An Observational Test of the Quasi-Geostrophic Relation between Eddy Momentum Flux Convergence and Eddy Fluxes of Potential Vorticity and Potential Temperature

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ABSTRACT

The quasi-geostrophic relation between the fluxes of momentum, potential vorticity and potential temperature is tested with atmospheric data by computing the convergence of momentum flux as a residual of of the potential temperature and potential vorticity flux and comparing it to the momentum flux convergence computed directly. It is shown that in the troposphere between 18 and 74°N the observed momentum flux convergence differs from the quasi-geostrophic convergence by 25–60% depending on tropospheric level and season.

1. Introduction

The poleward eddy transport of westerly momentum, crucial to the maintenance of the zonally averaged distribution of zonal winds, is to a large extent directed against the gradient of zonally averaged angular velocity. This countergradient flux cannot be described by the "diffusion" or "mixing" hypothesis because such a formulation either yields downgradient fluxes or, if one insists on countergradient fluxes, negative values of the exchange coefficients. The first option does not fit the data and the second is objectionable on physical grounds.

Various parameterizations of the momentum flux have been proposed. Williams and Davies (1965) parameterized the momentum flux in terms of the mean meridional temperature gradient using an eddy viscosity as the constant of proportionality. From their numerical results they concluded that this type of relation satisfactorily represented the main functions of the large eddies. Saltzman and Vernekar (1968) suggested a more elaborate expression based on the tilting of the trough lines in barotropic flow and a balance between the rate of convergence of momentum and its removal by friction. Their formulation also gave satisfactory agreement with observation.

A different approach to parameterizing the poleward eddy transport of westerly momentum is discussed by Green (1970). The chief characteristic of his theory is that the concept of "mixing" is avoided by finding the transfer of quantities, conserved during the eddy lifetime, directly in terms of trajectories. He shows that the transfer of entropy is related to the mean gradient and that this transfer can be formalized in terms of observable mean quantities.

White (1977) tested Green's parameterization in a spherical polar model of the zonal mean troposphere and in a steady-state β-plane model of the zonal mean troposphere. Using the realism of the surface zonal flow as an indicator, White concluded that a simple basis for estimating transfer coefficients leads to a satisfactory representation of the momentum flux in the steady-state β-plane model but to a less satisfactory one in the spherical polar model. He suggests that barotropic/baroclinic stability analyses seem likely to provide the most reliable basis, but that their use would amount to an essentially different parameterization technique. White speculates that less elaborate methods may be adequate.

An aspect of Green's parameterization that has not been tested is the quasi-geostrophic relation between the fluxes of momentum, potential vorticity and potential temperature. This relation between a flux that is not downgradient to two which are (in quasi-geostrophic motion) forms the heart of Green's parameterization. It is not clear to what extent it is satisfied in the atmosphere or in a primitive equation model of the atmosphere in which the large-scale eddy motion is explicit. For example, the numerical calculations of Simmons and Hoskins (1976) do show substantial differences between the quasi-geostrophic and primitive equation momentum fluxes accompanying the most unstable baroclinic waves. The physical implications of the quasi-geostrophic relation is discussed in Green (1970) and Held (1975). Examples of differences be-

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tween measured fluxes and their geostrophic counterparts are discussed by Lau *et al.* (1978).

Our purpose is to test the quasi-geostrophic relation for atmospheric data by computing the convergence of momentum flux as a residual of the potential temperature and potential vorticity fluxes and comparing it to the momentum flux convergence computed directly. If the large-scale eddy fluxes are nearly quasi-geostrophic, the convergence of momentum flux, computed as a residual of the potential temperature and potential vorticity fluxes, should equal the convergence of eddy momentum flux computed directly. We will show that the observed momentum flux convergence in the troposphere is quasi-geostrophic within 25–60%, depending on tropospheric level and season.

