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AMLT: anisotropic mixing length theory

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Summary. The standard MLT assumed a) that the largest eddies are the only ones that contribute to convection and b) that they are isotropic. These two requirements are internally inconsistent since it is experimentally known that only small eddies are isotropic, while large eddies exhibit large degrees of anisotropy. In this paper we present a new model: anisotropic mixing length theory, AMLT, together with a model that relates the anisotropy to other quantities of the problem. The new AMLT equations are solved for two cases of interest in stellar structure calculations.

In the first case, the gradients ∇_r and ∇_{ad} are considered known, and the resulting expression for ∇ and ∇' derived. The basic AMLT relations are given in two equivalent representations: Eqs. (49)–(52) and/or Eqs. (53)–(54). It is shown that:
1) The value of $\nabla - \nabla_{ad}$ from the AMLT is up to *two orders of magnitude larger* than the corresponding MLT value, Eq. (60).
2) In the physically important case of large degrees of anisotropy, the turbulent velocity in the AMLT is *lower* than the corresponding MLT value by a factor of 2, Eq. (67). This may help lessen the problem of overshooting.

In the second case, the gradients ∇ and ∇_{ad} are considered known and the resulting expressions for ∇_r and ∇' are derived, together with the expressions for the convective flux and the turbulent velocity, Eqs. (71), (73), (74) and (75). For the case of large anisotropies, the AMLT can be solved analytically, Eqs. (79)–(83). The convective flux, the convective velocities, the temperature excess, and the radiative gradient, are *smaller* than the MLT relations by the factors x^{-1}, $x^{-1/2}$, $x^{-1/2}$, and x^{-1}, respectively, where $x \gg 1$ is given in terms of known variables via Eq. (79).

The present formalism is valid only in the optically thick case.

Key words: convection – UV radiation – hydrodynamics

1. Introduction

The mixing length theory, MLT, despite its well known shortcomings (e.g. Renzini, 1987; Baker and Kuhfuss, 1987), remains a useful tool for the treatment of convective turbulence (Böhm-Vitense, 1958; Kippenhahn, 1963, referred to as K63; Cox and Giuli, 1968, referred to as CG68; Maeder, 1975; Maeder and Mermilliod, 1981; Langer, 1987). In this paper we point out and correct an internal contradiction of two basic assumptions of the MLT: on the one hand, it is assumed that not all turbulent eddies contribute to bulk properties like velocities and convective fluxes; only a subset of them, the large eddies, make the most significant contribution. (In practice, only one eddy in MLT is taken to dominate.) While this is a physically acceptable approximation (in turbulent flows the large eddies carry most of the energy), it is not correct to assume that these same large eddies are isotropic. In any turbulent flow, only the small eddies can be assumed to be isotropic, but they are not considered within the MLT framework since they contribute insignificantly to the bulk properties of the flow. The large eddies are experimentally known to be highly anisotropic and yet, in the standard MLT, they are treated as isotropic. This occurs since all the different length scales that enter into MLT are taken to be identical and equal to an unspecified mixing length, l, even though they represent different physical quantities.

A detailed derivation of the MLT expressions (see Appendix A) shows that there are at least three different lengths associated with the physical system:
1) The time derivative of the temperature and velocity fields is usually transformed as

$$\frac{\partial}{\partial t} \to w \frac{\partial}{\partial \Lambda_\xi}, \tag{1}$$

which introduces a length Λ_ξ. Here, w is the velocity in the z direction.
2) The three-dimensional Laplacian operator that enters in the temperature equation, Eq. (A8), is changed to

$$-\nabla^2 \to \Lambda^{-2}, \tag{2}$$

which introduces a second length, Λ. To be more precise, the action of the operator $-\nabla^2$ can be thought of as giving rise to k^2, where k is the total wavenumber, see Eq. (6) below.
3) The third length appears in the Taylor expansion of the temperature excess, see Eq. (A4), i.e.,

$$\Lambda_z = z - z_0. \tag{3}$$

In the standard MLT, it is assumed that

$$\Lambda_\xi \approx \Lambda \approx \Lambda_z \approx l, \tag{4}$$

where l is called the mixing length.

The identification (4) is tantamount to assuming isotropy. In fact, Λ^{-1}, which represents the magnitude of the total wavenumber k, see Eq. (B19), is taken to be equal to Λ_z^{-1} which represents only the z component of k, i.e., the inverse of the size of the eddy in the z-direction. Since the large eddies are expected to be considerably stretched in that direction, one expects a skinny eddy which would mean that

$$(k_x^2 + k_y^2) \gg k_z^2. \tag{5}$$

To properly account for the anisotropy of the eddies, it is

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therefore convenient to introduce an anisotropy parameter x,

$$x = (k_x^2 + k_y^2)/k_z^2, \tag{6}$$

which we expect to be much greater than unity when dealing with the large scale portion of the eddy spectrum. In Appendix B, it is shown that for the MLT expressions to be compatible with a more complete model of turbulence, the relation between the three lengths introduced earlier must be, Eqs. (B28) and (B29),

$$\Lambda \approx (1 + x)^{-1/2} \Lambda_z \tag{7}$$

$$\Lambda_\xi \approx \frac{x^{1/2}}{1 + x} \Lambda_z, \tag{8}$$

where

$$2\Lambda_z = l. \tag{9}$$

As expected, for large values of anisotropy, x, Λ, and Λ_ξ are considerably smaller than the mixing length l. Only for values of x of order unity, i.e., isotropy, are the three lengths essentially equal.

In what follows, we shall discuss the general expressions for the turbulent velocity, the convective flux, and the temperature excess, written as functions of Λ, Λ_ξ, and Λ_z. (The derivation of these expressions is given in Appendix A so as not to interrupt the flow of the presentation.)

The introduction of the parameter x clearly demands that a model be provided which relates this parameter to the other quantities in the system. Such a model will be given in Sect. 4. In Sect. 5, we will present the basic equations of the AMLT formalism. Interestingly enough, the well known cubic equation that must be solved in the standard MLT to derive the value of $\nabla - \nabla_{ad}$ reduces to a quadratic equation in the AMLT. Only one universal function $G(x)$ enters the problem which is then tabulated for different values of x. The values of the gradient $\nabla - \nabla_{ad}$ can then be obtained by solving a simple algebraic relation.

Before we enter the derivation and discussion of the equations, it must be mentioned that this is not the first time that anisotropy has been considered in the context of the MLT. For example, Gough's expressions of 1976 and 1977 (for a review, see Gough, 1978) included radiative losses but had an incorrect x dependence. Further work (Gough, 1978) has the correct x dependence but is valid only for the case of no radiative energy losses. Finally, the expressions in Cox and Giuli (1968) include radiative losses but assume isotropy, $x \approx$ unity, which as we have already discussed is intrinsically inconsistent.

2. The basic equations of the AMLT

Before presenting the equations to be solved, it is necessary to establish a system of notation. Following Cox and Guili (1968), Kippenhahn (1963), and Baker and Temesvary (1966; hereafter referred to as BT), T is defined to be the average temperature at any level in the medium and T' the temperature of the rising convective element. Furthermore,

$$\nabla = -\frac{H_p}{T}\frac{dT}{dz}, \quad H_p = \frac{P}{\rho g}, \tag{10}$$

where H_p is the pressure scale height. The radiative flux F_r is given by

$$F_r = -K\frac{dT}{dz} = \frac{4acT^4}{3\rho\kappa H_p}\nabla, \tag{11}$$

where K is the thermal conductivity. For future reference, we shall also introduce the thermometric conductivity, χ, defined as (where κ is the opacity)

$$\chi = \frac{K}{\rho c_p} = \frac{4acT^3}{3\rho^2\kappa c_p}. \tag{12}$$

Let us now consider the following expressions derived in Appendix A:

Convective velocity in the z direction, w (Eq. A5): with the volume expansion coefficient $\alpha = T^{-1}$, which is valid for a perfect gas, we have

$$w^2 = \frac{2x}{1+x}\frac{g}{T}\Lambda_z\Lambda_\xi\left(\frac{dT'}{dz} - \frac{dT}{dz}\right), \tag{13}$$

or

$$w = 2^{-1/2}x^{1/2}(1+x)^{-1/2}(\Lambda_\xi/\Lambda_z)^{1/2}(g/H_p)^{1/2}l(\nabla - \nabla')^{1/2} \tag{14}$$

Equation (14) can be compared with Eq. (14.28b) of CG68 and Eq. (69) of K63. See, however, the cautionary remark after Eq. (62).

