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Turbulent diffusivity

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Summary. A reliable quantification of many problems of astrophysical interest, e.g. mixing in the interior of the sun with implication for the solar neutrino problem, chemical evolution of the galaxy with cosmological consequences, grain sedimentation in the primitive solar nebula, etc., requires the knowledge of the turbulent diffusivity tensor D_{ij}. In this paper, we treat two problems.

First, given a passive scalar (e.g., a particle) embedded in an otherwise turbulent medium, and responding to it via frictional forces, we derive an expression for its turbulent velocity, Eq. (52), as well as for the components of D_{ij} for an axisymmetric case. These formulae are relevant to the problem of grain sedimentation in the primitive solar nebula.

Second, when the velocity of the contaminant coincides with that of the fluid, we show that the diffusivity tensor becomes diagonal, i.e., $D_{ij} = D_\mathrm{T}\delta_{ij}$ and we provide an expression for D_T, Eq. (65). This quantity is of interest, for example, to the astrophysical case of mixing in the interior of the sun.

In either case, one needs to know the turbulent energy spectral function $F(k)$, the eddy decorrelation rate $\sigma(k)$, and the growth rate $n_s(k)$ of the underlying instability. A model for them is also provided. We compute D_T in two cases: a constant $n_s(k) = n_0$ and the n_s corresponding to a convective instability with and without rotation. In both cases we find that, Eq. (83),

$$D_\mathrm{T} = \nu_\mathrm{t} Ko^3, \tag{a}$$

where $Ko = 1.5 \pm 20\%$ is the well-known Kolmogoroff constant of the theory of turbulence, and ν_t is the "eddy viscosity" defined in Eq. (82) and expressed in a variety of ways bý Eqs. (88) (for a full discussion of ν_t, see Canuto et al., 1987a, hereafter referred to as Paper I).

In addition to the general formula, Eq. (65), which involves the contribution of the whole spectrum of the turbulent eddies, we also derive a simplified formula for D_T, which yields a remarkably good approximation to the exact formula (see Sect. 7).

To test our results, we have compared our predicted values for D_T with recent results from full numerical simulation studies as well as with laboratory data on shear turbulence. The agreement is good.

In the case of astrophysics, we comment on the work of Schatzman and collaborators who, in the absence of a deterministic model for D_T, adopted an empirical relation of the type (ν is the kinematic viscosity)

$$D_\mathrm{T} = \nu\, Re^* \tag{b}$$

and fitted the empirical "effective" Reynolds number Re^* to astrophysical data. We show that the empirical Re^* is not to be identified with the "turbulence Reynolds number" R_T, but with the "microscale Reynolds number" R_λ. In fact, we compute R_λ and find it to be approximately 80, a value very close to the one empirically cetermined by Schatzman and Maeder in their study of the interior of the sun.

The application of our method to determine D_T from the knowledge of the underlying instability, together with the empirically determined value of D_T from Eq. (b) above, may help narrow down and hopefully single out, "the instability" that is most likely to operate. This would allow one to learn a great deal about the microscopic physical processes reflected macroscopically in D_T.

In summary, our method can be used in two ways: if the underlying instability generating turbulence is known, Eq. (65) allows D_T to be computed. If, on the other hand, one must consider more than one candidate for the instability, as it is probably the case in the astrophysical situations considered by Schatzman et al., our method may be used to narrow down the most likely candidate, thus revealing information about the underlying physics.

Finally, we consider the case of grains embedded in the turbulent gas characteristic of the primitive solar nebula. We compute the grain's turbulent velocity, Eq. (117), and the two components of the diffusivity tensor, D_n and D_p, Eqs. (60) and (61). The results are presented in Figs. (6–11) for a range of parameters characteristic of the primitive solar nebula (Cabot et al., 1987).

Key words: hydrodynamics – turbulence

1. Introduction

In many problems of astrophysical interest, turbulence is known to play an important role. Examples are turbulent heat transfer in stars (Cox and Giuli, 1968), turbulent transfer of momentum in accretion disks (Pringle, 1981; Colgate and Petschek, 1986; Cabot et al., 1987) and turbulent diffusion mixing in stars and in the galaxy (Schatzman, 1987).

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If one calls ν, χ and D the coefficients of molecular viscosity, thermal conductivity and molecular diffusivity, it can be shown that the presence of turbulence induces a "renormalization" of the type

$$\nu \to \nu + \nu_T, \quad \chi \to \chi + \chi_T, \quad D \to D + D_T \tag{1}$$

with

$$\nu_T \gg \nu, \quad \chi_T \gg \chi, \quad D_T \gg D, \tag{2}$$

where ν_T, χ_T, and D_T are the coefficients of turbulent viscosity, turbulent conductivity and turbulent diffusivity respectively.

The renormalization (1) comes about in the following way. Call $\boldsymbol{U}$ the fluid velocity and Ψ a passive scalar embedded in the fluid (e.g. temperature, particle concentration, etc.), whose presence does not affect the motion of the fluid. Using Reynolds decomposition, one first writes the velocity field as the sum of a mean quantity ($\langle U \rangle, \langle \Psi \rangle$) and a fluctuating component (u, ψ). The equations for the mean quantities are then as follows (see Hinze, 1959) (for ease of notation we shall temporarily omit the brackets on the mean quantities and use the convention $a_{i,j} \equiv (\partial/\partial x_j)\, a_i$)

$$\frac{\partial}{\partial t} U_i + U_j U_{i,j} + \langle u_i u_j \rangle_{,j} + \frac{1}{\varrho} P_{,i} = \nu\, U_{i,jj}, \tag{3a}$$

$$\frac{\partial}{\partial t} \Psi + U_j \Psi_{,j} + \langle \psi u_j \rangle_{,j} = D \Psi_{,jj}. \tag{3b}$$

As one can see, two new quantities have appeared in Eqs. (3), implying the correlations between fluctuating or turbulent quantities. Therefore, the solution of Eqs. (3) requires the knowledge of the turbulence equations also. If these correlations are thought to be caused by gradients in the mean quantities, one can write the Reynolds stress tensor $\tau_{ij} = -\langle u_i u_j \rangle$ and the flux $F_i = \langle \psi u_i \rangle$, as

$$\tau_{ij} = \nu_T \frac{\partial}{\partial x_j} U_i, \tag{4a}$$

$$F_i = -D_{ij} \frac{\partial}{\partial x_j} \Psi, \tag{4b}$$

where D_{ij} is the diffusivity tensor. In this case Eqs. (3) take the form

$$\frac{\partial}{\partial t} U_i + U_j U_{i,j} + \frac{1}{\varrho} P_{,i} = (\nu + \nu_T)\, U_{i,jj}, \tag{5a}$$

$$\frac{\partial}{\partial t} \Psi + U_j \Psi_{,j} = (D\delta_{ij} + D_{ij})\, \Psi_{,ij}, \tag{5b}$$

which justify Eq. (1) when $D_{ij} = D_T \delta_{ij}$. The next problem is the determination of the transport coefficients ν_T and D_{ij}.

For a long time, the lack of a reliable model to describe fully developed turbulence greatly hindered the determination of these transport coefficients. The problem was circumvented by the use of a variety of methods all of which are however beset by shortcomings. For example, in the treatment of accretion disks, ν_T has been traditionally written as (Pringle, 1981)

$$\nu_T = \alpha\, c_s H, \tag{6}$$

where c_s is the sound speed and H a typical scale height. The unknown coefficient α is then determined by fitting astrophysical data. The disk models constructed using Eq. (6) are known as α-models (see Pringle, 1981; Colgate and Petscheck, 1986; Cabot et al., 1987). In the absence of a calculation of τ_{ij}, it has been suggested that ν_T be identified with ν_t (Canuto et al., 1987a, Paper I).

As for χ_T, the traditional approach has been that of using the Mixing Length Theory (MLT) (Cox and Giuli, 1968). In Canuto et al. (1987a, Paper I), χ_T was computed and the MLT results found as a particular case. The computation of χ_T can easily be extended to include rotation and magnetic fields (Canuto and Hartke, 1986).

Finally, let us consider the turbulent diffusion coefficient, D_T. Schatzman (1987) has repeatedly stressed the importance of this quantity in the context of stellar evolution. Recently, he has shown that a reliable determination of D_T is instrumental to the determination of the abundance of ^{3}He in galaxies, a problem with important cosmological consequences.

In the present paper, we study D_T and derive an expression for it, Eq. (65), in terms of quantities that can be evaluated using the model of turbulence described in Paper I. Numerical values for D_T are presented for the case of convective turbulence with and without rotation. The results we obtain are also compared with numerical simulation studies as well as laboratory data. The astrophysical problems raised by Schatzman are also addressed.

2. General formalism

2.1. General formulae

Consider a turbulent fluid. Its motion can be described using two complementary points of view. One is the so called "Eulerian" point of view, in which the velocity of the fluid is specified (at a given time t) by a function $\boldsymbol{U}(\boldsymbol{x}, t)$ at each point $\boldsymbol{x}$ in a fixed frame of reference. The other is a "Lagrangian" point of view in which one specifies the position $\boldsymbol{X}(\boldsymbol{x}, t)$ of a fluid point which at time $t = 0$ is located at the space point $\boldsymbol{x}$.

While an Eulerian description of turbulence is more accessible both theoretically and experimentally, a Lagrangian approach is more natural for problems dealing with turbulent transport and, as we shall see in the following, of turbulence induced motion of small particles suspended in the fluid.

The relation between Lagrangian and Eulerian statistics is not straightforward (for a discussion see, Lumley, 1957, 1964), but, under certain, quite general conditions, some approximations (discussed in the following) are justified (Weinstock, 1976; Lundgreen and Pointin, 1976; Corrsin, 1959; Taylor and McNamara, 1971) that considerably simplify the problem of relating the two statistics.

Making use of the above mentioned approximation, we calculate the average velocity of small particles embedded in a turbulent fluid. The discussion is based on the works of Pismen and Nir (1978), Nir and Pismen (1979), and Lundgren and Pointin (1976). Different, but substantially equivalent approaches are described in Phytian (1975), Reeks (1977), and Roberts (1961).

