu{}^{\mathrm{QD}}}{\pi}\,\frac{4^\nu\Gamma(\nu+\alpha+3/2)\Gamma(\nu+1/2)}{(2\alpha\langle\theta^2\rangle_\infty)^{\nu+1/2}}$$
$$\times\,{}_2F_1\left(\nu+\alpha+\frac{3}{2},\nu+\frac{1}{2};\frac{1}{2};-\frac{\theta^2}{2\alpha\langle\theta^2\rangle_\infty}\right),\qquad(69)$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Now one can set $\theta^2=\langle\theta^2\rangle_\infty$ in Eq. (69) and obtain the relation between α and ν that does not contain any other parameters of the problem:

$${}_2F_1\left(\nu+\alpha+\frac{3}{2},\nu+\frac{1}{2};\frac{1}{2};-\frac{1}{2\alpha}\right)=0\,.\qquad(70)$$

One can solve this equation numerically and find the explicit dependence $\alpha(\nu)$. The graph of this function is plotted in Fig. 3. Estimation of the asymptotic behavior of $\alpha(\nu)$ gives $\alpha(\nu\to1)\simeq1.21/(1-\nu)$, i.e., $\alpha\to\infty$ while $\nu\to1$. One can easily verify that in this limit the angular function [formula (62)] transforms into the normed Gaussian function:

$$\Phi_{\mathrm{as}}{}^D(\theta)=\frac{1}{\sqrt{2\pi\langle\theta^2\rangle_\infty{}^D}}\,\exp\left(-\frac{\theta^2}{2\langle\theta^2\rangle_\infty{}^D}\right),\qquad(71)$$

that is, the solution to the transfer equation in the diffusion approximation (SADA).

Equation (69) does not hold at all θ, so the value of $\langle\theta^2\rangle_\infty$ obtained from it depends on the method of calculation. However, it follows from the homogeneity of that equation that an expression for angular dispersion in the depth mode always has the form

$$\langle\theta^2\rangle_\infty=\left(2d_\nu\,\frac{D_\nu{}^{\mathrm{QD}}}{\kappa}\right)^{1/1+\nu}.\qquad(72)$$

Here $D_\nu{}^{\mathrm{QD}}$ is the coefficient of angular quasi-diffusion [Eq. (68)], and the coefficient d_ν can be determined by one of several methods. We shall consider two of the most obvious of them. First, one can demand that the equality in Eq. (69) hold at some fixed θ; e.g., at $\theta=0$, it gives us the value

$$d_\nu{}^{\mathrm{I}}=\frac{2^\nu}{\sqrt{\pi}}\,\alpha^{-\nu}\,\frac{\Gamma(\alpha+\nu+3/2)\Gamma(\nu+1/2)}{\Gamma(\alpha+3/2)}\,.\qquad(73a)$$

Another way is to require that the equality in Eq. (69) hold on average, i.e., that the integrals on the left-hand side and the right-hand side of Eq. (69) be equal. This condition is completely equivalent to the 2D analog of normalization condition (37). In this way we obtain

$$d_\nu{}^{\mathrm{II}}=\frac{2^\nu}{\alpha^{\nu+1}}\,\frac{\Gamma(\alpha+\nu+2)\Gamma(\nu+1)}{\Gamma(\alpha+1)}$$
$$\times\,{}_2F_1(\alpha+\nu+2,\,\nu+1;\,3/2;\,-1/2\alpha)\,.\qquad(73b)$$

[In Eqs. (73) it is assumed that $\alpha=\alpha(\nu)$, as described above.] Graphs of $d_\nu{}^{\mathrm{I}}$ and $d_\nu{}^{\mathrm{II}}$ are plotted in Fig. 4. In the case $\nu=1/2$, Eqs. (73) take the following simple form:

$$d_{\nu=1/2}{}^{\mathrm{I}}=\sqrt{\frac{2}{\pi\alpha}}\,\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+3/2)}\,,\qquad(74a)$$

$$d_{\nu=1/2}{}^{\mathrm{II}}=\sqrt{\frac{\pi}{2\alpha^3}}\,\frac{\Gamma(\alpha+5/2)}{\Gamma(\alpha+1)}\left(1+\frac{1}{2\alpha}\right)^{-(\alpha+5/2)}.\qquad(74b)$$

After substitution of the value $\alpha(\nu=1/2)\approx1.89$ into Eqs. (74) we get $d_{\nu=1/2}{}^{\mathrm{I}}\approx1.03$ and $d_{\nu=1/2}{}^{\mathrm{II}}\approx0.95$.