Convective flux, F_c: from Eq. (A7), we obtain

$$F_c = c_p\rho w\Lambda_z\left(\frac{dT'}{dz} - \frac{dT}{dz}\right) = \tfrac{1}{2}c_p\rho T w\frac{l}{H_p}(\nabla - \nabla'). \tag{15}$$

Substituting the expression for w yields

$$F_c = 8^{-1/2}c_p\rho x^{1/2}(1+x)^{-1/2}(\Lambda_\xi/\Lambda_z)^{1/2} \times (l/H_p)^2 T(P/\rho)^{1/2}(\nabla - \nabla')^{3/2}, \tag{16}$$

to be compared with Eq. (14.18) of CG68 and Eq. (A1) of BT. As one can see, the AMLT expressions formally have the same structure as the MLT expressions which are valid for $x \approx 1$, even though this appearance is deceptive since x is a function of all the other variables within the system.

For the temperature excess, $\nabla - \nabla'$, we have from Eq. (A11),

$$(\nabla - \nabla')^{1/2} = A(x)\frac{\nabla' - \nabla_{ad}}{1 + B(x)(\nabla - \nabla')^{1/2}}, \tag{17}$$

where the new quantities $A(x)$ and $B(x)$ are defined as

$$A(x) = \frac{9}{4U}x^{1/2}(1+x)^{-1/2}(\Lambda_\xi/\Lambda_z)^{1/2}(\Lambda/\Lambda_z)^2, \tag{18a}$$

$$B(x) = A(x)\left[\frac{\Lambda_z}{\Lambda_\xi} - 1\right], \tag{18b}$$

$$U = \frac{24\sigma T^3}{c_p\rho^2\kappa l^2}(2H_p/g)^{1/2}, \tag{19}$$

where U is the parameter defined in Eq. (71) of K63. Note that $U^{-1} = 2A$, where A is defined in Eq. (14.70) of CG68. If Eq. (17) is compared to the corresponding MLT expression, Eqs. (14.71) and (14.73) of CG68, i.e.,

$$(\nabla - \nabla')^{1/2} = A(\nabla' - \nabla_{ad}), \tag{20}$$

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or Eq. (A7) of BT, it is noticed that (17) is structurally different because of the appearance of the new function $B(x)$ whose presence is due to the difference between the various length scales.

Having derived the basic equations, we shall proceed to solve them in two different cases. In Sect. 4 we present a model for x.

3. Solution of the AMLT equations when the total flux is given

Following standard procedure, we write for the total flux F, with $\sigma = ac/4$,

$$F = F_r + F_c = \sigma T_e^4 = \frac{4acT^4}{3\rho\kappa H_p}\nabla_r. \tag{21}$$

Using now Eqs. (11) and (16), we obtain

$$\nabla_r - \nabla = C(x)(\nabla - \nabla')^{3/2}, \tag{22}$$

where the function $C(x)$ is given by

$$C(x) = \Lambda_z^2 \Lambda^{-2} A(x). \tag{23}$$

Equation (22) should be compared with Eq. (14.72) of CG68 and Eq. (A6) of BT.

Equations (17) and (22) contain the four gradients ∇, ∇', ∇_r, and ∇_{ad}. The standard procedure is that of eliminating ∇' between the two relations so as to obtain one equation for ∇ as a function of the two gradients ∇_r and ∇_{ad}, which are considered to be known. The solution of (17) is

$$\nabla - \nabla' = \frac{1}{1+\sigma(x)}[\xi(x) - U(x)]^2, \tag{24}$$

where

$$\sigma(x) = B(x)/A(x) = \frac{\Lambda_z}{\Lambda_\xi} - 1. \tag{25}$$

The functions $\xi(x)$ amd $U(x)$ are generalizations of Kippenhahn's ξ and U, namely

$$\xi(x) = [U^2(x) + \nabla - \nabla_{ad}]^{1/2} \tag{26}$$

$$U(x)^{-1} = 2A(x)[1 + \sigma(x)]^{1/2}. \tag{27}$$

Substituting Eq. (24) in (22), we obtain the cubic equation,

$$(\Lambda_\xi^2/2\Lambda^2)[\xi(x) - U(x)]^3 + \xi^2(x)U(x) - U^3(x) - U(x)(\nabla_r - \nabla_{ad}) = 0, \tag{28}$$

which generalizes the well known cubic equation of the standard MLT, Eq. (72) of Kippenhahn (1963). In order to proceed, we need a model for the anisotropy x as a function of the other variables.

For completeness, let us remark that in the standard MLT, if one introduces the variable Γ,

$$1 + 2\Gamma = \frac{\xi}{U}, \tag{29}$$

Eq. (28) becomes

$$\frac{9}{4}\Gamma^3 + \Gamma^2 + \Gamma - \frac{1}{4U^2}(\nabla_r - \nabla_{ad}) = 0, \tag{30}$$

which is Eq. (14.82) of Cox and Giuli after use is made of (14.80) and (14.81).

4. The degree of anisotropy x

The reason why there is convective flux is because there is a convective instability that can be characterized by a growth rate $n_s(k)$, i.e., a stability analysis indicates that disturbances grow exponentially like

$$\exp[n_s(k)t], \qquad n_s(k) > 0. \tag{31}$$

In the linear analysis, the instability causes the kinetic energy to increase exponentially at the rate $2n_s(k)$. The system can reach stationarity only if the non-linear interactions are taken into account so as to distribute the energy injected at the largest scales to all the remaining eddies (scales). By imposing that balance, one finds the resulting turbulent energy spectral function. The way one sets up the non-linear interactions is determined by the turbulence model [see Canuto and Goldman (1985), Eq. (1) and Eq. (5) for the turbulent viscosity].

The growth rate n_s of the convective instability is known to be given by the solution of (Cabot et al., 1987)

$$n_s^2(k) + \chi k^2 n_s(k) - g\alpha\beta\left[\frac{k_\perp}{k}\right]^2 = 0, \tag{32}$$

where α is the volume expansion coefficient and β is the superadiabatic temperature gradient,

$$g\alpha\beta = gH_p^{-1}(\nabla - \nabla_{ad}), \quad k^2 = k_\perp^2 + k_z^2. \tag{33}$$

The growth rate given by Eq. (32) is the *eigenvalue* resulting from the stability analysis. As in many other problems, the specific form of n_s is the result of having imposed specific boundary conditions. Furthermore, since the structure of the equations is such that the velocity in the z-direction (L stands for linear) is given by

$$W_L(z) = W_0 \sin k_z z, \tag{34}$$

$k_z d$ is defined so that

$$W_L(d) = 0, \tag{35}$$

i.e.,

$$k_z d = n\pi. \tag{36}$$

Introducing the anisotropy parameter x, Eq. (6), one has for the total wavenumber k

$$k^2 = k_z^2(1 + x) = (\pi n/d)^2(1 + x). \tag{37}$$

The smallest wavenumber $k = k_0$, corresponds to taking $n = 1$, so that Eq. (B7) follows. Furthermore,

$$\frac{k_\perp^2}{k^2} = \frac{x}{1+x}. \tag{38}$$

Introducing (37) with $n = 1$ and (38) into (32), and requiring that

$$\frac{d}{dx}n_s(x) = 0, \tag{39}$$

which corresponds to determining the anisotropy that yields the fastest growing mode (the shortest timescale), one can work out the relation

$$(x-1)(x+1)^3 = (81/32)\,U^{-2}\,(\nabla - \nabla_{ad}). \tag{40}$$

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Eliminating the quantity $\nabla - \nabla_{ad}$ via Eq. (26) Eq. (40) becomes

$$(x-1)(x+1)^3 U^2 = \frac{81}{2\pi^4}(d/l)^4[\xi^2(x) - U^2(x)]. \quad (41)$$

If we eliminate U using (18a) and (27) and employ (B29), Eq. (41) further simplifies to

$$(2x-1)U(x) = \xi(x), \quad (42)$$

which is the extra relation needed to solve the problem. Introducing (42) into (28), the function $\xi(x)$ can be eliminated in terms of $U(x)$ and the equation becomes of second order in $U(x)$. Using then the definition of $U(x)$ from Eq. (27), one finally obtains

$$G(x) = (\nabla_r - \nabla_{ad})U^{-2}, \quad (43)$$

where $G(x)$ is a universal function of the variable x given by

$$G(x) = \frac{16}{81}\frac{1+x}{x}(\Lambda_z/\Lambda)^4[\Lambda_\xi^2\Lambda^{-2}(x-1)^3 + x(x-1)]. \quad (44)$$

Equation (43) has considerably simplified the problem. The right hand side of the equation contains only known quantities, while the left hand side is a universal function of x that can be tabulated once and for all.