Before we proceed, it is important to introduce some basic quantities. We begin by considering a passive contaminant, i.e. a scalar quantity, (such as a concentration of chemical species, a number of solid particles, a concentration of dye, etc.) that does not affect the motion of the fluid, and is transported along by it. We shall call $\boldsymbol{x}$ the space coordinate of the contaminant particle located at the point $\boldsymbol{x}$ at $t = 0$. If so, the quantity

$$\boldsymbol{Y}(\boldsymbol{x}, t) = \boldsymbol{X}(\boldsymbol{x}, t) - \boldsymbol{x} \tag{7}$$

represents the displacement of the contaminant from the initial point $\boldsymbol{x}$. The Lagrangian velocity is defined as

$$\boldsymbol{V}(\boldsymbol{x}, t) = \frac{\partial}{\partial t} \boldsymbol{X}(\boldsymbol{x}, t) = \boldsymbol{U}(\boldsymbol{X}(\boldsymbol{x}, t), t), \tag{8}$$

where $U(x, t)$ is the Eulerian velocity field. Now, since

$$X(x, 0) = x \tag{9}$$

we have, from (7) and (8) that

$$Y(x, t) = \int_0^t V(x, t)\, dt. \tag{10}$$

As customary, we will take all quantities as the sum of an averaged component plus a fluctuating part, whose ensemble average is zero; i.e.

$$Y(x, t) = \langle Y(x, t) \rangle + y(x, t), \tag{11}$$

$$V(x, t) = \langle V(x, t) \rangle + v(x, t) \tag{12}$$

with

$$\langle y(x, t) \rangle = \langle v(x, t) \rangle = 0. \tag{13}$$

Substituting (11) and (12) in (10), taking the average of the equation so obtained and subtracting the averaged equation from the unaveraged one, yields

$$y(x, t) = \int_0^t v(x, t')\, dt'. \tag{14}$$

Let us now define the tensor $\Lambda_{ij}(t)$ (Monin and Yaglom, 1975)

$$\Lambda_{ij}(t) = \langle y_i(x, t)\; y_j(x, t) \rangle \tag{15}$$

or, using (14)

$$\Lambda_{ij}(t) = \int_0^t dt_1 \int_0^t dt_2 \langle v_i(x, t_1)\; v_j(x, t_2) \rangle. \tag{16}$$

From the definition (15), we see that each diagonal element (no summation over i)

$$\Lambda_{ii}(t) = \langle y_i^2(x, t) \rangle \tag{17}$$

has the physical meaning of the particle's mean square displacement from its initial position along the i direction. Clearly, the sum of the diagonal terms Λ_{ii} represent the mean square total displacement. If we consider homogeneous isotropic stationary turbulence, the x dependence disappears, and time correlations, that is, correlations between variables at different times, depend only on the time separation.

Next, we define a Lagrangian velocity correlation tensor

$$B_{ij}(\tau) = \langle v_i(t)\; v_j(t+\tau) \rangle. \tag{18}$$

It can be shown that (Monin and Yaglom, 1975; Batchelor, 1949)

$$\frac{d\Lambda_{ij}(t)}{dt} = \int_0^t [B_{ij}(\tau) + B_{ji}(\tau)]\, d\tau \tag{19}$$

and that the contaminant's turbulent velocity v and the turbulent diffusivity tensor D_{ij} [introduced in Eq. (4b)] are given by (repeated indices are summed over)

$$\langle v^2 \rangle = B_{ii}(t=0) = \frac{1}{2}\left(\frac{d^2 \Lambda_{ii}(t)}{dt^2}\right)_{t=0}, \tag{20}$$

$$D_{ij} = \lim_{t\to\infty} \left(\frac{1}{2}\frac{d\Lambda_{ij}(t)}{dt}\right). \tag{21}$$

While relations (20) and (21) can be formulated for any passive contaminant in turbulent motion, in the following we shall concentrate on the special case in which the contaminant is a set of solid particles embedded in the fluid and responding through frictional forces to velocity fluctuations in the surrounding fluid.

The particle's equation of motion is (Volk et al., 1980)

$$\frac{d}{dt} V(x, t) = g(X(x, t)) + \eta\,(U(X(x, t), t) - V(x, t)), \tag{22}$$

where $V(x, t)$. defined in (8) is the particle's velocity, $U(x, t)$ is the fluid's Eulerian velocity field, $\eta \equiv \tau_f^{-1}$ is the inverse of the frictional response time of the particles, and g is the acceleration of gravity. For a detailed discussion of the possible forms of τ_f, see Weidenschilling (1977).

Note that in the limit $\eta \to \infty$, Eq. (22) yields

$$V(x, t) = U(X(x, t), t), \tag{23}$$

i.e. the contaminant's velocity coincides with the Lagrangian velocity of the fluid. Contrary to its seemingly innocuous form, Eq. (22) is actually a rather complicated non-linear integro-differential equation. In fact, the velocity $V(x, t)$ on the left hand side appears under an integral sign in the argument of both g and U since, according to (7) and (10)

$$X(x, t) = \int_0^t V(x, t)\, dt + x. \tag{24}$$

Substituting now the decomposition (11) into Eq. (22) and taking an ensemble average gives

$$\frac{d}{dt}\langle V(x, t) \rangle = g(X(x, t)) + \eta\,(\langle U(X(x, t), t) \rangle - \langle V(x, t) \rangle). \tag{25}$$

Subtracting (25) from (22), yields the equation for the particle's random, i.e. fluctuating, velocity

$$\frac{d}{dt} v(x, t) = \eta\,[u(X(x, t), t) - v(x, t)]. \tag{26}$$

For the specific goal of deriving the diffusion coefficient, we can consider the case in which the fluid has no mean motion, i.e. $\langle U \rangle = 0$, and the particles frictional response time τ_f is much shorter than the typical turbulence time scale. The latter approximation is well satisfied for example in a typical primitive solar nebula (Nakagawa et al., 1981). In such case, the particles can be considered as always moving with their terminal velocity, given by [using (25)]

$$\langle V \rangle \equiv v_0 = \frac{g}{\eta} \tag{27}$$

which implies

$$\langle Y \rangle = v_0 t. \tag{28}$$

Taking $v(0) = 0$, and using (28), we can rewrite Eq. (26) in integral form as

$$v_i(t) = \eta \int_0^t e^{-\eta(t-s)}\, u_i(v_0 s + y(s), s)\, ds, \tag{29}$$

where, as discussed above, we neglected the x dependence in the argument of v.

We can now calculate the tensor B_{ij} [defined in Eq. (18)]. Since B_{ij} is defined for steady state turbulence, we have

$$B_{ij}(s) = \lim_{t\to\infty} \langle v_i(t)\; v_j(t+s) \rangle. \tag{30}$$

Using Eq. (29), and calling G_{ij} the Eulerian velocity correlation along the Lagrangian path of the particle, i.e.

$$G_{ij}(t) = \langle u_i(0, 0)\; u_j(v_0 t + y(t), t) \rangle \tag{31}$$

we get (Pismen and Nir, 1978)

$$B_{ij}(s) = \frac{1}{2}\eta \int_{-\infty}^{\infty} e^{-\eta|s-s'|}\, G_{ij}(s')\, ds' . \tag{32}$$

If we now Fourier transform the turbulent velocity field

$$u_i(\boldsymbol{x}, t) = \int d\boldsymbol{k}\; u_i(\boldsymbol{k}, t)\, e^{-ikx} \tag{33}$$

and substitute in Eq. (31), we get

$$G_{ij}(t) = \iint d\boldsymbol{k}\, d\boldsymbol{k}'\, e^{-i\boldsymbol{k}'\boldsymbol{v}_0 t} \langle u_i(\boldsymbol{k}, 0)\, u_j(\boldsymbol{k}', t)\, e^{-i\boldsymbol{k}'\boldsymbol{y}(t)}\rangle . \tag{34}$$

In order to render Eq. (34) more manageable, we introduce the following approximation

$$\langle u_i(\boldsymbol{k}, 0)\, u_j(\boldsymbol{k}', t)\, e^{-i\boldsymbol{k}'\boldsymbol{y}(t)}\rangle \simeq \langle u_i(\boldsymbol{k}, 0)\, u_j(\boldsymbol{k}', t)\rangle \langle e^{-i\boldsymbol{k}'\boldsymbol{y}(t)}\rangle . \tag{35}$$

The conditions under which this approximation is justified are thoroughly discussed by Weinstock (1976). It is interesting to remark that the above approximation was also derived by Roberts (1961) in the frame of the "Direct Interaction Approximation" to turbulence (Leslie, 1973). Under the assumption that the random function $v_i(t)$ is jointly normal, we have (Lundgren and Pointin, 1976; Pismen and Nir, 1977; Lumley, 1970)

$$\langle e^{-i\boldsymbol{k}\boldsymbol{y}(t)}\rangle = e^{-(1/2) k_i \Lambda_{ij}(t) k_j} \tag{36}$$

with Λ_{ij} defined in Eq. (16).

Finally, for homogeneous turbulence, modes $\boldsymbol{k}$ and $\boldsymbol{k}'$ are uncorrelated unless $\boldsymbol{k} + \boldsymbol{k}' = 0$ (Batchelor, 1982; Monin and Yaglom, 1975), so that

$$\langle u_i(\boldsymbol{k}, 0)\, u_j(\boldsymbol{k}', t)\rangle = \delta(\boldsymbol{k} + \boldsymbol{k}')\, \Phi_{ij}(\boldsymbol{k}, t) \tag{37}$$

and Eq. (34) can be rewritten as

$$G_{ij}(t) = \int d\boldsymbol{k}\; e^{i\boldsymbol{k}\boldsymbol{v}_0 t}\, \Phi_{ij}(\boldsymbol{k}, t)\, e^{-(1/2) k_i \Lambda_{ij}(t) k_j} . \tag{38}$$

Substituting Eq. (38) back into Eq. (32), we obtain

$$B_{ij}(s) = \frac{1}{2}\eta \int_{-\infty}^{\infty} ds'\, e^{-\eta|s-s'|} \int d\boldsymbol{k}\, e^{i\boldsymbol{k}\boldsymbol{v}_0 s'}\, \Phi_{ij}(\boldsymbol{k}, s')\, e^{-\Lambda(\boldsymbol{k}, s')} , \tag{39}$$

where

$$\Lambda(\boldsymbol{k}, s') \equiv \tfrac{1}{2} k_i k_j \Lambda_{ij}(s') . \tag{40}$$

Once the tensor $B_{ij}(s)$ is known, Eqs. (19)–(21) yield the desired physical quantities.