Now, observing from Eq. (68) that $D_{\nu=1/2}{}^{\mathrm{QD}}=\sigma\gamma_{\mathrm{ef}}$, one can obtain expressions for the angular dispersion and the damping index in the depth mode defined by Eqs. (60):

$$\langle\theta^2\rangle_\infty=(d_{\nu=1/2})^{2/3}\eta^{2/3}\gamma_{\mathrm{ef}}{}^2,\qquad k_0=(d_{\nu=1/2})^{2/3}\eta^{-1/3}\sigma\,.\qquad(75)$$

Then we plug in the numerical values and get

$$\langle\theta^2\rangle_\infty{}^{\mathrm{I}}\approx1.019\eta^{2/3}\gamma_{\mathrm{ef}}{}^2,\qquad\langle\theta^2\rangle_\infty{}^{\mathrm{II}}\approx0.966\eta^{2/3}\gamma_{\mathrm{ef}}{}^2,\qquad(76)$$
$$k_0{}^{\mathrm{I}}\approx1.019\eta^{-1/3}\sigma,\qquad k_0{}^{\mathrm{II}}\approx0.966\eta^{-1/3}\sigma\,,\qquad(77)$$

which are in very good agreement with the exact QDA expressions (61).

Graphs of the angular spectra $f(\psi)$ derived from formula (62) are given in Fig. 5 (solid curves).

One can calculate the value of the constant a in Eq. (55) in a similar manner, as was done in a 3D medium [by Eq. (23b)]:

$$a=\frac{\Phi_{\mathrm{as}}(\theta=0)}{\int_{-\infty}^{\infty}\Phi_{\mathrm{as}}(\theta)\mathrm{d}\theta}=\frac{\Gamma(\alpha+1)\Gamma(2\alpha+3)}{\Gamma(\alpha+3/2)\Gamma(2\alpha+5/2)}\,.\qquad(78)$$

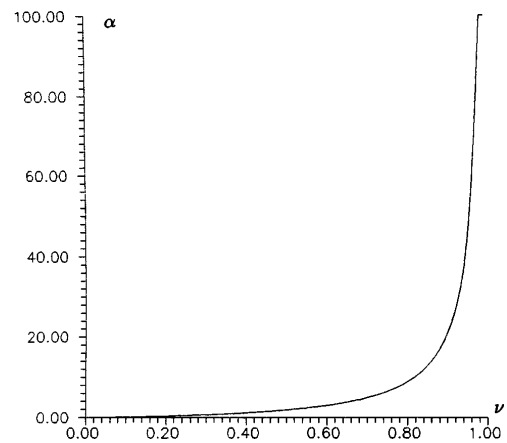

Fig. 3. Dependence of the parameter α of the depth mode angular function [formula (62)] on the parameter ν of the phase function [Eq. (49)] in a 2D medium, obtained by numerical solution of Eq. (70).

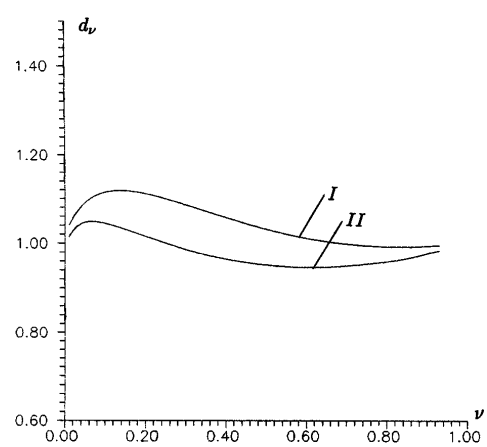

Fig. 4. Graphs of the coefficients $d_\nu{}^{\mathrm{I}}$ and $d_\nu{}^{\mathrm{II}}$ as functions of the phase function parameter ν in the 2D case. These coefficients appear in Eq. (72) for angular dispersion in the depth mode and are defined by Eqs. (73a) and (73b), respectively.

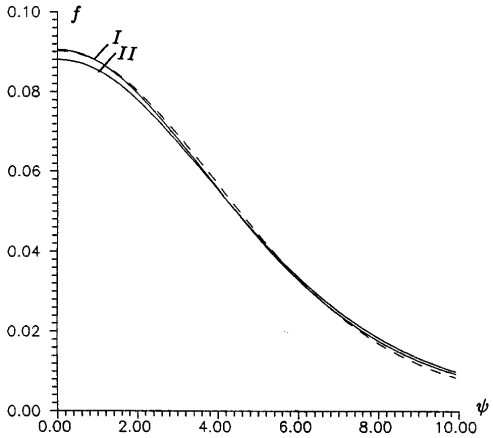

Fig. 5. Plots of the 2D depth angular spectra $f^{\mathrm{I}}(\psi)$ and $f^{\mathrm{II}}(\psi)$ derived from formula (62) in the case $\nu = 1/2$ with the use of $\langle\theta^2\rangle_\infty{}^{\mathrm{I}}$ and $\langle\theta^2\rangle_\infty{}^{\mathrm{II}}$, respectively, from formulas (76). Here $\psi = \theta/\gamma_{\mathrm{ef}}$, and the value of the parameter [Eq. (59)] is $\eta = 150$. For comparison the exact QDA spectrum [Eqs. (57)] is plotted by the dashed curve. Good coincidence between the approximate and the exact functions is seen.