5. Solution of the AMLT equations when the total flux is given

The basic equations of the AMLT formalism are Eqs. (42)–(44), (26)–(27), and (18)–(19). Before we collect them together in their final form, we shall transform them so that only the x variable appears. To do this, the relations (B28)–(B30) are used to eliminate the Λ's in terms of the variable x. When that is done, the basic equations of the AMLT can be written as

$$\nabla_r - \nabla_{ad} = U^2 G(x), \quad (45)$$

$$U(x) = (8^{1/2}/9)x^{-1/2}(1+x)^{3/2}U, \quad (46a)$$

$$\xi(x) = (2x-1)U(x), \quad (46b)$$

$$\xi^2(x) = U^2(x) + \nabla - \nabla_{ad}, \quad (46c)$$

$$(82/32)G(x) = \gamma^{-1}(1+x)^2(x-1)^3 + (1+x)^3(x-1). \quad (47)$$

Given the simple structure of the above equations, which we have written in detail to simplify the comparison with the standard case, one can reduce the set of Eqs. (46) to one equation, i.e.,

$$\nabla - \nabla_{ad} = (32/81)(1+x)^3(x-1)U^2, \quad (48)$$

so that the three basic equation of AMLT are now given by (45), (47), and (48).

5.1. First representation

Eliminating U^2 between (48) and (45), the complete system of equations, including Eq. (22), becomes

$$G(x) = (32/81)(x-1)(x+1)^2[x+1+\gamma^{-1}(x-1)^2] \quad (49)$$

$$\nabla_r - \nabla_{ad} = U^2 G(x) \quad (50)$$

$$\frac{\nabla_r - \nabla_{ad}}{\nabla - \nabla_{ad}} = 1 + \gamma^{-1}(x-1)^2/(x+1), \quad (51)$$

$$\nabla - \nabla' = (2^{17/12}\gamma^{1/6}/81^{1/3}) \times (1+x)^{2/3}x^{-1/2}U^{2/3}(\nabla_r - \nabla)^{2/3}. \quad (52)$$

The procedure is therefore as follows: *given a value of U and $\nabla_r - \nabla_{ad}$, Eqs. (49)–(50) are solved for x; values of the function $G(x)$ are listed in Table 1 and Fig. 1. When the value of x is substituted in (51) and (52), the values of $\nabla - \nabla_{ad}$ and $\nabla - \nabla'$ are obtained.*

5.2. Second representation

To render the comparison between AMLT and MLT more transparent, we shall employ the variables ζ and B introduced by Cox and Giuli, Eqs. (14.78) and (14.81), i.e.,

$$\zeta = \frac{\nabla_r - \nabla}{\nabla_r - \nabla_{ad}}, \quad B = [(\nabla_r - \nabla_{ad})/9U^2]^{1/3}, \quad (53)$$

in which case Eqs. (50)–(51) become

$$G(x) = 9B^3$$

$$1 - \zeta = (32/729)B^{-3}(1+x)^3(x-1)$$

$$\nabla = \nabla_{ad} + (1-\zeta)(\nabla_r - \nabla_{ad}), \quad (54)$$

where $G(x)$ is given by (49). The procedure is now: *given a value of B, the first of (54) is solved for x, which is then used in the second relation to obtain the variable ζ and in the third relation to obtain the gradient ∇. The temperature excess $\nabla - \nabla'$ is obtained from Eq. (52).*

For completeness, we shall also give the expressions for the convective flux F_c and convective velocity, w. They are,

$$F_c = \zeta(1 - \nabla_{ad}/\nabla_r)F \quad (55)$$

where F is the total (known) flux. The velocity w, derived from Eqs. (14) and (22), is, with v_s the sound speed and Γ the ratio of specific heats,

$$\Gamma^{1/2}w/v_s = (2^{1/12}/\gamma^{1/6})x^{1/2}(1+x)^{-2/3}(l/H_p)UB\zeta^{1/3}, \quad (56)$$

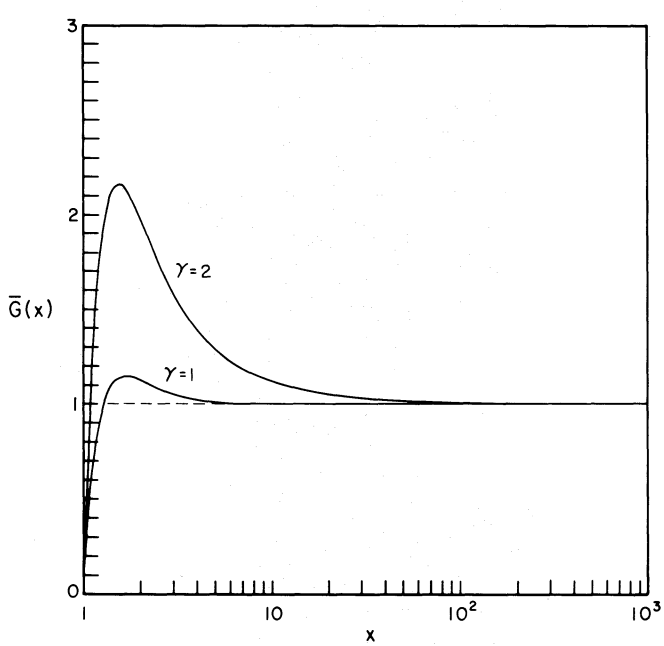

Fig. 1. These are plots of the function $\bar{G}(x)$ for $1 \le x \le 10^3$ and for $\gamma = 1$ and 2 as indicated. $\bar{G}(x)$ is defined to be $G(x)$, Eq. (49), divided by the value of $G(x)$ as $x \to \infty$, i.e., $\bar{G}(x) = G(x)/G(x \to \infty)$, where $G(x \to \infty) = 32x^5/(81\gamma)$, Eq. (64)

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Table 1. Values of the function $G(x)$, Eq. (47)

x	G(x) $\gamma=1$	G(x) $\gamma=2$
2	1.422×10^1	1.244×10^1
3	1.011×10^2	7.585×10^1
4	4.148×10^2	2.815×10^2
5	1.252×10^3	7.964×10^2
6	3.097×10^3	1.887×10^3
7	6.675×10^3	3.944×10^3
8	1.299×10^4	7.504×10^3
9	2.339×10^4	1.327×10^4
10	3.958×10^4	2.216×10^4
20	1.265×10^6	6.670×10^5
30	9.601×10^6	4.971×10^6
40	4.046×10^7	2.076×10^7
50	1.235×10^8	6.301×10^7
60	3.072×10^8	1.563×10^8
70	6.640×10^8	3.369×10^8
80	1.295×10^9	6.556×10^8
90	2.333×10^9	1.180×10^9
100	3.951×10^9	1.996×10^9
150	3.000×10^{10}	1.510×10^{10}
200	1.264×10^{11}	6.353×10^{10}
250	3.858×10^{11}	1.937×10^{11}
300	9.600×10^{11}	4.816×10^{11}
350	2.075×10^{12}	1.041×10^{12}
400	4.045×10^{12}	2.028×10^{12}
450	7.290×10^{12}	3.653×10^{12}
500	1.235×10^{13}	6.185×10^{12}
550	1.988×10^{13}	9.960×10^{12}
600	3.072×10^{13}	1.539×10^{13}
650	4.584×10^{13}	2.296×10^{13}
700	6.640×10^{13}	3.325×10^{13}
750	9.375×10^{13}	4.694×10^{13}
800	1.295×10^{14}	6.481×10^{13}
850	1.753×10^{14}	8.775×10^{13}
900	2.333×10^{14}	1.168×10^{14}
950	3.057×10^{14}	1.530×10^{14}
1000	3.951×10^{14}	1.977×10^{14}

For the MLT case, we shall adopt Eq. (14.86) of CG68, i.e. (the equation satisfied by ζ is given below, Eq. 58),

$$\Gamma^{1/2} w/v_s = 2^{-1/2}(l/H_p) U B \zeta^{1/3}. \tag{57}$$

However, *a cautionary remark* is in order. Within the AMLT formalism, w is the convective velocity in the z direction, while v_t is the average convective velocity; the two are in principle different, their relation being given by Eq. (B8). Within the MLT formalism, the difference between the two is not so clear: for example, in Eq. (14.18) of CG68, the velocity v is *defined* by CG68 as the vertical velocity which therefore coincides with our w; on the other hand, the same velocity is then identified with the one in Eq. (14.23) which is defined by CG68 as the average speed, i.e., our v_t. For this reason, one must be careful in interpreting the ratio (56) to (57), computed in Table 2 below, *especially* for small values of x, where the difference between w and v_t is significant, see Eq. (B8).