2. The quasi-geostrophic eddy momentum flux[2]

The quasi-geostrophic potential vorticity equation on a β-plane in isobaric coordinates (Holton, 1972) is

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)q=0, \tag{1}$$

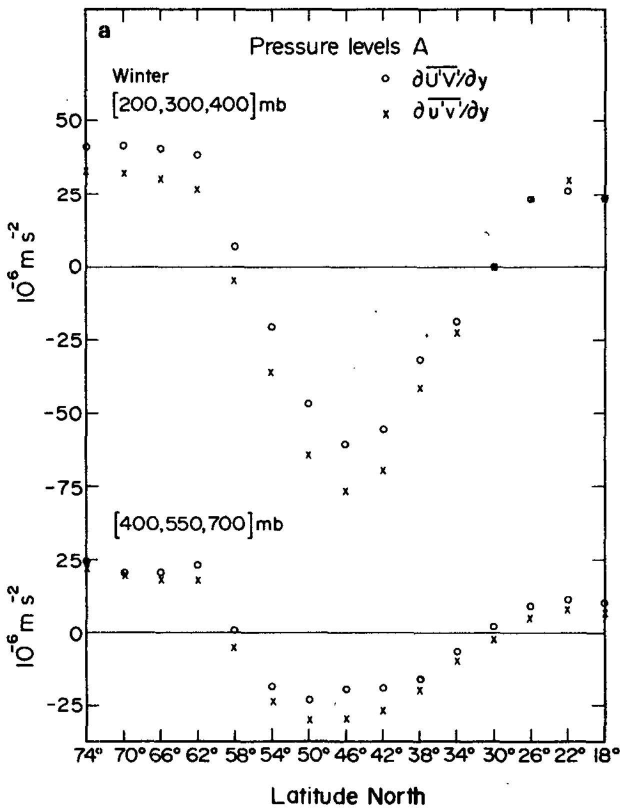

FIG. 1a. Pressure levels A: The winter average of the observed and quasi-geostrophic momentum flux convergence as a function of latitude for the indicated pressure levels.

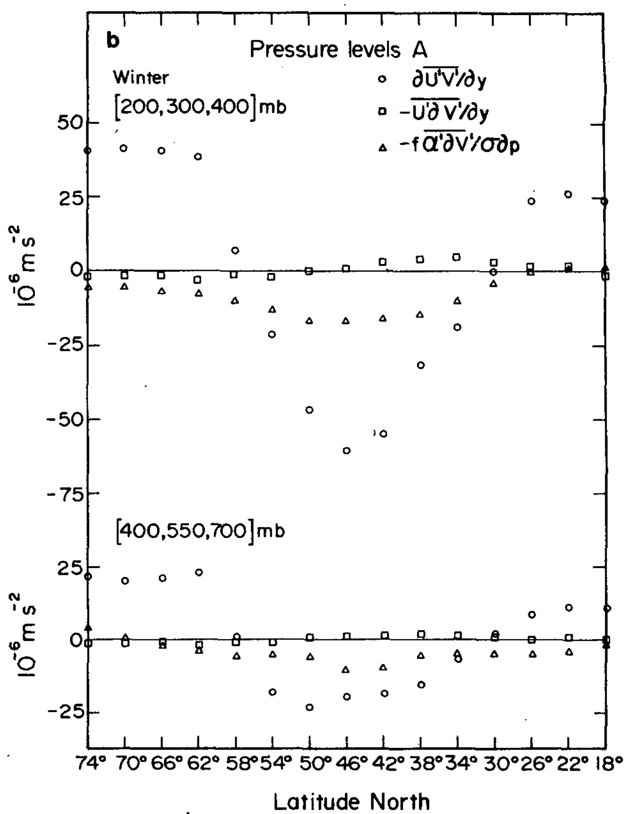

FIG. 1b. Pressure levels A: The winter average of the terms in inequality (11) as a function of latitude for the indicated pressure levels.

where

$$q=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f+f_0^2\frac{\partial}{\partial p}\left(\frac{1}{\sigma}\frac{\partial\psi}{\partial p}\right) \tag{2}$$

is the quasi-geostrophic potential vorticity. Multiplying (2) by v', averaging zonally, and making use of the geostrophic approximations, one obtains an expression for the convergence of eddy momentum flux as a function of the potential vorticity and potential temperature flux (Holton, 1972):

$$\frac{\partial\overline{u'v'}}{\partial y}=-\overline{v'q'}+f_0^2\frac{\partial}{\partial p}\left(\overline{\frac{v'}{\sigma}\frac{\partial\psi'}{\partial p}}\right). \tag{3}$$

(Note that $\partial\psi'/\partial p$ is proportional to θ' on an isobaric surface.)