2.2. *Solution of the equations*

In what follows, we shall take the fluid's turbulent field to be homogeneous and isotropic. For such field the function $\Phi_{ij}(\boldsymbol{k}, t)$ has the form (Batchelor, 1982)

$$\Phi_{ij}(\boldsymbol{k}, t) = \frac{E(k, t)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right), \tag{41}$$

where $k \equiv |\boldsymbol{k}|$ and δ_{ij} is the Kronecker delta symbol.

From (33), (37), and (41), we can see that

$$\langle u_i(\boldsymbol{x}, 0)\, u_i^*(\boldsymbol{x}, 0)\rangle \equiv \langle u^2\rangle = 2\int_0^{\infty} E(k, 0)\, dk \tag{42}$$

that is, $E(k, 0)$ represents the fluid's turbulent kinetic energy (per unit mass per unit wavenumber) associated with the mode k. In the following, we will use the function $F(k, t)$, defined as

$$F(k, t) = 2E(k, t) . \tag{43}$$

Since solid particles are subject to gravity, the tensor D_{ij} will be axially symmetric, with the axis of symmetry parallel to the direction of $\boldsymbol{g}$. We can therefore write (Buckingham, 1967)

$$\begin{aligned} \Lambda_{ij} &= \Lambda_{\rm n}\delta_{ij} + (\Lambda_{\rm p} - \Lambda_{\rm n})\, \lambda_i \lambda_j \\ B_{ij} &= B_{\rm n}\delta_{ij} + (B_{\rm p} - B_{\rm n})\, \lambda_i \lambda_j \\ D_{ij} &= D_{\rm n}\delta_{ij} + (D_{\rm p} - D_{\rm n})\, \lambda_i \lambda_j , \end{aligned} \tag{44}$$

where the subscripts "p" and "n" represent quantities respectively parallel and normal to the axis of symmetry and $\boldsymbol{\lambda}$ is a unit vector parallel to the same axis.

As explained in detail in Appendix A, we have constructed a set of two coupled integro-differential equation for $\Lambda_{ii}(t)$ and $\Lambda_{\rm p}(t)$ and solve them numerically by an iteration method. We found that starting with $\Lambda_{ii}^{(0)} = \Lambda_{\rm p}^{(0)} = 0$ as the initial guesses in the exponential Λ, the iterative procedure converges very rapidly, i.e. the results obtained after one or two iterations do not differ much from the fully iterated result. This allows an enormous simplification since by taking $\Lambda \to 0$ in Eq. (39) we can carry out the calculations analytically.

If we take $\boldsymbol{g}$ along the z-direction, i.e. $\lambda_{ij} = \delta_{i3}$, the only non-vanishing components of the tensor B_{ij} are

$$B_{xx}(s) = B_{yy}(s) = B_{\rm n} = \frac{1}{4}\eta \int_{-\infty}^{+\infty} ds'\, e^{-\eta|s-s'|} \int_0^{\infty} F(k, s')\, g(as')\, dk , \tag{45}$$

$$B_{zz}(s) = B_{\rm p} = \frac{1}{2}\eta \int_{-\infty}^{+\infty} ds'\, e^{-\eta|s-s'|} \int_0^{\infty} F(k, s')\, h(as')\, dk , \tag{46}$$

where

$$\begin{aligned} a &\equiv k v_0 \\ g(x) &= j_0(x) - h(x) \\ h(x) &= \frac{1}{x} j_1(x) . \end{aligned} \tag{47}$$

Here, $j_n(x)$ is the spherical Bessel function

$$j_n(x) \equiv (-)^n \frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right), \tag{48}$$

whose limit for $x \ll 1$ is

$$j_n(x) \cong \frac{x^n}{(2n+1)!!} . \tag{49}$$

Therefore in the limit $v_0 \to 0$ $(\eta \to \infty)$

$$g \to \tfrac{2}{3}, \quad h \to \tfrac{1}{3} \tag{50}$$

and

$$B_{\rm n} = B_{\rm p} , \tag{51}$$

i.e. the tensor B_{ij} is diagonal and so will be the turbulent diffusivity tensor D_{ij}.

From Eq. (20) we finally obtain

$$\langle v^2\rangle = \eta \int_0^{\infty} ds\, e^{-\eta s} \int_0^{\infty} F(k, s)\, \frac{\sin as}{as}\, dk \tag{52}$$

which provides the expression for the solid particle's turbulent velocity. Note that in the limit $\eta \to \infty$, Eq. (52) becomes

$$\langle v^2\rangle = \langle u^2\rangle \tag{53}$$

that is, the fluid's Lagrangian velocity equals its Eulerian velocity, a result obtained earlier by Lumley (1964).

Also, from Eq. (21) we obtain

$$D_{\mathrm{n}}=\frac{1}{4}\eta\int_0^\infty ds\int_{-\infty}^{+\infty} ds'\, e^{-\eta|s-s'|}\int_0^\infty F(k,s')\, dk\, g(as'), \tag{54}$$

$$D_{\mathrm{p}}=\frac{1}{2}\eta\int_0^\infty ds\int_{-\infty}^{+\infty} ds'\, e^{-\eta|s-s'|}\int_0^\infty F(k,s')\, dk\, h(as'). \tag{55}$$

Using the condition

$$F(k,t)=F(k,-t), \tag{56}$$

we can integrate Eqs. (54) and (55) to get

$$D_{\mathrm{n}}=\frac{1}{2}\int_0^\infty dk\int_0^\infty ds\, F(k,s)\, g(as) \tag{57}$$

$$D_{\mathrm{p}}=\int_0^\infty dk\int_0^\infty ds\, F(k,s)\, h(as). \tag{58}$$

At this point, we must specify the spectral function $F(k,s)$. As explained in Appendix B, the results of the numerical solution of the DIA model of turbulence (Kraichnan, 1964), can be parameterized in the form

$$F(k,t)=e^{-(1/2)\sigma^2(k)t^2}\, F(k) \tag{59}$$

where the function $\sigma(k)$ will be discussed in Sect. 3 below.

Inserting (59) into (57) and (58) and performing the integral over the time variable s, yields ($D_{ii}=2D_{\mathrm{n}}+D_{\mathrm{p}}$)

$$D_{ii}=\frac{\pi}{2}\int_0^\infty dk\,\frac{F(k)}{a}\,\Phi\left(\frac{a}{2^{1/2}\sigma(k)}\right), \tag{60}$$

$$3D_{\mathrm{p}}=\left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty dk\,\frac{F(k)}{\sigma(k)}\,{}_1F_1\left(\frac{1}{2},\frac{5}{2};-\frac{a^2}{2\sigma^2(k)}\right). \tag{61}$$

Here the error function $\Phi(x)$ and the confluent hypergeometric function ${}_1F_1$ are defined as (Abramowitz and Stegun, 1972)

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt. \tag{62}$$

$${}_1F_1\left(\frac{1}{2},\frac{5}{2};-x^2\right)=\frac{3}{4x^2}\left[e^{-x^2}-\frac{\pi^{1/2}}{2x}(1-2x^2)\,\Phi(x)\right]. \tag{63}$$

When the contaminant's velocity equals the fluid velocity, which, by Eq. (23) is attained by putting $\eta\to\infty$ [i.e. $v_0\to 0$ by Eq. (27)], the tensor D_{ij} has only diagonal elements, and thus

$$D_{\mathrm{T}}\equiv D_{\mathrm{n}}=D_{\mathrm{p}}=\tfrac{1}{3}D_{ii} \tag{64}$$

which, using Eq. (60), yields

$$D_{\mathrm{T}}=\frac{1}{3}\left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty \frac{F(k)}{\sigma(k)}\, dk, \tag{65}$$

which is our final formula for the fluid's turbulent diffusivity. Except for the factor $(\pi/2)^{1/2}$, originating from the gaussian from (59), Eq. (65) is of the same form as Eq. (8.69) of Leslie (1972), since $F(k)=2E(k)$. Our D_{T} is also equivalent to D_m of Eq. (2.1) in Moffat (1981).

3. How to compute D_{T}

Equation (65) for D_{T} requires the knowledge of two quantities that must be derived from a model for turbulence, i.e. $F(k)$ and $\sigma(k)$. Since a model has already been presented in Paper I, we shall quote here only the results.

The function $F(k)$ is obtained from

$$F(k)=\frac{1}{k^2}\frac{d}{dk}y(k), \tag{66}$$

where the function $y(k)$ (which physically represents the vorticity squared) is given by

$$y(k)=V(k)-\tfrac{1}{2}\gamma n_c^2(k). \tag{67}$$

The auxiliary function $V(k)$ is solution of the equation

$$\frac{dV(k)}{dk}=\frac{2\gamma}{k}\, n_c^2(k) \tag{68}$$

with

$$3\gamma\, n_c(k)\equiv n_s(k)+(n_s^2(k)+6\gamma\, V(k))^{1/2}, \tag{69}$$

where $n_s(k)$ is the growth rate of the underlying instability generating turbulence. Finally (see Appendix B)

$$\sigma(k)=2\gamma\, n_c(k)-n_s(k). \tag{70}$$

The procedure is therefore as follows: chosen an instability, the growth rate is known (e.g., from linear stability analysis). Equations (68) and (69) yield $V(k)$; $n_c(k)$ is derived from (69), $y(k)$ from (67) and $\sigma(k)$ from (70); $F(k)$ is then derived from (66). Although the equation for $V(k)$ is non-linear, its numerical solution presents no problems. The integral in (65) begins ostensibly at $k=0$; however, one should actually begin the integration where $n_s(k)$ is non-zero, i.e. when there is a growth function; such minimum value is, say, k_0. It can be shown that k_0 can be determined by the relation

$$\frac{d}{dk}\left(\frac{n_s(k)}{k^2}\right)_{k=k_0}=0 \tag{71}$$

that is, k_0 corresponds to the point where $n_s(k)/k^2$ has an extremum. At $k=k_0$, we also have

$$V(k_0)=\frac{1}{2\gamma}\, n_s^2(k_0) \tag{72}$$

which must therefore be considered the initial condition associated with Eq. (68).

Finally, the value of the constant γ is determined to be (see Paper I)

$$\gamma=\left(\frac{2}{3\,Ko}\right)^3, \tag{73}$$

where Ko is the Kolmogoroff constant whose values is (Monin and Yaglom, 1975)

$$Ko=1.5\pm 20\% \tag{74}$$

4. The growth rate $n_s(k)$

The specific form of $n_s(k)$ clearly depends on the specific problem at hand. A variety of forms of n_s can found in the book by Chandrasekhar (1960), where a large variety of flows is considered and the stability analysis carried out in detail in each case.