In order to compare the AMLT results with those of the MLT, we solved the relations (54) together with the cubic equation that in the MLT relates the variables B and ζ. The latter is given by Eq. (14.82) of CG68, i.e., with $a_0 = 9/4$,

$$\zeta^{1/3} + B\zeta^{2/3} + a_0 B^2 \zeta - a_0 B^2 = 0. \tag{58}$$

The results are presented in Table 2 and in Figs. 2 and 3.

In each of the two Tables 2, for $\gamma = 1$ and 2 and a given value of B, we present the value of x, as from the first of Eqs. (54), the value of $1 - \zeta$, as from the second of Eqs. (54) and the value of $1 - \zeta$ as from the MLT, Eq. (58). In the fifth column, we present the value of the ratio

$$\frac{(\nabla - \nabla_{ad})_{AMLT}}{(\nabla - \nabla_{ad})_{MLT}}. \tag{59}$$

As one can see from the numerical results from Fig. 2, the ratio is of order unity for $B \approx 1$, i.e., $x \approx 1$, which as we shall see means inefficient convection. On the other hand, it becomes significantly larger than unity for $B \gg 1$ and $x \gg 1$, i.e., efficient convection. Numerically, one can see from Fig. 2 that for $\gamma = 1$ to 2, we have for $B = 10^4$ the result

$$\frac{(\nabla - \nabla_{ad})_{AMLT}}{(\nabla - \nabla_{ad})_{MLT}} \approx 50 - 80, \tag{60}$$

i.e., *the MLT underestimates the superadiabaticity by almost two orders of magnitude*, the cause being the lack of a proper treatment of anisotropy.

The numerical result (60) can be generalized to an analytical expression by using the asymptotic relations for $1 - \zeta$ for large values of x, see Eq. (65). Since in the MLT, $1 - \zeta = (4/9B)$, we derive

$$\frac{(\nabla - \nabla_{ad})_{AMLT}}{(\nabla - \nabla_{ad})_{MLT}} = 1.2\gamma^{4/5} B^{2/5}, \tag{61}$$

which represents the numerical results quite accurately.

Finally, in column 6 of Table 2 we give the ratio of the turbulent velocities w from the AMLT and MLT, Eqs. (56) and (57). As one can see from Fig. 3,

$$w_{AMLT} < w_{MLT}, \tag{62}$$

a result for which we shall derive an analytical expression, see Eq. (66). (In deriving these numerical results, we assumed that the

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Table 2. Results of the solution of the AMLT and MLT equations, Eqs. (53)–(54), (56)–(57) and (58) of the text

$\gamma = 1$					
B	x	$1-\zeta_{AMLT}$	$1-\zeta_{MLT}$	$\frac{(\nabla-\nabla_{ad})_{AMLT}}{(\nabla-\nabla_{ad})_{MLT}}$	$\frac{w_{AMLT}}{w_{MLT}}$
0.1	1.003	1.000	1.000	1.000	0.6685
0.2	1.022	9.998×10^{-1}	9.993×10^{-1}	1.000	0.6663
0.3	1.069	9.977×10^{-1}	9.931×10^{-1}	1.005	0.6632
0.4	1.146	9.902×10^{-1}	9.700×10^{-1}	1.021	0.6652
0.5	1.245	9.739×10^{-1}	9.220×10^{-1}	1.056	0.6765
0.6	1.357	9.488×10^{-1}	8.544×10^{-1}	1.110	0.6957
0.7	1.473	9.169×10^{-1}	7.796×10^{-1}	1.176	0.7184
0.8	1.591	8.812×10^{-1}	7.067×10^{-1}	1.247	0.7412
0.9	1.707	8.442×10^{-1}	6.401×10^{-1}	1.319	0.7625
1.0	1.820	8.074×10^{-1}	5.813×10^{-1}	1.389	0.7817
2.0	2.798	5.403×10^{-1}	2.784×10^{-1}	1.940	0.8859
3.0	3.588	4.065×10^{-1}	1.765×10^{-1}	2.304	0.9216
4.0	4.275	3.297×10^{-1}	1.280×10^{-1}	2.577	0.9365
5.0	4.894	2.799×10^{-1}	1.000×10^{-1}	2.799	0.9431
6.0	5.464	2.450×10^{-1}	8.197×10^{-2}	2.988	0.9456
7.0	5.996	2.189×10^{-1}	6.938×10^{-2}	3.156	0.9461
8.0	6.498	1.987×10^{-1}	6.011×10^{-2}	3.306	0.9453
9.0	6.976	1.826×10^{-1}	5.301×10^{-2}	3.444	0.9439
10.0	7.432	1.693×10^{-1}	4.740×10^{-2}	3.572	0.9420
20.0	1.127×10^{1}	1.042×10^{-1}	2.298×10^{-2}	4.533	0.9185
30.0	1.438×10^{1}	7.912×10^{-2}	1.516×10^{-2}	5.220	0.8984
40.0	1.709×10^{1}	6.532×10^{-2}	1.130×10^{-2}	5.778	0.8822
50.0	1.954×10^{1}	5.639×10^{-2}	9.013×10^{-3}	6.257	0.8688
60.0	2.180×10^{1}	5.007×10^{-2}	7.494×10^{-3}	6.682	0.8574
70.0	2.391×10^{1}	4.531×10^{-2}	6.413×10^{-3}	7.066	0.8476
80.0	2.590×10^{1}	4.158×10^{-2}	5.604×10^{-3}	7.419	0.8390
90.0	2.780×10^{1}	3.855×10^{-2}	4.977×10^{-3}	7.746	0.8312
100.0	2.961×10^{1}	3.604×10^{-2}	4.476×10^{-3}	8.053	0.8243
200.0	4.489×10^{1}	2.327×10^{-2}	2.230×10^{-3}	10.43	0.7776
300.0	5.725×10^{1}	1.808×10^{-2}	1.485×10^{-3}	12.17	0.7502
400.0	6.804×10^{1}	1.513×10^{-2}	1.113×10^{-3}	13.59	0.7309
500.0	7.779×10^{1}	1.319×10^{-2}	8.901×10^{-4}	14.81	0.7161
600.0	8.678×10^{1}	1.179×10^{-2}	7.416×10^{-4}	15.90	0.7041
700.0	9.519×10^{1}	1.073×10^{-2}	6.356×10^{-4}	16.88	0.6940
800.0	1.031×10^{2}	9.885×10^{-3}	5.560×10^{-4}	17.78	0.6853
900.0	1.107×10^{2}	9.198×10^{-3}	4.942×10^{-4}	18.61	0.6777
1000.0	1.179×10^{2}	8.626×10^{-3}	4.448×10^{-4}	19.39	0.6710
2000.0	1.787×10^{2}	5.658×10^{-3}	2.223×10^{-4}	25.45	0.6278
3000.0	2.279×10^{2}	4.426×10^{-3}	1.482×10^{-4}	29.87	0.6036
4000.0	2.709×10^{2}	3.719×10^{-3}	1.111×10^{-4}	33.47	0.5869
5000.0	3.097×10^{2}	3.250×10^{-3}	8.890×10^{-5}	36.56	0.5742
6000.0	3.455×10^{2}	2.911×10^{-3}	7.408×10^{-5}	39.30	0.5640
7000.0	3.789×10^{2}	2.653×10^{-3}	6.350×10^{-5}	41.78	0.5555
8000.0	4.106×10^{2}	2.448×10^{-3}	5.556×10^{-5}	44.05	0.5483
9000.0	4.406×10^{2}	2.280×10^{-3}	4.939×10^{-5}	46.16	0.5419
10000.0	4.694×10^{2}	2.140×10^{-3}	4.445×10^{-5}	48.14	0.5363