In order to test the relation given by (3), we must evaluate q in terms of observable quantities. When the potential vorticity is computed from the observed eastward wind speed U and the observed northward wind speed V, it will be denoted by Q:

$$Q=\frac{\partial V}{\partial X}-\frac{\partial U}{\partial y}+f+f_0\frac{\partial}{\partial p}\left(\frac{-\alpha}{\sigma}\right), \tag{4}$$

[2] The notation for this and subsequent sections is explained in the Appendix.

where we have used

$$\psi=\frac{\Phi}{f_0},\tag{5}$$

$$\frac{\partial\Phi}{\partial p}=-\alpha.\tag{6}$$

Multiplying (4) by V' and averaging zonally gives

$$\frac{\partial\overline{U'V'}}{\partial y}=-\overline{V'Q'}-f_0\frac{\partial}{\partial p}\left(\frac{\overline{V'\alpha'}}{\sigma}\right)+\overline{U'\frac{\partial V'}{\partial y}}+f_0\overline{\frac{\alpha'}{\sigma}\frac{\partial V'}{\partial p}}.\tag{7}$$

Each of the last two terms on the right-hand side (RHS) of (7) are zero if atmospheric motions are quasi-geostrophic (Holton, 1972, p. 229). In that case, (7) simplifies to

$$\frac{\partial\overline{U'V'}}{\partial y}=-\overline{V'Q'}-f_0\frac{\partial}{\partial p}\left(\frac{\overline{V'\alpha'}}{\sigma}\right).\tag{8}$$

Comparing the RHS of (3) and (8), we find that for quasi-geostrophic flow

$$-\overline{V'Q'}-f_0\frac{\partial}{\partial p}\left(\frac{\overline{V'\alpha'}}{\sigma}\right)=\frac{\partial\overline{u'v'}}{\partial y}.\tag{9}$$

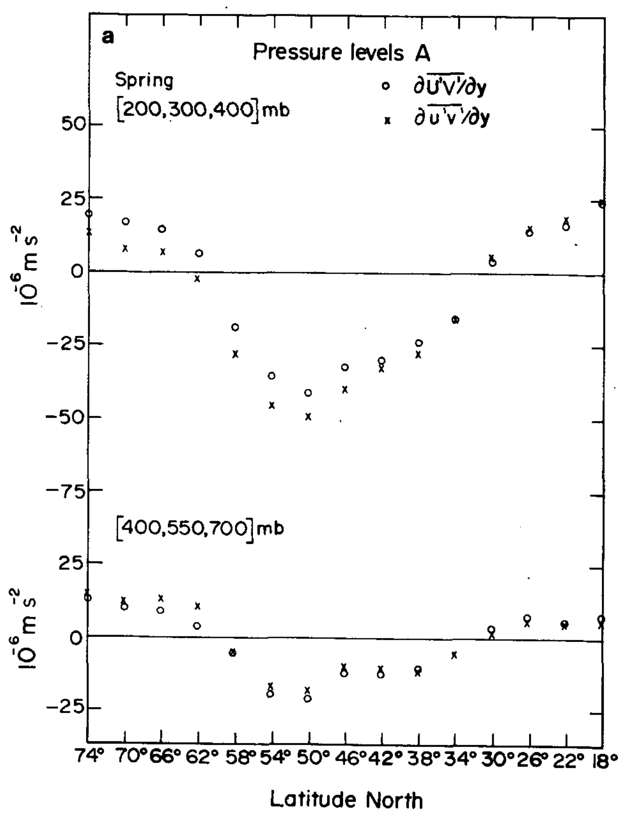

FIG. 2a. As in Fig. 1a, but for spring.