For differentially rotating stars, Goldreich and Schubert (1967) derived a general expression for $n_s(k)$, their Eqs. (32) or (39). One could therefore carry out a detailed analysis of the turbulent diffusion coefficient D_{T} for example in the convective region of the sun. We shall however not perform such an analysis here since the main purpose of this paper was to derive the general formalism to be used for the evaluation of D_T. We shall nevertheless work out two examples of interest.

5. First example: $n_s = n_0 =$ const

Using Eqs. (66)–(70) and the fact that

$$y(k) = \gamma n_c^2(k) - n_s(k)\, n_c(k) \tag{75}$$

one can in this case transform (65) to the form

$$D_T = \frac{1}{3}\left(\frac{\pi}{2}\right)^{1/2} \int_0^\infty \frac{1}{k^2}\, dn_c(k)\,. \tag{76}$$

Consider now Eq. (16) of Paper I. Divide by k^2, differentiate with respect to k and use the definition for $y(k)$; the result is

$$\frac{2y'}{n_c k^2} + y\left(\frac{1}{k^2 n_c}\right)' + (n_s/k^2)' = 0 \tag{77}$$

where the symbol $'$ represents differentiation with respect to k. Substitute now Eq. (75) above for $y(k)$; the result is

$$3\frac{n_c'}{n_c} - \frac{2}{k} - \frac{(n_c n_s)'}{\gamma n_c^2} = 0 \tag{78}$$

Equations (77) and (78) are still general. For a constant $n_s = n_0$, Eq. (78) can be integrated to yield

$$(k/k_0)^2 = \left(\frac{\gamma n_c}{n_0}\right)^3 e^{\frac{n_0}{\gamma n_c} - 1} \tag{79}$$

which can be substituted into Eq. (76). Carrying out the integration gives

$$D_T = \frac{1}{3}\left(\frac{\pi}{2}\right)^{1/2} \left(\frac{n_0}{k_0^2}\right) \frac{(e-2)}{\gamma} \tag{80}$$

or, using Eq. (73)

$$D_T = \left(\frac{n_0}{k_0^2}\right) Ko^3\,, \tag{81}$$

At this point we notice that in Paper I, it was shown that the ratio

$$\frac{n_s(k_0)}{k_0^2} = \nu_t\,, \tag{82}$$

is the "eddy viscosity", ν_t, i.e., the viscosity exerted by all the eddies on the largest ones. Using Eq. (82), we thus have

$$D_T = \nu_t Ko^3\,. \tag{83}$$

It may be of interest to note that Eq. (145) of Paper I, reproduced here as Eq. (88), expresses ν_t in terms of other quantities characteristic of a turbulent flow, e.g. the turbulent kinetic energy and the energy input rate, thus allowing D_T to be expressed in terms of the same quantities.

6. Second example: convection with and without rotation

In the case of a thermally driven convective layer of thickness L rotating at a rate Ω, the growth rate is given by the cubic equation (Cabot et al., 1987)

$$(n_s + \nu k^2)(n_s + \chi k^2) = \frac{g\alpha\beta x}{1+x}\left[1 - T\frac{Pr}{Rx}\frac{n_s + \chi k^2}{n_s + \nu k^2}\right], \tag{84}$$

where ν and χ are the kinematic viscosity and thermometric conductivity, $g \equiv g_z$ is the z-component of gravity, α the coefficient of thermal expansion, and β the temperature gradient excess over the adiabatic gradient. Furthermore, the Taylor number T, the Rayleigh number R and the Prandtl number Pr are defined by:

$T = 4L^4 \Omega_*^2 \nu^{-2}$, $\quad \Omega_*^2 \equiv \Omega^2 [1 + (r/2)\, d\Omega/dr]$,

$R = g\alpha\beta L^4/\chi\nu$, $Pr = \nu/\chi$.

The parameter x represents the degree of anisotropy of the eddies sizes: $x = (k_x^2 + k_y^2)/k_z^2$. A proper combination of the roots of Eq. (84), useful in numerical studies, has been worked out be Cabot et al. (1987). In the case of zero rotation, Eq. (84) reduces to Eq. (67) of Paper I, the well known form of n_s first derived by Rayleigh. As expected, rotation is seen to stabilize the problem. We have solved Eq. (84) for n_s and used the result in Eq. (69). The $F(k)$, $n_c(k)$ and $\sigma(k)$ so derived were then used to compute the integral in Eq. (65). The resulting D_T is presented in Fig. 1, where we plot D_T in units of $\chi(1+\sigma)/2$ vs. $\boldsymbol{S} \equiv R\,Pr$, a quantity frequently used to express the convective flux [see Eq. (86) of Paper I, which, once multiplied by $c_p \varrho \beta \chi$ coincides with Eq. (14.108) of Cox and Giuli (1968). In Cox and Giuli's notation, $\boldsymbol{S} \equiv 160\, A^2 (\nabla - \nabla_{ad})$, where the coefficient A is given by Eq. (14.99) of the same authors; $A^2(\nabla - \nabla_{ad})$ is called the convective efficiency.]

From recent work on rotating disks (Cabot et al., 1987), one deduces typical values of $\Omega^2 \approx 10\, g\alpha\beta$ and $\boldsymbol{S} \approx 10^6$–$10^7$. In Fig. 2, we show D_T in units of $n_s(k_0)/k_0^2 = \nu_t$, Eq. (82), vs. $\boldsymbol{S}$ for different values of the parameter $\Omega^2/g\alpha\beta$, which is just the inverse of the square of the Rossby number. We note that D_T is constant and equal to about $3\nu_t$, in agreement with Eq. (83).

7. A simplified expression for D_T

The fact that the previous cases have yielded the same result, namely the constancy of the ratio D_T/ν_t and that the constant is

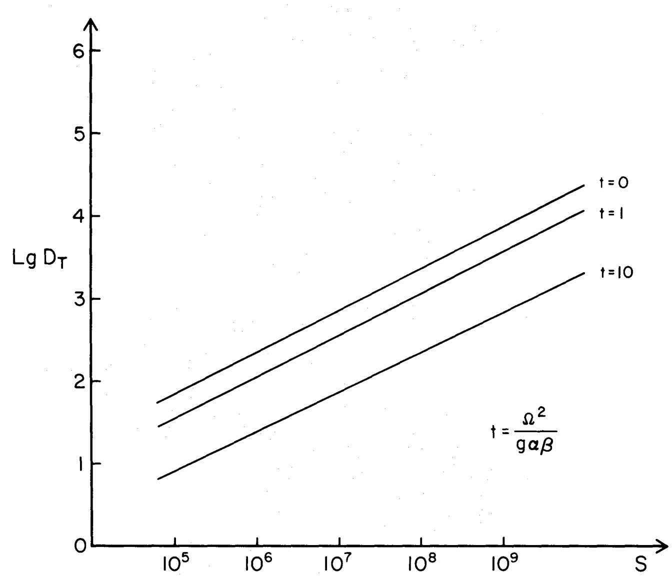

Fig. 1. Plot of the turbulent diffusivity D_T, solution of Eq. (65), in units of $(\chi+\nu)/2$, vs. $S = Pr\,R$ (see Sect. 6), for different values of the parameter $t = \Omega^2/g\alpha\beta$. The growth rate $n_s(k)$ is solution of Eq. (84). As expected, the effect of rotation is that of hindering turbulence and thus producing a lower D_T

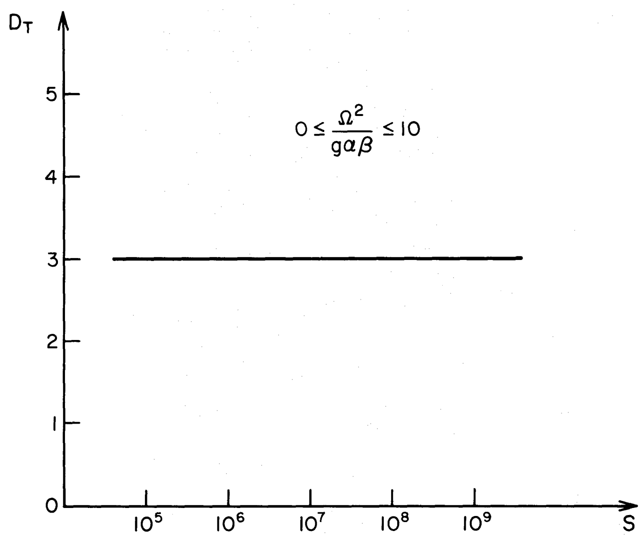

Fig. 2. Plot of the turbulent diffusivity D_{T}, solution of Eq. (65), for the same set of parameters as in Fig. 1, but normalized to the eddy viscosity ν_t defined by Eq. (82). As one can see, the ratio is constant and equal to approximately 3 [see Eq. (87)]

virtually identical suggests that one may try to get a simplified form of the general expression (65). Using simple algebraic manipulations, one can rewrite Eq. (65) in the following form

$$D_{\mathrm{T}} = \frac{1}{6\gamma}\left(\frac{\pi}{2}\right)^{1/2} \int_{k_0}^{\infty} \frac{F(k)}{n_c(k)} Z(k)\, dk$$

$$Z(k) = \left[1 - \frac{1}{2}\left(1 - \frac{y(k)}{n_s n_c}\right)\right]^{-1} \tag{85a}$$

a form that is particularly useful for our purposes. In fact, in the low k part of the spectrum, $k \approx k_0$, $y(k_0) = 0$ and so $Z(k) = 2$; at larger k, the universal Heisenberg-Kolmogoroff inertial region sets in, see Figs. 1 of Paper I. Since this region is characterized by the inequality $y(k) \gg |n_s(k)| n_c(\mathrm{k})$, it follows that there $Z(k) \approx 1$. We therefore conclude that $1 \leqq Z(k) \leqq 2$. As an example, we shall take $Z(k) = 3/2$. Equation (85a) then becomes

$$D_{\mathrm{T}} = \frac{1}{4\gamma}\left(\frac{\pi}{2}\right)^{1/2} \int_{k_0}^{\infty} \frac{F(k)}{n_c(k)}\, dk\,. \tag{85b}$$

At this point, we recall Eq. (2) of Paper I, i.e., the definition of the "eddy viscosity" at an arbitrary k

$$\nu_t(k) = \int_{k}^{\infty} \frac{F(k)}{n_c(k)}\, dk\,. \tag{86}$$

Using Eq. (73), we finally obtain from (85b)

$$D_{\mathrm{T}} = \nu_t Ko^3\,, \quad \nu_t = \nu_t(k_0) \tag{87}$$

which confirms Eq. (83) and the result of Fig. (2). Recalling Eq. (145) of Paper I, ($\xi_1 = 0.05$, $\xi_3 = 0.09$)

$$\nu_t = \frac{n_s(k_0)}{k_0^2}\,, \quad \nu_t = \xi_1 \varepsilon^{1/3} \Delta^{4/3}\,, \quad \nu_t = \xi_3 K^2 \varepsilon^{-1}\,, \tag{88}$$

where ε is the rate of energy input, K the turbulent kinetic energy and Δ the size of the largest eddy, we conclude that Eq. (87) allows us to express D_{T} in terms of other quantities characterizing turbulence.