$\gamma = 2$					
B	x	$1-\zeta_{AMLT}$	$1-\zeta_{MLT}$	$\frac{(\nabla-\nabla_{ad})_{AMLT}}{(\nabla-\nabla_{ad})_{MLT}}$	$\frac{w_{AMLT}}{w_{MLT}}$
0.1	1.003	1.000	1.000	1.000	0.4727
0.2	1.022	9.999×10^{-1}	9.993×10^{-1}	1.001	0.4712
0.3	1.069	9.988×10^{-1}	9.931×10^{-1}	1.006	0.4695
0.4	1.147	9.950×10^{-1}	9.700×10^{-1}	1.026	0.4724
0.5	1.247	9.866×10^{-1}	9.220×10^{-1}	1.070	0.4835
0.6	1.363	9.729×10^{-1}	8.544×10^{-1}	1.139	0.5016
0.7	1.486	9.547×10^{-1}	7.796×10^{-1}	1.225	0.5233
0.8	1.611	9.332×10^{-1}	7.067×10^{-1}	1.321	0.5456
0.9	1.737	9.097×10^{-1}	6.401×10^{-1}	1.421	0.5670
1.0	1.861	8.853×10^{-1}	5.813×10^{-1}	1.523	0.5868
2.0	2.965	6.725×10^{-1}	2.784×10^{-1}	2.415	0.7051
3.0	3.874	5.413×10^{-1}	1.765×10^{-1}	3.067	0.7520
4.0	4.666	4.574×10^{-1}	1.280×10^{-1}	3.575	0.7745
5.0	5.380	3.994×10^{-1}	1.000×10^{-1}	3.994	0.7865
6.0	6.037	3.568×10^{-1}	8.197×10^{-2}	4.353	0.7933
7.0	6.651	3.239×10^{-1}	6.938×10^{-2}	4.669	0.7970
8.0	7.230	2.978×10^{-1}	6.011×10^{-2}	4.954	0.7990
9.0	7.781	2.764×10^{-1}	5.301×10^{-2}	5.214	0.7998
10.0	8.307	2.585×10^{-1}	4.740×10^{-2}	5.454	0.7999
20.0	1.273×10^{1}	1.664×10^{-1}	2.298×10^{-2}	7.241	0.7878
30.0	1.630×10^{1}	1.288×10^{-1}	1.516×10^{-2}	8.497	0.7734
40.0	1.941×10^{1}	1.075×10^{-1}	1.130×10^{-2}	9.507	0.7610
50.0	2.223×10^{1}	9.345×10^{-2}	9.013×10^{-3}	10.37	0.7504
60.0	2.483×10^{1}	8.340×10^{-2}	7.494×10^{-3}	11.13	0.7413
70.0	2.725×10^{1}	7.577×10^{-2}	6.413×10^{-3}	11.82	0.7333
80.0	2.955×10^{1}	6.974×10^{-2}	5.604×10^{-3}	12.45	0.7262
90.0	3.172×10^{1}	6.484×10^{-2}	4.977×10^{-3}	13.03	0.7198
100.0	3.381×10^{1}	6.075×10^{-2}	4.476×10^{-3}	13.57	0.7140
200.0	$5.136D \times 10^{1}$	3.966×10^{-2}	2.230×10^{-3}	17.78	0.6749
300.0	6.556×10^{1}	3.095×10^{-2}	1.485×10^{-3}	20.84	0.6515
400.0	7.795×10^{1}	2.597×10^{-2}	1.113×10^{-3}	23.34	0.6350
500.0	8.915×10^{1}	2.268×10^{-2}	8.901×10^{-4}	25.48	0.6223
600.0	9.948×10^{1}	2.030×10^{-2}	7.416×10^{-4}	27.37	0.6120
700.0	1.091×10^{2}	1.849×10^{-2}	6.356×10^{-4}	29.09	0.6033
800.0	1.183×10^{2}	1.705×10^{-2}	5.560×10^{-4}	30.67	0.5958
900.0	1.269×10^{2}	1.588×10^{-2}	4.942×10^{-4}	32.13	0.5893
1000.0	1.352×10^{2}	1.490×10^{-2}	4.448×10^{-4}	33.49	0.5835
2000.0	2.051×10^{2}	9.799×10^{-3}	2.223×10^{-4}	44.08	0.5461
3000.0	2.616×10^{2}	7.674×10^{-3}	1.482×10^{-4}	51.79	0.5252
4000.0	3.109×10^{2}	6.453×10^{-3}	1.111×10^{-4}	58.06	0.5107
5000.0	3.555×10^{2}	5.641×10^{-3}	8.890×10^{-5}	63.46	0.4997
6000.0	3.966×10^{2}	5.055×10^{-3}	7.408×10^{-5}	68.23	0.4908
7000.0	4.351×10^{2}	4.607×10^{-3}	6.350×10^{-5}	72.56	0.4834
8000.0	4.714×10^{2}	4.252×10^{-3}	5.556×10^{-5}	76.52	0.4771
9000.0	5.059×10^{2}	3.961×10^{-3}	4.939×10^{-5}	80.20	0.4716
10000.0	5.390×10^{2}	3.718×10^{-3}	4.445×10^{-5}	83.64	0.4668

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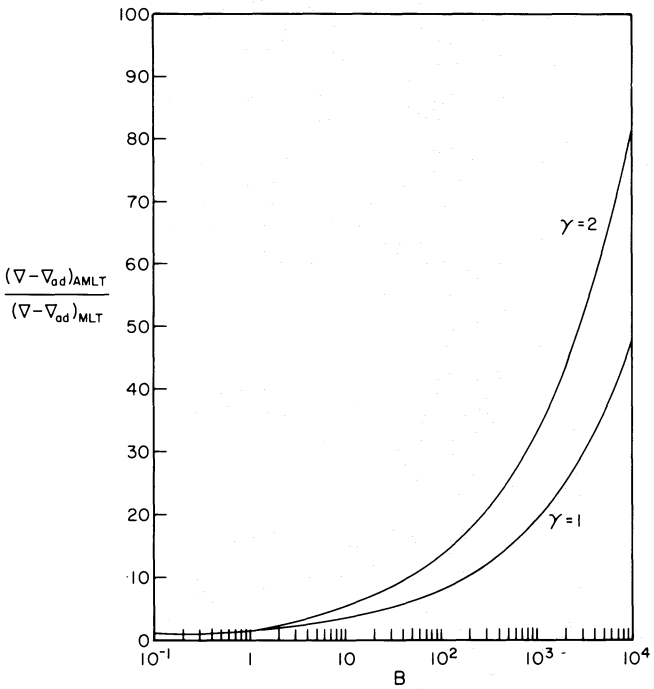

Fig. 2. Plot of $(\nabla - \nabla_{ad})_{AMLT}/(\nabla - \nabla_{ad})_{MLT}$, versus B for $\gamma = 1$ and 2. See text for details

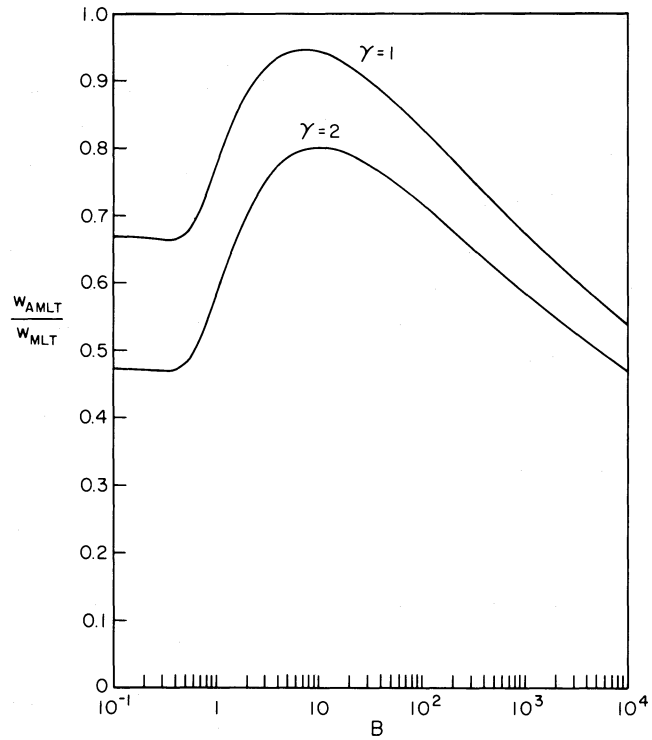

Fig. 3. Plot of w_{AMLT}/w_{MLT}, Eqs. (56)–(57), versus B for $\gamma = 1$ and 2. See text for details, especially the cautionary remark following Eq. (57)

two mixing lengths l, for the AMLT, and Λ used by CG68 for the MLT, are identical. That is why the parameters U are identical.)

Having established these results, it is useful to discuss some of the general features of the AMLT relations. First, in order to reduce the AMLT equations to the ones corresponding to the MLT, it is advisable to go back to the general formulation, i.e., Eq. (28), which is structurally identical to Kippenhahn's 1963 relation once the x dependence is neglected. At the level of Eqs. (49)–(52), and/or (54), one must be careful because a physical model has already been introduced to relate x to the other variables of the problem, i.e., Eq. (40), which has no direct physical analog in the MLT since it is based on a physical ingredient, the maximization of the growth rates, that does not exist in the standard framework.