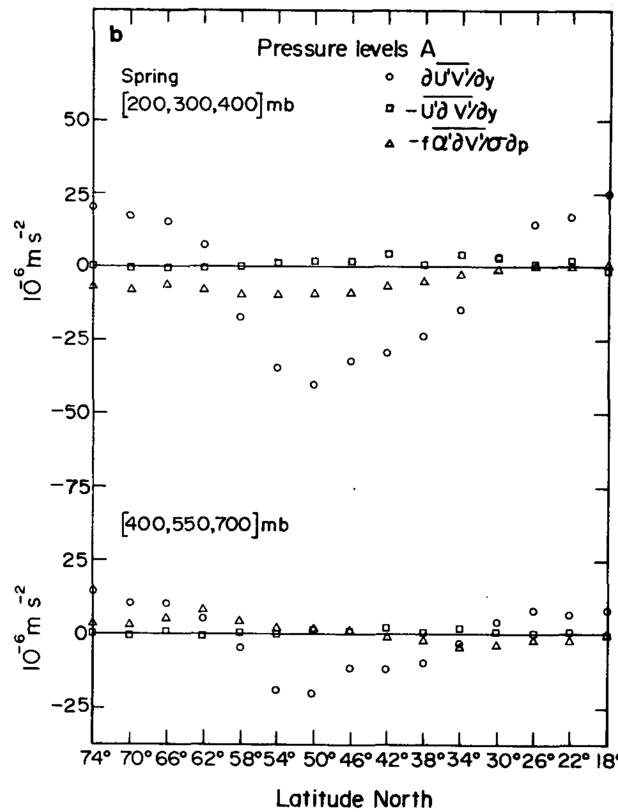

FIG. 2b. As in Fig. 1b, but for spring.

Substituting (9) into (7), we obtain an expression which allows us to test the quasi-geostrophy of the momentum flux in terms of observable quantities:

$$\frac{\partial\overline{u'v'}}{\partial y}=\frac{\partial\overline{U'V'}}{\partial y}-\overline{U'\frac{\partial V'}{\partial y}}-f_0\overline{\frac{\alpha'}{\sigma}\frac{\partial V'}{\partial p}}.\tag{10}$$

Throughout this paper we will refer to the terms in (1)–(10) in their β-plane formulation, although the numerical calculations were performed in spherical coordinates.

3. The data set and analysis procedures

The basic data set, covering the period July 1976–June 1977, consists of twice daily (0000 and 1200 GMT) analyses of the wind and temperature fields for the Northern Hemisphere, as made by the National Meteorological Center (NMC) and archived at the National Center for Atmospheric Research (NCAR). The analyses are based on Flattery's global analysis scheme with 12 h operational forecasts used as the first-guess field. With the exception of the zonally averaged meridional wind (which is forced to zero), the values produced by Flattery's analysis are presumed unbiased. Computed values of $\partial\overline{U'V'}/\partial y$, $\overline{U'\partial V'/\partial y}$ and $\overline{f\alpha'\partial V'/\sigma\partial p}$ are also presumed free of systematic errors related to the analysis. Even if the Flattery

TABLE 1. Pressure levels for sets A and B.

Set A Level number	Set A Pressure (mb)	Set B Level number	Set B Pressure (mb)
1	200	1	150
2	300	2	200
3	400	3	250
4	550	4	300
5	700		

analysis systematically underestimates the ageostrophic terms, it would result in an underestimate of only the $\overline{U'\partial V'/\partial y}$ term and not the $\overline{\alpha'\partial V'/\partial p}$ term. In Section 4 we will show that the magnitude of $\overline{\alpha'\partial V'/\partial y}$ is larger than the magnitude of $\overline{U'\partial V'/\partial y}$, and since the conclusion of this paper is based on the magnitude of the former term, we don't expect the conclusion to be affected by any bias NMC may have put in the analysis.

The values of $\partial\overline{U'V'}/\partial y$, $\overline{U'\partial V'/\partial y}$ and $f\overline{\alpha'\partial V'/\sigma\partial p}$ were computed for a 4°×5° latitude-longitude grid resolution. With this horizontal resolution fixed, the quantities were computed from pressure level data for two sets of pressure levels (A and B) having different vertical resolutions. The pressure levels for sets A and B are listed in Table 1. With the exception of the 550 mb level, all the levels shown are standard pressure

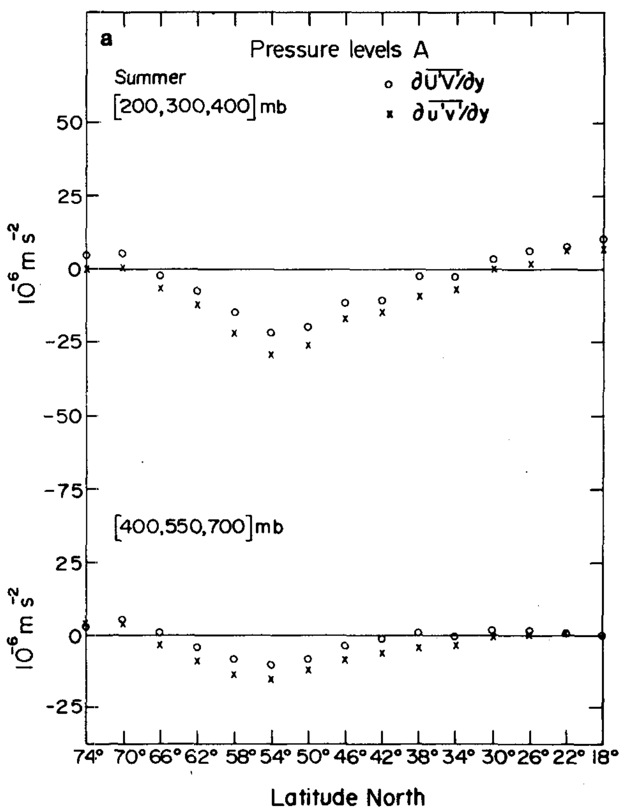

FIG. 3a. As in Fig. 1a. but for summer.