8. Comparison with laboratory and numerical simulation data

Since both the constant $n_s(k)$ as well as the case of convective instability seem to indicate that ratio of D_{T} to the "eddy viscosity" ν_t is constant, it is important to assess the validity of this result on independent grounds.

Fortunately, both experimental data by Tavoularis and Corrsin (1981, 1985) as well as extensive numerical simulations by Rogers et al. (1986) have recently become available regarding the tensor D_{ij} in turbulent shear flows.

In the case of isotropic turbulence, Rogers et al. (1986) were able to parameterize their numerical results by the relation [see Eq. (5.4.1) of their work].

$$D_{ij} = C_2 \frac{q^4}{\varepsilon} \delta_{ij} \tag{89}$$

i.e.,

$$D_{\mathrm{T}} = C_2 \frac{q^4}{\varepsilon}\,. \tag{90}$$

Here, $q^2 = 2K$ is twice the turbulent kinetic energy, and ε ($\mathrm{erg\,g^{-1}\,s^{-1}}$) is the rate of energy input or, equivalently, the energy dissipation rate. Finally, C_2 is a numerical constant plotted in their figure (5.9b) and whose value is between 0.05 and 0.07. Equation (90) can thus be written as

$$D_{\mathrm{T}} = 4\,C_2\, K^2/\varepsilon\,. \tag{91}$$

Using the third expression for ν_t given in Eq. (88), we also have

$$D_{\mathrm{T}} = \frac{4\,C_2}{\xi_3}\, \nu_t \tag{92}$$

which is precisely of the form (83) derived earlier, provided that

$$\frac{4\,C_2}{\xi_3} = Ko^3 \tag{93}$$

which we see is indeed satisfied.

We may therefore conclude that our expression for D_{T}, Eq. (83), is not only in agreement with laboratory and simulation data, but it also expresses D_{T} in terms of ν_t for which there exist relation like Eq. (88) expressing it in terms of other quantities characteristic of turbulence and with perhaps a more direct physical interpretation.

9. Astrophysics: the work of Schatzman and Maeder

For many years the school of Schatzman (for a recent review, see Schatzman, 1987) has stressed the importance of turbulent diffusion in a large variety of astrophysical scenarios varying from stellar evolution to the chemical evolution of the Galaxy. In particular, Schatzman and Maeder (1981), Schatzman et al. (1981), and Schatzman (1987) have recently analyzed the implications of a putative D_{T} on stellar evolution and in particular on the neutrino problem and on the determination of the primordial ^{3}He abundance. They find that even a relatively moderate value of D_{T} may have major implications. Several comments are in order:

1) By their own admission, the above authors did not attempt to show that an instability with the right properties exists to generate a D_{T} of the desired strength. In the terminology of the present paper, the growth rate of the underlying instability, $n_s(k)$, is therefore not known.

2) Even if an $n_s(k)$ could be derived, i.e., a particular instability singled out as the dominant one (a choice that may be impossible to make a priori), the formalism needed to translate that information into a D_T, Eq. (65), was not available to these authors.

The situation confronted by Schatzman and collaborators is thus similar to the one confronted by the accretion disks modellers: not only they had to assume that turbulence exists but, lacking a formalism to translate an $n_s(k)$ into a ν_T, they had to adopt an empirical expression of the form (6) and then use astrophysical data to put limits on the parameter α.

Schatzman et al. (1981) proceeded in a similar manner. Following early suggestions by Schatzman (1969, 1977), D_T was written on dimensional grounds as

$$D_T = \nu\, Re^*\,, \tag{94}$$

where ν is the kinematic viscosity and Re^* is an "effective" Reynolds number, encompassing the effect of turbulence. The latest value of Re^* was found to be (Schatzman, 1987)

$$30 < Re^* < 40\,. \tag{95}$$

If we now combine (94) and (95) with Eq. (65), we obtain the condition

$$\frac{1}{3\nu}\left(\frac{\pi}{2}\right)^{1/2} \int_0^\infty \frac{F(k)}{\sigma(k)}\, dk = Re^*\,, \tag{96}$$

or, if we use the simplyfied form, Eq. (87)

$$\frac{\nu_t}{\nu} Ko^3 = Re^* \tag{97}$$

two constraints that may be useful in the following sense. Since it is likely that there is more than one type of physically plausible mechanisms leading to instabilities, one may have to consider a variety of $n_s(k)$. The implementation of the program to calculate D_T outlines in Sects. 3 or 7 above would thus yield a set of values of D_T. The constraint represented by (96) and/or (97) may help to isolate a small subset of the most probable $n_s(k)$, ideally one. The retrieval of "the" $n_s(k)$ is ultimately the quantity of real physical interest for through it, one can learn about the underlying physics. In that spirit, the empirical determination of Re^* represents a useful complement, i.e., a constraint, to our model to calculate D_T.

Given the variety of astrophysical phenomena affected by D_T, it seems that the "retrieval" program of $n_s(k)$ just outlined could lead to important information about the physics in the deep interior of the sun that would otherwise be unaccessible. [The retrieval of $n_s(k)$ from Eqs. (96) and/or (97) can be likened in spirit to the retrieval of say the nucleon-nucleon potential from the Schroedinger equation from the measured phase-shifts or/and cross sections.] An example of this retrieval process has been recently carried out to extract information about the nature of the physics underlying the turbulence observed in molecular clouds (Canuto and Battaglia, 1986).

While we believe that there is no substitute for a detailed analysis of D_T, using for example the $n_s(k)$ of Goldreich and Schubert (1967) or other forms of n_s if one wants to study the central region of the sun, it is nevertheless intriguing that the empirical value of Re^* is relatively low, Eq. (95). This makes one suspect that it may be related to the "turbulence Reynolds number" R_T or to the "microscale Reynolds number" R_λ (Kraichnan, 1964)

$$R_T = \frac{q^4}{\varepsilon\,\nu}, \quad R_\lambda = \frac{q\,\lambda}{\nu}, \tag{98}$$

where q^2 is twice the kinetic energy, $2K$ and λ is the Taylor microscale

$$\lambda = \left(5\,\nu\,\frac{q^2}{\varepsilon}\right)^{1/2} = \left(10\,\nu\,\frac{K}{\varepsilon}\right)^{1/2}. \tag{99}$$

Clearly

$$R_\lambda^2 = 5\,R_T, \quad R_\lambda^2 = 20\,\frac{K^2}{\varepsilon\,\nu}. \tag{100}$$

Let us analyze R_T. Using the third of Eqs. (88), we derive

$$R_T = \left(\frac{4}{\xi_3}\right)\frac{\nu_t}{\nu}. \tag{101}$$

Eliminating ν_t via Eq. (83) we have

$$D_T = \tfrac{1}{4}\,\xi_3\, Ko^3\, R_T\, \nu\,. \tag{102}$$

Consistency with Schatzman's Eq. (94) demands that Re^* be identified with

$$Re^* = \tfrac{1}{4}\,\xi_3\, Ko^3\, R_T\,. \tag{103}$$

Because of the values of ξ_3 $(=0.09)$ and of Ko $(=1.5)$.

$$Re^* = \tfrac{1}{13}\, R_T\,. \tag{104}$$

and thus Re^* cannot be identified with R_T.

Next, consider R_λ. Using the first of Eqs. (100) in (104), we obtain

$$Re^* = \tfrac{1}{65}\, R_\lambda^2 \tag{105}$$

Clearly, Eqs. (104) and (105) do not yield the value of Re^*: they only relate Re^* to two other quantities commonly used in the theory of turbulence, i.e. a model of turbulence is still needed to compute R_T (or R_λ). We may however note that *if* we adopt the "experimental" value of Re^*, Eq. (95), relation (105) yields

$$Re^* \approx R_\lambda \tag{106}$$

thus allowing us to conclude that the phenomenological Re^* introduced by Schatzman et al. is rather closely related to the "microscale Reynolds number". Other relations of interest may also be derived. Using Eq. (88) and the combination of (83) with (94), i.e.

$$\nu_t = \nu\,\frac{Re^*}{Ko^3} \tag{107}$$

we obtain:

a) Typical turbulent velocity $u\,(K = \tfrac{1}{2}u^2)$

$$u = \left(\frac{4\,Re^*}{\xi_3\, Ko^3}\right)^{1/4} (\nu\,\varepsilon)^{1/4}. \tag{108}$$

b) Typical size Δ of a large eddy

$$\Delta = \left(\frac{Re^*}{\xi_1\, Ko^3}\right)^{3/4} \nu^{3/4}\,\varepsilon^{-1/4}. \tag{109}$$

c) Typical size l and lifetime t_l of a small eddy (Kolmogoroff eddy)

$$l = \nu^{3/4}\,\varepsilon^{-1/4}, \quad t_l = \left(\frac{\nu}{\varepsilon}\right)^{1/2} \tag{110}$$

d) Typical time scale for the instability $t_s = n_s(k_0)^{-1}$,

$$t_s = \left(\frac{4\,Re^*}{9\pi^4\,\xi_1^3\, Ko^3}\right)^{1/2} \left(\frac{\nu}{\varepsilon}\right)^{1/2}. \tag{111}$$

e) Reynolds number Re for the large scale eddies

$$Re = \frac{u\Delta}{\nu} = \left(\frac{4}{\xi_3 \xi_1^3}\right)^{1/4} \frac{Re^*}{Ko^3}. \tag{112}$$

As an example, let us consider the physical conditions characteristic of the interior of the sun,

$$T = 1.5\ 10^7\,\mathrm{K}, \quad \varrho = 150\,\mathrm{g\,cm^{-3}}, \quad \nu = 2.2\ 10^{-3}\,\mathrm{cm^2\,s^{-1}}. \tag{113}$$

Using $\xi_1 = 0.05$, $\xi_3 = 0.09$, $Ko = 1.5$ and $Re^* = 40$, we derive from Eqs. (108)–(112):

$$\begin{aligned} u &= 1.04\,\varepsilon^{1/4} \\ \Delta &= 0.61\,\varepsilon^{-1/4} \\ l &= \Delta/130 \\ t_s &= 0.97\,\varepsilon^{-1/2} \\ t_l &= 5\ 10^{-2}\,\varepsilon^{-1/2} \\ Re &= 7.23\,Re^* \end{aligned} \tag{114}$$

where u is in $\mathrm{cm\,s^{-1}}$, Δ and l in cm, t_s and t_l in seconds and ε in $\mathrm{erg\,g^{-1}\,s^{-1}}$. As discussed earlier, ε represents the energy pumped into the system to generate turbulence or, equivalently, the energy dissipated by visous forces at the smallest scales, i.e., (see Paper I),

$$\varepsilon = \int_{k_0}^{\infty} F(k)\,[n_s(k) + \nu k^2]\,dk = \nu \int_{k_0}^{\infty} F(k)\,k^2\,dk. \tag{115}$$

Both expressions require the knowledge of the turbulence spectral function $F(k)$ unless, of course, one can make a reliable estimate of ε based on other physical arguments.