As for the physical implications, let us note that from Eq. (51) it follows that

$$x \to 1, \qquad \text{small anisotropy}, \qquad \nabla \to \nabla_r$$

$$x \gg 1, \qquad \text{large anisotropy}, \nabla \to \nabla_{ad}, \tag{63}$$

which implies that *small anisotropies are equivalent to inefficient convection, while large anisotropies are equivalent to very efficient convection.*

From the physical point of view, these results are in accord with the picture of a turbulent medium being composed of large and small eddies. In a given turbulent fluid, most of the bulk properties (like heat fluxes) are contributed by a relatively small group of large and very anisotropic eddies whose spectral energy function depends strongly on both the specific nature of the stirring source (the instability), as well as on the geometry of the system. Large eddies do not become isotropic until they have undergone a significant number of non-linear interactions that induce successive break-ups leading to a subset of eddies, the so-called Kolmogoroff eddies, that are indeed small and isotropic. It is known that these eddies contain a small fraction of the total turbulent energy but a large fraction of the total vorticity; they are therefore particularly useful for the study of the dynamics of the non-linear interactions (which can be formulated in terms of vorticity), for any type of turbulent flow, since these eddies exist in any turbulent flow independently of the specific nature of the stirring mechanism. They are called universal because, being removed in wavenumber space from the largest eddies, they are no longer sensitive to the details of the source; they *only* sense the amount of energy that is being transferred to them by the largest eddies, but not the specific nature of it. On the contrary, the largest eddies are not universal, i.e., their spectrum depend sensitively on the specific nature of the source, they contain little vorticity, they are highly anisotropic, and they contribute to most of the bulk properties.

Given these fundamental differences, the two classes of eddies must be treated differently, i.e., the correlation time scale that characterizes the non-linear interactions cannot be the same. When the argument was studied quantitatively (Canuto et al., 1985), it was found that if one assumes that the time scale suggested by Kolmogoroff for the small scale eddies is valid also for the large eddies, the resulting bulk properties, for example, heat flux, are not in agreement with the convective formulae calibrated by Gough and Weiss (1976) using stellar structure models. The result implies that the large scale eddies must be treated differently than the Kolmogoroff eddies. One such treatment was proposed by Canuto and Goldman (1985) who were able to derive for the first time the MLT formulae from a model of turbulence, i.e., from a closure for the non-linear interactions.

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6. The case of large anisotropies

When $U \ll 1$, the corresponding value of x is also large, as one can surmise from Table 1 and Fig. 1. In this limit, the function $G(x)$ becomes, to the highest order, see Eq. (49),

$$G(x) = G_0 x^5, \qquad \gamma G_0 = 32/81, \tag{64}$$

and so, from Eqs. (54), we obtain

$$x = x_0 B^{3/5}, \qquad 1 - \zeta = \zeta_0 B^{-3/5}, \tag{65}$$

where $x_0 = (729\gamma/32)^{1/5}$ and $\zeta_0 = (32\gamma^4/729)^{1/5}$. Using (65) in (56), as well as the MLT relation (57), and noting that for large x, $\zeta_{\mathrm{AMLT}} \approx \zeta_{\mathrm{MLT}} \approx 1$ (see Table 2), we derive

$$\frac{w_{\mathrm{AMLT}}}{w_{\mathrm{MLT}}} = 1.35\gamma^{-1/5} B^{-1/10}, \tag{66}$$

i.e., for the largest value of B in Table 2 we have

$$\frac{w_{\mathrm{AMLT}}}{w_{\mathrm{MLT}}} = 0.54\gamma^{-1/5}, \tag{67}$$

which reproduces rather well the asymptotic results of Table 2.

If we use the *first representation*, Eqs. (49)–(51) then give

$$x = G_0^{-1/5}(\nabla_r - \nabla_{\mathrm{ad}})^{1/5} U^{-2/5}, \qquad \frac{\nabla_r - \nabla_{\mathrm{ad}}}{\nabla - \nabla_{\mathrm{ad}}} = \gamma^{-1} x. \tag{68}$$

Eliminating the variable x and taking logarithms, we obtain

$$\log(\nabla_r - \nabla_{\mathrm{ad}}) = -\tfrac{1}{2}\log U + \tfrac{5}{4}\log(\nabla - \nabla_{\mathrm{ad}}) + C, \tag{69}$$

where $4C = -\log(32\gamma^4/81)$. In the standard MLT, one can derive an analogous relation from Figs. (14.3) of Cox and Giuli and Table A1 of Baker and Temesvary. The result is, with η of order unity,

$$\log(\nabla_r - \nabla_{\mathrm{ad}}) = -\log U + \eta \log(\nabla - \nabla_{\mathrm{ad}}). \tag{70}$$

For the standard MLT, the slope of $\log(\nabla_r - \nabla_{\mathrm{ad}})$ is steeper; instead of $\frac{1}{2}$, the coefficient in front of $\log U$ is 1.

7. Solutions of the AMLT when the actual gradient is known

In the previous sections, we have solved the AMLT equations for the case in which ∇_r is specified. In this section, we shall consider the reverse case in which the actual gradient ∇ is specified at some point in the convective zone. Since ∇_{ad} is also known, we shall therefore calculate ∇_r. The needed relations have already been worked out in Appendix A. Making use of the relations between the various Λ's as given in Appendix B, we have the following results: From Eq. (A12)

$$\nabla - \nabla' = \frac{8U^2}{81\gamma^{1/2}2^{1/4}}(1+x)^2 x^{-1/2}[(1+\Sigma)^{1/2} - 1]^2, \tag{71}$$

$$\Sigma = (81/8)U^{-2}x(1+x)^{-3}(\nabla - \nabla_{\mathrm{ad}}), \tag{72}$$

which should be compared with the standard MLT expression given by Eq. (14.111) of CG68, where $2A = U^{-1}$. Substituting Eq. (71) into Eq. (22), we obtain

$$\nabla_r - \nabla = \frac{4}{81\gamma} U^2 (1+x)^2 [(1+\Sigma)^{1/2} - 1]^3, \tag{73}$$

to be compared with the MLT expression, Eq. (14.107) of CG68.

The convective flux F_c and the turbulent velocity w are given by Eqs. (A14) and (A15), i.e.,

$$F_c = \frac{4acT^4 g}{3\kappa p}\frac{4}{81\gamma} U^2 (1+x^2)[(1+\Sigma)^{1/2} - 1]^3, \tag{74}$$

which we have written in a way that facilitates the comparison with the standard expressions, Eq. (14.108) of CG68. For the velocity, we have

$$w/v_s = 2^{3/4}(9\gamma^{1/2}\Gamma^{1/2})^{-1} U (l/H_{\mathrm{p}}) x^{1/2}[(1+\Sigma)^{1/2} - 1], \tag{75}$$

which can be compared with CG68 Eq. (14.110), once the cautionary remarks following Eq. (62) have been taken into account.

The procedure is thus as follows: *given a value of* ∇ *and* ∇_{ad}, *Eq. (40) is solved for x which is then substituted in Eqs. (71)–(75) to obtain the desired results.*

While the form of the previous equations is useful for physical interpretation as well as comparison with the standard MLT expressions, for the purpose of their numerical solution an even simpler representation can be devised. Combining (40) with (72), yields

$$\Sigma = 4x(x-1), \tag{76}$$

and so F_c and w can be expressed entirely as functions of x, which is given by the solution of (40). The results are

$$F_c = \frac{4acT^4 g}{3\kappa p}\frac{32}{81\gamma} U^2 (x+1)^2 (x-1)^3, \tag{77}$$

$$w/v_s = 2^{7/4}(9\gamma^{1/2}\Gamma^{1/2})^{-1} U (l/H_{\mathrm{p}}) x^{1/2}(x-1). \tag{78}$$

To obtain the model, it is only necessary to solve Eq. (40) for x and substitute the result in Eqs. (77) and (78).