As well as in a 3D medium, the value of a depends only on the parameter ν: $a = a(\nu)$. The function $a(\nu)$ varies in the range from $a_{\min} = \sqrt{2} \approx 1.4$ at $\nu = 0$ to $a_{\max} = 16/3\pi \approx 1.7$, when $\nu = 1$ and $\alpha \to \infty$. The value $a_{\min} = \sqrt{2}$ exactly coincides with a^D in the Fokker–Planck approximation in a 2D medium.

After substitution of the value of $\alpha(\nu = 1/2)$ into Eq. (78) we obtain $a(\nu = 1/2) \approx 1.51$, which is very close to the exact QDA value [relation (56)].

From Fig. 5 one can see that the deviations of our approximate functions from the exact QDA spectra do not exceed 1–2%.

5. CONCLUSIONS

It is interesting to compare the values of $\langle\theta^2\rangle_\infty$ computed by Eq. (38) with the results for the same quantity obtained in Ref. 18. In the latter paper an original method, a self-consistent Gaussian approximation, was used for computation of some important characteristics of light fields in strongly absorbing anisotropically scattering media. This approximation is based on a generalization of well-known results of the diffusion theory in the cases of both normal and oblique incidence of light upon the surface of a medium. (In the latter case the effect of the brilliancy body rotation, as the depth is increasing, should be taken into account.) A heuristic procedure for renormalization of the angular diffusion coefficient that depended on the depth of the light beam penetration into the medium was used.[18] In the case of power-law phase functions analogous to the ones used in the present paper, the following expression for $\langle\theta^2\rangle_\infty$ in the depth mode was obtained:

$$\langle\theta^2\rangle_\infty = \rho_\nu\left(8\,\frac{D_\nu}{\kappa}\right)^{1/1+\nu}, \qquad \rho_\nu = \left(\frac{\xi}{2}\right)^{2(1-\nu/1+\nu)}, \qquad \nu < 1. \tag{79}$$

Here ξ is a free parameter the value of which can be determined, for example, by comparison with results of numerical calculations. For instance, the value $\xi = 1.8$ is given in Ref. 18 in the case of the Henyey–Greenstein phase function and is $\rho = 0.93$. And it was pointed out there that for other phase functions one also usually takes $\xi \approx 2$. In this case $\rho \approx 1$, and Eq. (79) exactly coincides with formula (42) obtained in the present paper, which, however, does not contain any fitting parameters. (Let us note that in Ref. 18 the angular spectrum was *a priori* assumed to have Gaussian form, which is certainly not justified in the case of slowly decreasing phase functions when $\nu < 1$.)

The values of $\langle\theta^2\rangle_\infty$ obtained with the aid of formulas (33) and (42) permit calculation of the depth damping index k by Eq. (22). The obtained values were compared with the results of Monte Carlo calculations.[4] The results of such comparison are presented in Table 1. The accuracy of coincidence turned out to be of the order of magnitude of $D/\kappa \sim l_a/l_{\mathrm{tr}}$, which was to be expected and which is directly related to the applicability of the small-angle approximation.

As far as a 2D medium is concerned, one is presented with an extra possibility to compare the results for $\langle\theta^2\rangle_\infty$ obtained in the present paper for the case of the Henyey–Greenstein phase function with the results of analogous calculations in the framework of the QDA from Ref. 23. One can see that a near coincidence of formulas (76) and (61) occurs.

Table 1. Values of the Depth Damping Index k Computed by Formulas (22), (33), and (42) Compared with the Results of the Numerical Calculations

$\langle\cos\gamma\rangle$ ($\langle\gamma^2\rangle$)	Photon-Survival Probability $\Lambda_* = \dfrac{\sigma}{\sigma + \kappa_*}$	$\dfrac{D}{\kappa_*}$	$\Delta = \dfrac{k - \kappa_*}{\kappa_* + \sigma}$	Δ Numerical Calculation[a]
0.95	0.2	6.25×10^{-3}	0.0542	0.0518
(0.1)	0.4	1.67×10^{-2}	0.0783	0.0750
	0.8	1.00×10^{-1}	0.0862	0.0812
0.97	0.2	3.75×10^{-3}	0.0386	0.0379
(0.06)	0.4	1.00×10^{-2}	0.0557	0.0545
	0.8	6.00×10^{-2}	0.0613	0.0587
0.98	0.2	2.5×10^{-3}	0.0295	0.0294
(0.04)	0.4	6.67×10^{-3}	0.0425	0.0422
	0.8	4.00×10^{-2}	0.0468	0.0453

[a]Ref. 4.

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