10. Primitive solar nebula: the grains

One of the major problems in trying to construct a reliable scenario for the formation of planets concerns the sedimentation process of the (heavy) grains embedded in the nebula's turbulent gas (of solar composition). Weidenschilling (1984) has recently shown that a particular model of turbulence would not allow the sedimentation process to proceed past the first stage because even ~km size "protoplanetesimals" would collide so disruptively as to keep the nebula consistently opaque, and therefore turbulent. Since this difficulty can now be avoided with an improved model of turbulence (Cabot et al., 1987), that predicts lower turbulent velocities, the grain sedimentation process must be reinvestigated. In such an analysis, two important ingredients are the grain's turbulent velocity, Eq. (52), and the diffusivity tensor, Eqs. (60) and (61). Both quantities depend on the turbulence spectral function $F(k)$ of the gas (see Sect. 3), as well as on the quantities

$$a \equiv k\,v_0, \quad \eta = \tau_{\mathrm{f}}^{-1}, \tag{116}$$

where v_0 is the grain's terminal velocity and τ_{f} the grain's frictional response time, which can be expressed in terms of other physical properties of the grain, e.g., mass and size (Wiedenschilling, 1977).

The grain's turbulent velocity v and the gas turbulent velocity u are given by

$$\begin{aligned} \langle v^2 \rangle &= \int_0^{\infty} F(k)\,I(k,\eta)\,dk \\ \langle u^2 \rangle &= \int_0^{\infty} F(k)\,dk, \end{aligned} \tag{117}$$

where

$$I(k,\eta) = \eta \int_0^{\infty} e^{-\eta s - \frac{1}{2}\sigma^2(k)s^2}\,\frac{\sin(as)}{as}\,ds \tag{118}$$

with $I(k,\infty) = 1$. While the integral in Eq. (118) can in principle be performed analytically, the result is given in terms of parabolic cylinder functions of complex arguments, which are not easy to handle numerically. Rather, we transform (118) in such a way as to express it in a numerically more convenient way. Some algebraic manipulations lead to

$$I(k,\eta) = \frac{\eta\,\pi^{1/2}}{a} \int_0^{t(k)} [1 - \Phi(z)]\,e^{-x^2}\,dx, \tag{119}$$

where

$$t(k) = \frac{a}{2^{1/2}\sigma(k)}, \quad z = \frac{\eta\,x}{(a^2 - 2x^2\sigma^2(k))^{1/2}}. \tag{120}$$

We have computed Eqs. (117) using the $F(k)$ that corresponds to a convective instability, Eq. (84), with and without rotation. The results are presented in Figs. 6–9, where one sees that for zero response time, $\tau_{\mathrm{f}} \to 0$, the grain's turbulent velocity coincides with that of the gas, while it goes to zero for $\tau_{\mathrm{f}} \to \infty$, which physically reflects the fact that very massive grains are unaffected by the turbulent medium.

The two normal and parallel components of the diffusivity tensor, D_{n} and D_{p}, Eqs. (60) and (61), were also computed for the same type of instability. The results are presented in Figs. 10–12 for the case of zero rotation and for the case $\Omega^2/g\,\alpha\beta = 10$. As one can see, the tensor D_{ij} becomes diagonal when v_0 goes to zero.

11. Conclusions

To describe the behavior of a passive scalar embedded in a turbulent fluid, one needs to know the turbulent diffusivity tensor D_{ij}. We have derived expressions for D_{ij} in two cases: a) the contaminant's velocity coincides with that of the fluid, and b) the contaminant velocity does not coincide with that of the fluid.

Case a) In this case $D_{ij} = D_{\mathrm{T}}\delta_{ij}$, where D_{T} is called the coefficient of turbulent diffusivity. We have derived a general expression for D_{T}, Eq. (65), in terms of the turbulence energy spectral function $F(k)$ and of the eddy decorrelation rate $\sigma(k)$. Together with the model of turbulence described in Paper I, and summarized here in Sect. 3, Eq. (65) allows the explicit calculation of D_{T} once the growth rate of the instability generating turbulence has been identified.

We also derived a simplified expression for D_{T} and express the result in terms of the "eddy viscosity" ν_{t}, Eq. (87). One of the advantages of Eq. (87) is that it allows D_{T} to be expressed, using Eq. (88), in terms of other quantities characteristic of a turbulent medium.

Finally, using the empirical determination of D_{T} by Schatzman and Maeder (1981), we show that our formalism to calculate D_{T} provides a method to gain useful insight on the mechanism (instabilities) that generate turbulence for example in the interior of the sun. The results for this case are shown in Figs. 1 and 2.

Case b) In this case the grain's turbulent velocity v is given by Eq. (117), and the diffusivity tensor by Eqs. (60) and (61). We have computed $\langle v^2 \rangle$, D_{n} and D_{p} for the case of a convective instability which is believed to have been one of the likely sources of turbulence in the primitive solar nebula. The result for this case, for a wide range of parameters characteristic of the primitive solar nebula (see Cabot et al., 1987), are presented in Figs. 6–11. With the help of these results, one can now carry out a detailed calculation of the coagulation and sedimentation of grains in the primitive solar nebula.

Appendix A: numerical solution of the equations for Λ_{ii} and Λ_p

Using Eq. (19), the fact that $B_{ij}(t) = B_{ji}(t)$, and that the imaginary part of the right hand side of Eq. (39) integrates to zero, we can rewrite Eq. (39) as

$$\frac{d^2}{dt^2}\Lambda_{ij}(t) = \eta \int_{-\infty}^{\infty} dt'\, e^{-\eta|t-t'|} \cdot \int d\boldsymbol{k} \cos(\boldsymbol{k}\, \boldsymbol{v}_0\, t')\, \Phi_{ij}(\boldsymbol{k}, t')\, e^{-(1/2)k_i \Lambda_{ij}(t) k_j}. \quad \text{(A1)}$$

The initial conditions associated with (A1) are

$$\Lambda_{ij}(0) = 0\,, \quad \text{(A2)}$$

$$\frac{d}{dt}\Lambda_{ij}(0) = 0 \quad \text{(A3)}$$

which can be derived from (16) and (19).

In general, (A1) represents a system of 9 coupled integro-differential equations; by use of Eq. (41) and the first of (44), we can reduce this system to the following set of two equations for the two independent quantities Λ_{ii} and Λ_p

$$\frac{d^2}{dt^2}\Lambda_{ii}(t) = \eta \int_{-\infty}^{\infty} e^{-\eta|t-t'|} dt' \int_0^{\infty} F(k, t')\, dk \int_0^1 \cos(v_0 k t' x) \quad \text{(A4)}$$
$$\cdot \exp\{-(1/4) k^2 [(1-x^2)\Lambda_{ii}(t') - (1-3x^2)\Lambda_p(t')]\}\, dx\,,$$

$$\frac{d^2}{dt^2}\Lambda_p(t) = \frac{\eta}{2} \int_{-\infty}^{\infty} e^{-\eta|t-t'|} dt' \int_0^{\infty} F(k, t')\, dk \int_0^1 (1-x^2) \cos(v_0 k t' x)$$
$$\cdot \exp\{-(1/4) k^2 [(1-x^2)\Lambda_{ii}(t) - (1-3x^2)\Lambda_p(t)]\}\, dx\,, \quad \text{(A5)}$$

where $x \equiv k_z/k$.

It is convenient to put Eqs. (A4) and (A5) in non-dimensional form. We introduce the following variables

$$\begin{aligned} q &= kL \\ f(q, t) &= F(k, t)/F_0 \\ y(\tilde{t}) &= \Lambda_{ii}(t)/L^2 \\ z(\tilde{t}) &= \Lambda_p(t)/L^2 \\ \tilde{t} &= u\, t/L \\ \lambda &= \eta\, L/u \\ \beta &= u/v_0\,, \end{aligned} \quad \text{(A6)}$$

where L was defined in Eq. (84) and we have called

$$u = (\langle u^2 \rangle)^{1/2} \quad \text{(A7)}$$

and

$$F_0 \equiv u^2 L\,. \quad \text{(A8)}$$

Equations (A4) and (A5) can then be rewritten as

$$\frac{d^2}{dt^2} y(t) = \lambda \int_{-\infty}^{\infty} e^{-\lambda|t-t'|} dt' \int_0^{\infty} f(q, t')\, dq \int_0^1 \cos\left(\frac{q t' x}{\beta}\right)$$
$$\cdot \exp\{-(1/4) q^2 [(1-x^2)\, y(t') - (1-3x^2)\, z(t')]\}\, dx\,, \quad \text{(A9)}$$

$$\frac{d^2}{dt^2} z(t) = \frac{\lambda}{2} \int_{-\infty}^{\infty} e^{-\lambda|t-t'|} dt' \int_0^{\infty} f(q, t')\, dq \int_0^1 (1-x^2) \cos\left(\frac{q t' x}{\beta}\right)$$
$$\cdot \exp\{-(1/4) q^2 [(1-x^2)\, y(t') - (1-3x^2)\, z(t')]\}\, dx\,, \quad \text{(A10)}$$

where, to simplify the notation, we omitted the tilde in the non-dimensional time variables.