8. The case of large anisotropy

As in Sect. 6, we shall analyze here the case of large anisotropy since it allows a complete analytical solution of the problem. In that case, Eqs. (40) and (76) give

$$\Sigma = 4x^2$$

$$x = (81/32)^{1/4} U^{-1/2}(\nabla - \nabla_{\mathrm{ad}})^{1/4}, \tag{79}$$

and so Eqs. (71), (73)–(75) yield

$$\nabla - \nabla' = 2^{-1/4}\gamma^{-1/2}(\nabla - \nabla_{\mathrm{ad}}) x^{-1/2} \tag{80}$$

$$\nabla_r - \nabla = 9(2^{5/2}\gamma)^{-1} U^{-1}(\nabla - \nabla_{\mathrm{ad}})^{3/2} x^{-1} \tag{81}$$

$$F_c = (4\gamma)^{-1} c_{\mathrm{p}} \rho T g^{1/2} l^2 H_{\mathrm{p}}^{-3/2}(\nabla - \nabla_{\mathrm{ad}})^{3/2} x^{-1}, \tag{82}$$

$$w^2 = (8^{1/2}\gamma)^{-1} g l^2 H_{\mathrm{p}}^{-1}(\nabla - \nabla_{\mathrm{ad}}) x^{-1}. \tag{83}$$

The explicit x dependence makes the comparison with the MLT expressions, see for example Langer (1986), easy. In fact, since $x \gg 1$, it follows that

$$F_c(\mathrm{AMLT}) \ll F_c(\mathrm{MLT}),$$

$$w(\mathrm{AMLT}) \ll w(\mathrm{MLT}),$$

$$(\nabla - \nabla')_{\mathrm{AMLT}} \ll (\nabla - \nabla')_{\mathrm{MLT}},$$

$$(\nabla_r - \nabla)_{\mathrm{AMLT}} \ll (\nabla_r - \nabla)_{\mathrm{MLT}}. \tag{84}$$

9. Conclusions

In this paper, we have presented a mixing length formalism that incorporates an arbitrary degree of anisotropy in the eddies sizes.

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The new expressions for the convective velocities in the z-direction and the convective flux are given by Eqs. (14) and (16), together with relation (B28). The temperature excess $\nabla - \nabla'$ is given by Eq. (17) which differs *structurally* from the standard MLT expressions because of the presence of the functions $A(x)$ and $B(x)$.

A model to determine the degree of anisotropy x is proposed in Sect. 4 where the basic relation is Eq. (40), which expresses x as a function of the other physical variables of the system.

The new AMLT equations are solved for two cases of interest for stellar structure calculations.

In the first case, the gradients ∇ and $\nabla_{\rm ad}$ are given and the gradients ∇_r and ∇' are solved for. The AMLT relations are presented in two different representations, Eqs. (49)–(52) and (53)–(54). Numerical solutions are presented in Tables 1 and 2 and in Figs. 1, 2, and 3. The main physical results, see Figs. 2 and 3, can be expressed by saying that

$$\frac{(\nabla - \nabla_{\rm ad})_{\rm AMLT}}{(\nabla - \nabla_{\rm ad})_{\rm MLT}} \approx 10^2, \qquad \frac{w_{\rm AMLT}}{w_{\rm MLT}} \approx \frac{1}{2}, \tag{85}$$

i.e., that the MLT formalism *underestimates the adiabaticity* by a factor which can be as high as one hundred, while it *overestimates the convective velocities* by a factor of two. These results may lessen the problem of overshooting.

In the *second case*, the gradients ∇ and $\nabla_{\rm ad}$ are given and the gradients ∇_r and ∇' are solved for. The value of x is determined by solving Eq. (40). Equations (71) and (73) yield ∇' and ∇_r, while Eqs. (74) and (75) yield F_c and the convective velocity w. In the case of large anisotropies, or, equivalently, efficient convection, the model can be solved analytically and the results are given by Eqs. (79)–(83).

In this case, it is shown that the MLT framework *overestimates* F_c, w, $\nabla - \nabla'$ and $\nabla_n - \nabla$ in various degrees of the anisotropy x, as indicated by Eqs. (80)–(83).

Having presented the extension of the classical MLT model to include anisotropies in the large eddies spectrum, it would be interesting to carry out stellar structure calculations so as to quantify the astrophysical implications of this new model.

Appendix A

We shall employ the MLT formalism (Spiegel, 1963; Gough, 1977, 1978). Consider the Navier–Stokes equations for the velocity field. Eliminating the pressure terms, making use of the compressibility relation, and taking the z component of the resulting equation, the equation for w is

$$\frac{\partial w}{\partial t} = \frac{x}{1+x} g\alpha\theta, \tag{A1}$$

where x has been defined in Eq. (6) and α is the volume expansion coefficient. For a perfect gas, $\alpha = T^{-1}$. The fluctuation temperature θ is defined in terms of the total temperature $T'(x, y, z)$ and the average temperature $T(z)$ as

$$T'(x, y, z) = T(z) + \theta(x, y, z). \tag{A2}$$

Taking

$$\partial/\partial t = w\partial/\partial\Lambda_\xi, \tag{A3}$$

where Λ_ξ is the eddy displacement, integrating (A1), and assuming that the fluctuations at the point (x, y, z) are caused by the arrival of a convective element from the point (x, y, z_0), one can show that

$$\theta = \Lambda_z\left(\frac{dT'}{dz} - \frac{dT}{dz}\right), \qquad \Lambda_z \equiv z - z_0, \tag{A4}$$

where dT'/dz is the total temperature gradient as experienced by the convective element. We then obtain from (A1)

$$w = \left(\frac{2x}{1+x} g\alpha\Lambda_\xi\Lambda_z\right)^{1/2}\left(\frac{dT'}{dz} - \frac{dT}{dz}\right)^{1/2}, \tag{A5}$$

and from the definition of the convective flux

$$F_c = c_{\rm p}\rho\langle w\theta\rangle, \tag{A6}$$

we obtain

$$F_c = c_{\rm p}\rho w\Lambda_z\left(\frac{dT'}{dz} - \frac{dT}{dz}\right) = \tfrac{1}{2}c_{\rm p}\rho w l T H_{\rm p}^{-1}(\nabla - \nabla') \tag{A7}$$

(cf. Cox and Giuli, 1968, Eqs. 14.18).

Next, we must eliminate dT'/dz. This is done using the MLT equation for θ,

$$\frac{\partial\theta}{\partial t} - \chi\nabla^2\theta - \beta w = 0, \tag{A8}$$

where β is defined as

$$\beta \equiv \left(\frac{dT}{dz}\right)_{\rm ad} - \left(\frac{dT}{dz}\right), \tag{A9}$$

and where χ is defined in Eq. (12). From (A8) and (A4) we obtain the desired result,

$$\frac{dT'}{dz} - \frac{dT}{dz} = \left[\left(\frac{dT}{dz}\right)_{\rm ad} - \frac{dT'}{dz}\right]\frac{w\Lambda^2\Lambda_z^{-1}}{\chi + w\Lambda^2\Lambda_\xi^{-1}}, \tag{A10}$$

or alternatively,

$$\frac{dT'}{dz} - \frac{dT}{dz} = \left[\left(\frac{dT}{dz}\right)_{\rm ad} - \frac{dT'}{dz}\right]\frac{w\Lambda^2\Lambda_z^{-1}}{\chi + w\Lambda^2(\Lambda_\xi^{-1} - \Lambda_z^{-1})}. \tag{A11}$$

Substituting (A5) into (A10) and solving for dT'/dz yields

$$\frac{dT'}{dz} - \frac{dT}{dz} \equiv \frac{\Lambda_\xi}{\Lambda_z}\left[\left(\frac{dT}{dz}\right)_{\rm ad} - \frac{dT}{dz}\right]\Sigma^{-1}[(1+\Sigma)^{1/2} - 1]^{1/2}, \tag{A12}$$

with

$$\Sigma = \frac{8x}{1+x}\frac{g\alpha\beta\Lambda^4}{\chi^2}. \tag{A13}$$

Substituting (A12) into (A5) and (A6), we obtain

$$F_c = \tfrac{1}{2}c_{\rm p}\rho\beta\chi\left(\frac{\Lambda_\xi}{\Lambda}\right)^2\Sigma^{-1}[(1+\Sigma)^{1/2} - 1]^3, \tag{A14}$$

$$w = \tfrac{1}{2}\chi\frac{\Lambda_\xi}{\Lambda^2}[(1+\Sigma)^{1/2} - 1]. \tag{A15}$$

The above procedure does not determine the correct x dependence, a problem that is by no means trivial within MLT (see main text).

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Appendix B

The model for large scale turbulence to be used here has been described in detail elsewhere (Canuto and Goldman, 1985; in this Appendix we shall refer to this work as CG). Here, we will merely quote the relevant expressions.

The turbulent velocity v_t is defined as

$$v_t^2 = \int_{k_0}^{\infty} F(k)\,dk, \tag{B1}$$

where k_0 is the smallest wavenumber allowed by the system, see Eq. (B7), and where the turbulent energy spectral function $F(k)$ for the case of convection is given by Eq. (8) of the CG paper. The integration over all wavenumbers in (B1) can be carried out analytically. We have done so with the result

$$v_t = \pi\gamma^{-1/2} B_\lambda (1+x)^{1/2}[(1+bS)^{1/2} - 1]\chi d^{-1}. \tag{B2}$$

Here γ is a parameter that is shown to be of order unity, CG, Eq. (20). B_λ is a function of the parameter λ defined as

$$\lambda^2 = \frac{(1+x)^3}{x}\frac{\pi^4}{S}. \tag{B3}$$

The quantities S and b are defined as

$$S \equiv \frac{g\alpha\beta d^4}{\chi^2}, \tag{B4}$$

$$b \equiv \frac{4}{\pi^4}\frac{x}{(1+x)^3}. \tag{B5}$$

The quantity S is defined as the product of the Rayleigh number and the Prandtl number (ν/χ); for a perfect gas, $\alpha = 1/T$.