Equations (A9) and (A10) contain two independent parameters, λ and β. As can be seen from (A6), λ is equal to the ratio of the time scale of the largest turbulent eddies to the frictional timescale, while β is the ratio of sedimentation velocity to turbulent velocity.

Assuming the function $F(k, t)$ to be known, Eqs. (A9) and (A10) can be solved iteratively, i.e. by guessing an expression for $\Lambda_{ii}(t)$ and $\Lambda_p(t)$, substituting it in the right hand side, integrating in time twice, and substituting the new expression for $\Lambda_{ii}(t)$ and $\Lambda_p(t)$ in the right hand side again, until convergence is achieved.

We found that, using $\Lambda_{ii} = \Lambda_p = 0$ as initial starting values for the iteration, the iterative procedure converges very rapidly, the result obtained after one or two iterations does not differ much from the fully iterated result.

Pithian (1975) and Reeks (1977) used this property of Eqs. (A9) and (A10) to justify stopping the iteration scheme after two iterations, and provided expressions for $\Lambda_{ij}(t)$ corresponding to this case. Such expressions still involve complicate integrations over the time variable, and are not very easy to use.

In order to simplify the calculation of $\Lambda_{ij}(t)$, we propose to stop the iteration scheme after just one iteration, that is to use for $\Lambda_{ii}(t)$ and $\Lambda_p(t)$ the expressions obtained by substituting $\Lambda_{ij}(t) = 0$ on the right hand side of Eqs. (A9) and (A10), which allows us to express the grain's velocity and coefficient of diffusion in an analytical form. Comparison of the solution so obtained with the fully iterated solution shows an agreement within a few percent for the quantities of interest to the present work, namely (Eqs. (20 and (21) of the main text)

$$\langle v^2 \rangle = \frac{1}{2}\left(\frac{d^2}{dt^2}\Lambda_{ii}(t)\right)_{t=0}, \quad \text{(A12)}$$

$$D_{ii} = \lim_{t\to\infty}\left(\frac{1}{2}\frac{d}{dt}\Lambda_{ii}(t)\right). \quad \text{(A13)}$$

In Table 1 we show the results for $\Lambda''(0)$ and $\Lambda'(\infty)$, where $\Lambda \equiv \Lambda_{ii}$ and the prime indicates derivative with respect to time, for the trial function

$$f(q, t) = \frac{3}{8(\pi)^{1/2}}\, q^4 e^{-q^2} e^{-q^2 t^2}\,. \quad \text{(A14)}$$

Note that in all cases the iteration scheme converged to 10^{-3} accuracy within 3 iterations ($' = d/dt$; $'' = d^2/dt^2$).

Appendix B: derivation of Eq. (59)

Kraichnan's Direct Interaction Approximation (DIA, Leslie, 1973) has been used for many years as a theoretical tool to study

Table 1

$(\lambda; \beta)$	Iteration 1	Iteration 2	Iteration 3
(4; 11.5)	$\Lambda''(0) = 1.66$	1.64	1.64
	$\Lambda'(\infty) = 1.33$	1.27	1.27
(1.5; 2.33)	$\Lambda''(0) = 1.11$	1.09	1.09
	$\Lambda'(\infty) = 1.31$	1.27	1.27
(0.25; 1)	$\Lambda''(0) = 0.253$	0.249	0.25
	$\Lambda'(\infty) = 1.23$	1.22	1.22

unforced turbulence. In order to include forcing, we first extended the DIA equations in the following way:

$$\left(\frac{d}{dt} - n_s(k)\right) Q(k, t-s)$$
$$= 2\pi \iint^{\Delta} dp\, dr\, kp\, rb(k,p,r) \left[\int_{-\infty}^{s} ds'\, G(k, s-s') \cdot Q(r, t-s') - \int_{-\infty}^{t} ds'\, G(p, s-s')\, Q(r, t-s')\, Q(k, s-s') \right], \tag{B1}$$

$$\left(\frac{d}{dt} - n_s(k)\right) G(k, t-s)$$
$$= -2\pi \iint^{\Delta} dp\, dr\, kprb(k,p,r) \left[\int_{s}^{t} ds'\, G(p, t-s') \cdot Q(r, t-s')\, G(k, s-s') + \delta(t-s) \right], \tag{B2}$$

where the function $E(k)$ of Eq. (43) of the text is related to $Q(k, 0)$ by the relation

$$\frac{1}{2}\langle u^2 \rangle = 4\pi \int_0^\infty k^2\, Q(k,0)\, dk \equiv \int_0^\infty E(k)\, dk\,. \tag{B3}$$

Equations (B1) and (B2) were solved numerically with $n_s(k)$ given by Eq. (84) with $\Omega = 0$. Since the technical details of the work will be presented elsewhere (Hartke et al., 1987), we limit our discussion here to mention the fact that the numerical results for $E(k, t)$ were first fitted to a function of the type (Volk et al., 1981)

$$E(k, t) = e^{-\theta(k)t}\, E(k, 0)\,. \tag{B4}$$

The resulting fit was however not satisfactory. We then tried to fit the results with a function of the type (Leslie, 1973)

$$E(k, t) = e^{-(1/2)\sigma^2(k)t^2}\, E(k, 0)\,. \tag{B5}$$

The fit was by far superior. The resulting function $\sigma(k)$ is plotted in Figs. 3 and 4 for the cases $S = 10^5$ and 10^7, and a Prandtl number of 0.7.

Having derived a $\sigma(k)$ vs k from the full DIA model, we must now ask whether the turbulence model presented in Paper I and whose main results are presented in Sect. 3 of this paper, can provide the same function $\sigma(k)$ in terms of its main ingredients, namely $n_s(k)$ and $n_c(k)$. Physically, the quantity $\sigma(k)$ represents the decay rate of an arbitrary eddy of wavenumber k. Consider now Fig. (5): $n_s(k)$ is the rate at which energy is supplied to the eddy by the external sources and $\gamma n_c(k)$ (see Paper I) is the rate at which non linear interactions drain energy out of it. We should therefore write for $\sigma(k)$,

$$\sigma(k)_{\text{model}} = \gamma n_c(k) + \gamma n_c(k) - n_s(k) = 2\gamma n_c(k) - n_s(k)\,. \tag{B6}$$

The quantity $n_c(k)$ comes from solving the set of equations described in Sect. 3. For the case of the $n_s(k)$ given by Eq. (84) (with $\Omega = 0$), the resulting $\sigma(k)_{\text{model}}$ is also plotted in Figs. (3) and (4). As one can see, Eq. (A6) represents an excellent approximation to the $\sigma(k)$ obtained from the full DIA model (Hartke et al., 1987). Equation (B6), which can be constructed entirely with the ingredients discussed in Sect. 3, is therefore the form of $\sigma(k)$ that we propose to use in Eq. (65). We may note that a formula similar to (B6) but with $n_c(k)$ treated as a local function, i.e., only a function of k and $F(k)$, was proposed by Leith (1971).

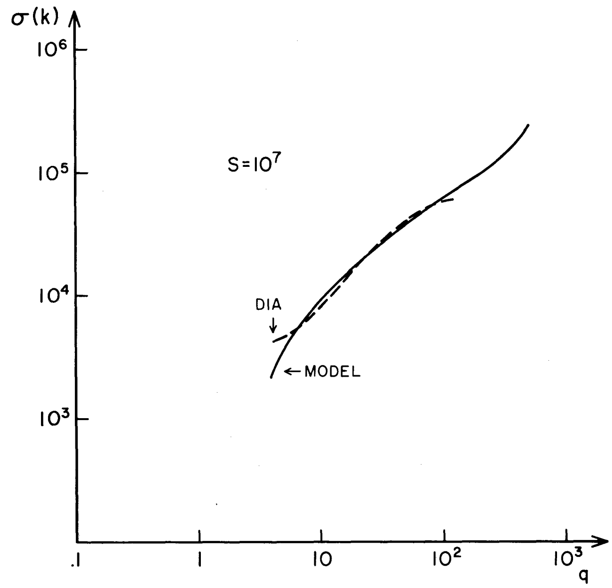

Fig. 3. Decorrelation rate $\sigma(k)$ defined in Eq. (B5) vs. $q\,(=kL$, see Sect. 6) for $S = 10^7$ and $Pr = 0.7$. The units of $\sigma(k)$ are $(\nu+\chi)/2\,L^2$. The quantity $\sigma(k)_{\text{DIA}}$ is obtained solving the full DIA set of equations described in Appendix B, while the $\sigma(k)_{\text{model}}$ is obtained using Eq. (B6). The growth rate $n_s(k)$ is solution of Eq. (84) for the case of no rotation. As one can see, the agreement between the two σ's is good

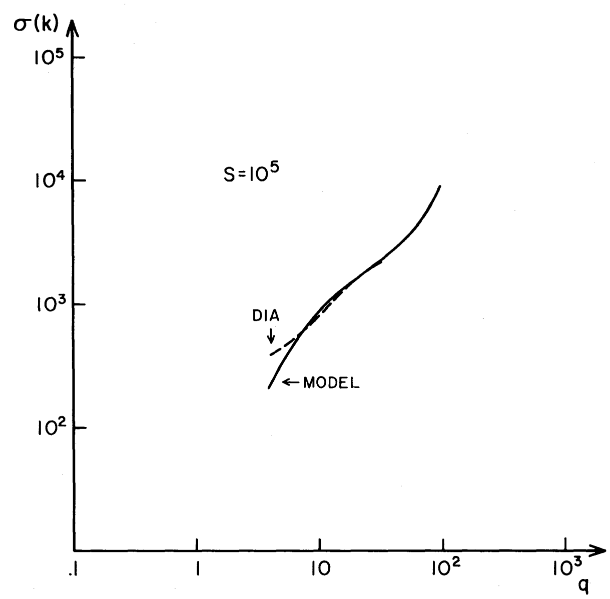

Fig. 4. The same as Fig. 3, but for $S = 10^5$

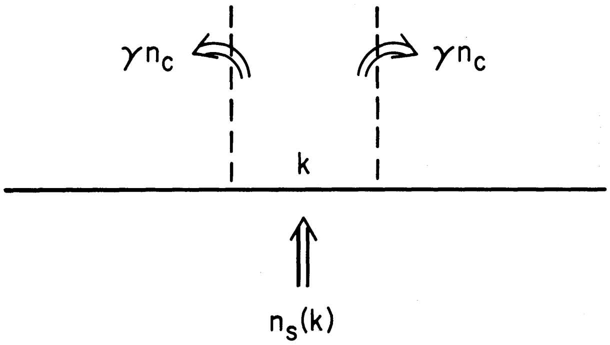

Fig. 5. The diagram represents the separate contributions to the decorrelation rate $\sigma(k)$, *using the ingredients of the model of turbulence described in Paper I;* $n_s(k)$ represents the rate at which energy is being pumped into an eddy of arbitary wavenumber k, while $\gamma n_c(k)$ represents the decay rate caused by the non-linear interactions. The model gives rise to Eq. (B6)

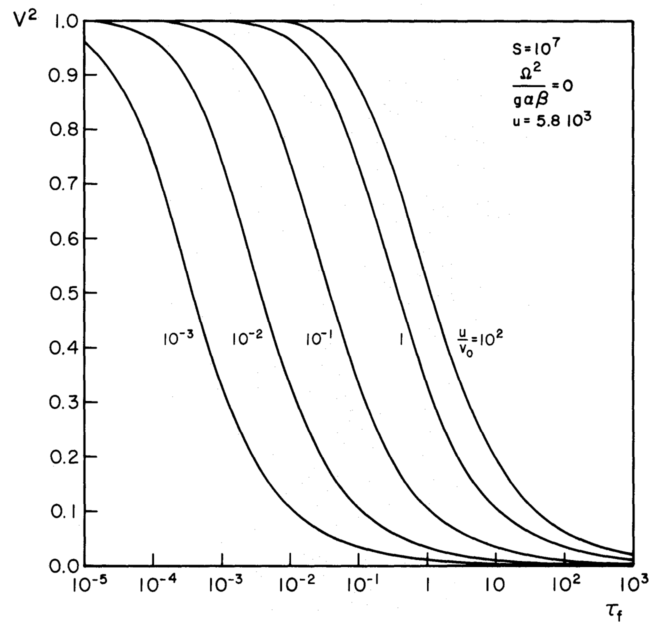

Fig. 7. Same as Fig. 6 for $\Omega = 0$

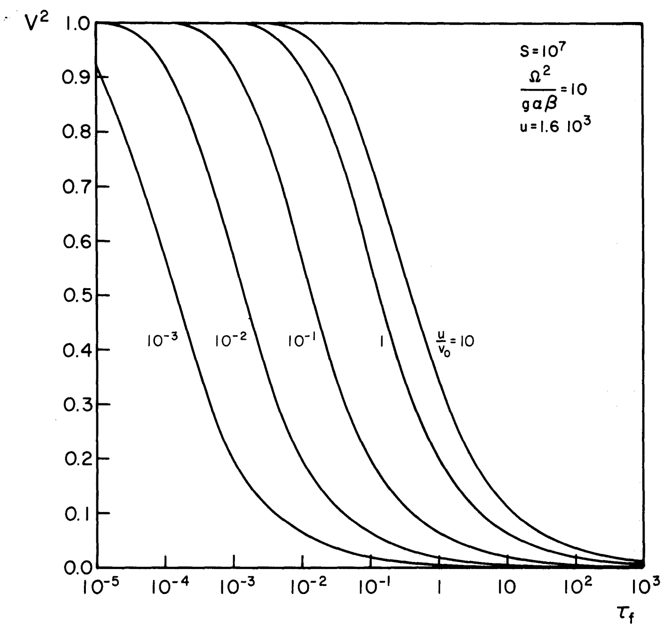

Fig. 6. The square of the grain's turbulent velocity v^2, Eq. (117) vs. the frictional response time τ_f, defined in Eq. (22). For the relation of τ_f to the mass and radius of the grain, see Weidenschilling (1977). The velocity v is normalized to the gas turbulent velocity u, Eq. (117): the value of the latter is given in the figure in units of $(\chi + \nu)/2\,L$, where the thickness of the convective layer, L, is introduced in Eq. (84). The time τ_f is given in units of L/u, which can be interpreted as the turnover time of the largest eddies. Values of S for the solar nebula can be found in Cabot et al. (1987), part II, Table 1. The quantity $\Omega^2/g\alpha\beta = Ro^{-2}$, where Ro is the Rossby number; values of Ro can be found in Cabot et al. The different curves correspond to different values of the ratio u/v_0, where v_0 is the grain's terminal velocity, $v_0 = g\tau_f$, Eq. (27)

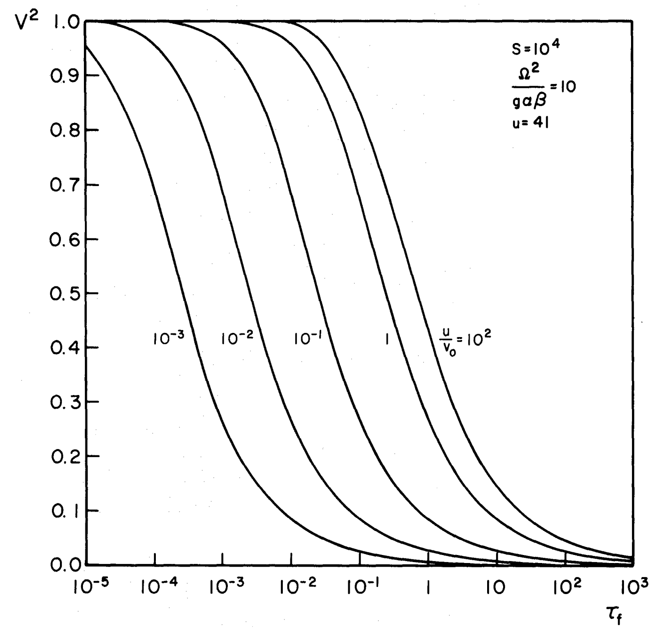

Fig. 8. Same as Fig. 6 for $S = 10^4$

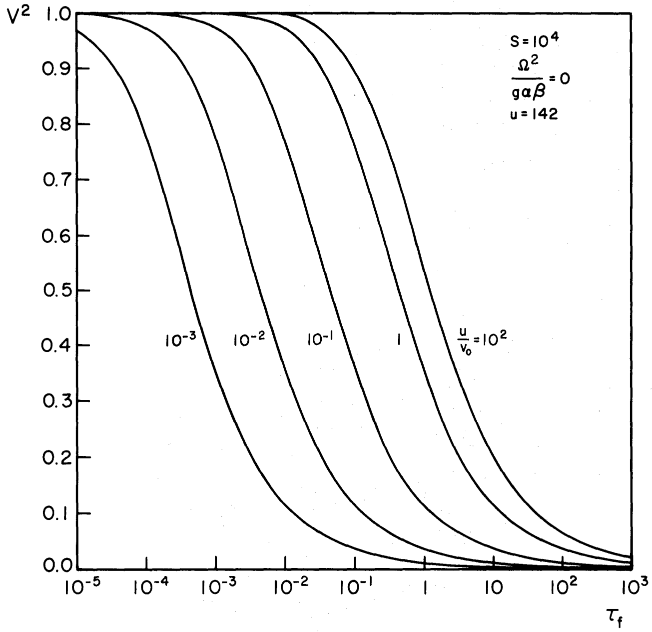

Fig. 9. Same as Fig. 7 for $S=10^4$

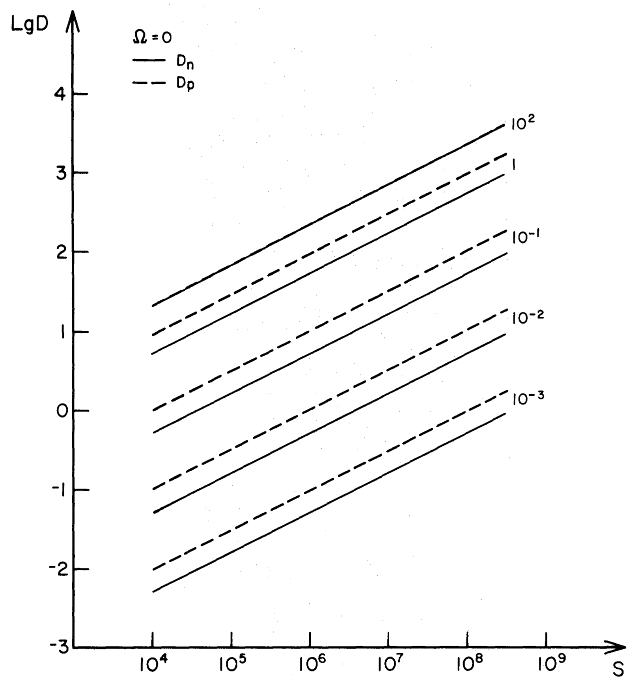

Fig. 10. The normal and perpendicular components of the diffusivity tensor D_n and D_p vs. S for different values of the ratio u/v_0, where u is the gas turbulent velocity and v_0 is the grain's terminal velocity. For u/v_0 greater than 100, the curves coincide with the last one. Typical values of S corresponding to the solar nebula can be found in Table 1, part II of Cabot et al. (1987). The units of D_n and D_p are $(\chi+\nu)/2\,L$. We have considered the case of zero kinematic viscosity ν, as appropriate for the solar nebula. As expected, the degree of anisotropy vanishes as v_0 goes to zero, i.e., when the contaminant's velocity coincides with that of the gas. In this figure, $\Omega=0$

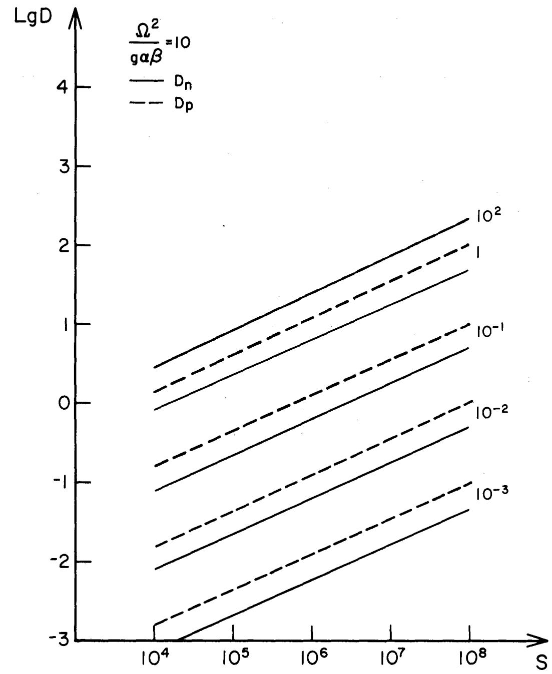

Fig. 11. Same as Fig. 10 for $\Omega^2/g\alpha\beta=10$

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