The quantity β is defined as (see Eq. A9)

$$\beta \equiv \left(\frac{dT}{dz}\right)_{\rm ad} - \left(\frac{dT}{dz}\right). \tag{B6}$$

The size of the convective zone is defined so that (see Eq. 37 of the text)

$$k_0 d = \pi(1+x)^{1/2}. \tag{B7}$$

The relation between v_t and its z-component w is

$$w = \left(\frac{x}{1+x}\right)^{1/2} v_t, \tag{B8}$$

so that, finally,

$$w = \pi\gamma^{-1/2} B_\lambda x^{1/2}[(1+bS)^{1/2} - 1]\chi d^{-1}. \tag{B9a}$$

Compare now this expression with Eq. (A5), i.e.,

$$w(\mathrm{MLT}) = \tfrac{1}{2}\left(\frac{\Lambda_\xi d}{\Lambda^2}\right)[(1+\Sigma)^{1/2} - 1]\chi d^{-1}. \tag{B9b}$$

First, for Σ, defined in Eq. (A13), and bS to be identical, we must have

$$2\pi^4(1+x)^2\Lambda^4 = d^4. \tag{B10}$$

Secondly, we must have

$$\Lambda_\xi \Lambda^{-2} d = 2\pi\gamma^{-1/2} B_\lambda x^{1/2}. \tag{B11}$$

Let us now consider the convective flux F_c. From Eq. (11) of the CG work, we have

$$F_c = c_p\rho\beta\chi\Phi = c_p\rho\beta\chi a S^{-1}[(1+bS)^{1/2} - 1]^3, \tag{B12}$$

where the parameter a is defined as

$$8\gamma a = \pi^4 A_\lambda^2 (1+x)^2. \tag{B13}$$

The function A_λ will be discussed momentarily. From the MLT treatment, we have also derived the following expression, Eq. (A14),

$$F_c(\mathrm{MLT}) = \tfrac{1}{2} c_p\rho\beta\chi\Lambda_\xi^2\Lambda^{-2}\Sigma^{-1}[(1+\Sigma)^{1/2} - 1]^3. \tag{B14}$$

Since (B10) assures us that $bS = \Sigma$, Eqs. (B12) and (B14) are identical if

$$\Lambda_\xi^2\Lambda^{-2} = 2ab = \frac{x}{1+x}\gamma^{-1}A_\lambda^2. \tag{B15}$$

Consider now relations (B10), (B11), and (B15). From the last two, we obtain

$$2\pi\Lambda_\xi = A_\lambda^2 B_\lambda^{-1}\gamma^{-1/2}x^{1/2}(1+x)^{-1}d, \tag{B16}$$

$$2\pi\Lambda = A_\lambda B_\lambda^{-1}(1+x)^{-1/2}d. \tag{B17}$$

When substituted (B10), they imply

$$A_\lambda = 2^{3/2}B_\lambda. \tag{B18}$$

Let us now recall the fact that the symbol Λ^2 actually stands for k^{-2} where k is the total wavenumber. Since we are dealing with a one wavenumber model and since in the CG turbulence model the smallest wavenumber k_0 corresponding to the largest eddy is related to the size of the convective zone by the relation (B7), we must take

$$\Lambda = k_0^{-1}. \tag{B19}$$

Before proceeding, we shall present a more quantitative derivation of (B19). Consider again the original MLT equations, (A1) and (A8). The time derivative can in principle be thought of as operating on w and θ which may be taken to vary in time as $\exp(nt)$ where n is the growth rate corresponding to a convective instability, see Eq. (7) of CG,

$$2n(k_0) = \chi k_0^2[(1+\lambda^{-2})^{1/2} - 1], \tag{B20}$$

with the parameter λ given by Eq. (B3). Since we have taken (A3), we must identify

$$w\Lambda_\xi^{-1} = n(k_0). \tag{B21}$$

Using (B9b) for w and the fact that $\Sigma = \lambda^{-2}$ because of (B10), Eq. (B21) can be satisfied only if (B19) is satisfied.

Let us now return to (B19). Using (B7) and (B17), we find

$$A_\lambda = 2B_\lambda, \tag{B22}$$

which, within the accuracy of the MLT one mode approximation, must be considered consistent with (B18). It is quite remarkable that the empirical relations (B18) and (B22) are actually satisfied by the explicit expressions for A_λ and B_λ computed with the CG turbulence model, i.e.,

$$A_\lambda^{-1} = 1 + (2\lambda)^{1/2}C_\lambda^{-1},$$
$$4\lambda^2\delta_\lambda^2 B_\lambda = [1 - (\tan^{-1}\delta_\lambda)/\delta_\lambda]^{1/2}, \tag{B23}$$

where

$$\delta_\lambda = (2\lambda)^{-1/2}C_\lambda^{-1}, \qquad C_\lambda^2 = \lambda + (1+\lambda^2)^{1/2}. \tag{B24}$$

One can easily check that for the case $\lambda \ll 1$, $A_\lambda = 1$, and $B_\lambda = 1/2$, in good agreement with the previous relations.

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We have related Λ and Λ_ξ to d, but nothing has yet been said about the functional relationship between Λ_z and d. Since this quantity does not enter in the MLT expressions for F_c and/or w, Eq. (B9b), it is not possible to relate it to d by comparing the MLT expressions with those derived from a model of turbulence. We shall therefore propose an alternative method. Much in the same way as Λ^2 is a symbol that was introduced to represent an average value of k^{-2}, in the same way one can make the argument that Λ_z is a symbol that stands for k_z^{-1} since it is derived from a Fourier expansion. We shall therefore take

$$\Lambda_z = k_z^{-1}. \tag{B25}$$

Because of (36), this implies that the relation between d and Λ_z is given by

$$d = \pi\Lambda_z, \tag{B26}$$

so that the mixing length $l = 2\Lambda_z$ is related to d by

$$\pi l = 2d. \tag{B27}$$

With this identification, and making use of (B18), Eqs. (B16) and (B17) become, with $A_\lambda = 1$,

$$\Lambda_\xi/\Lambda_z = 2^{-1/4}\gamma^{-1/2}\frac{x^{1/2}}{1+x}, \tag{B28}$$

$$\Lambda/\Lambda_z = 2^{-1/4}(1+x)^{-1/2}, \tag{B29}$$

$$(\Lambda_\xi/\Lambda)^2 = \gamma^{-1}\frac{x}{1+x}. \tag{B30}$$

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References

Baker, N.H., Temesvary, S.: 1966, *Tables of Convective Stellar Envelope Models*, NASA, Goddard Institute of Space Studies Report
Baker, N.H., Kuhfuss, R.: 1987, *Astron. Astrophys.* **185**, 117
Böhm-Vitense, E.: 1958, *Z. Astrophys.* **46**, 108
Canuto, V.M., Goldman, I.: 1985, *Phys. Rev. Lett.* **54**, 430
Canuto, V.M., Cabot, W., Hartke, G.J., Battaglia, A.: 1985, *Astrophys. Space. Sci.* **116**, 367
Cabot, W., Canuto, V.M., Hubickyj, O., Pollack, J.: 1987, *Icarus* **69**, 387
Cox, J.P., Giuli, R.T.: 1968, *Principles of Stellar Structure*, Gordon and Breach, New York
Gough, D.O., Weiss, N.O.: 1976, *Monthly Notices Roy. Astron. Soc.* **176**, 589
Gough, D.O.: 1977, *Astrophys. J.* **214**, 196
Gough, D.O.: 1978, in *Proc. of the Workshop on Stellar Rotation*, eds. G. Belvedere, L. Paterno, Oss. Astr. Catania, No. 162
Kippenhahn, R.: 1963, in *Star Evolution*, E. Fermi Summer School, Varenna, Italy, Course XXVIII
Langer, N.: 1987, *Astron. Astrophys.* **164**, 45
Maeder, A.: 1975, *Astron. Astrophys.* **40**, 303
Maeder, A., Mermilliod, J.C.: 1981, *Astron. Astrophys.* **93**, 136
Renzini, A.: 1987, *Astron. Astrophys.* **188**, 49
Spiegel, E.A.: 1963, *Astrophys. J.* **138**, 216

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