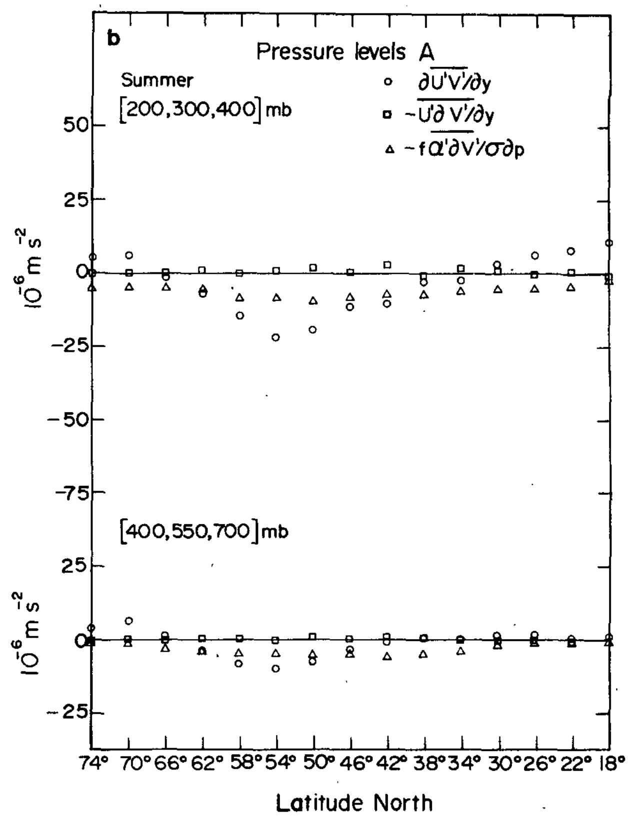

FIG. 3b. As in Fig. 1b, but for summer.

levels. The 500 mb level, a standard pressure level, was not used directly because it would have resulted in unequal pressure increments between 400 and 700 mb. The 500 mb data was replaced by the 550 mb data, which was computed from linear interpolation of the 500 and 700 mb data. (The sensitivity of the computed fluxes to quadratic interpolation will be discussed briefly in a later section). The momentum flux convergences that we compute are averages for three pressure levels.

4. Results and discussion

The main computational results will be presented in terms of seasonal averages: winter (December, January, February), spring (March, April, May), summer (June, July, August) and fall (September, October, November).

a. Pressure levels A

Figs. 1a–4a show the observed and quasi-geostrophic momentum flux convergences as a function of latitude and pressure for pressure levels A for winter, spring, summer and fall, respectively. Qualitatively, the observed and quasi-geostrophic convergences are similar during all seasons for both the mid (400–700 mb) and upper (200–400 mb) troposphere. Quantitatively, the difference between the observed and quasi-geostrophic momentum flux convergence is largest at latitudes where the convergence is a maximum.

The numerical difference between the observed and quasi-geostrophic momentum flux convergence is given by the sum of the last two terms on the RHS of Eq. (10). When these terms are small, the observed momentum flux convergence is quasi-geostrophic. Each of these terms is shown as a function of latitude and pressure in Figs. 1b–4b for winter, spring, summer and fall, respectively. During all seasons, the terms are of opposite sign at most latitudes south of 60°; at some latitudes between 45° and 25° the terms are both opposite in sign and equal in order of magnitude. By examining only the sum of these terms, we could conclude falsely that the observed fluxes are more quasi-geostrophic than they actually are. Therefore, we must also examine the magnitude of each of the terms, not just the magnitude of the sum to determine the degree of quasi-geostrophy of the fluxes.

We will consider the last two terms on the RHS of (10) small when the following inequalities are satisfied:

$$\left|\overline{U'\frac{\partial V'}{\partial y}}\right| \ll \left|\frac{\partial \overline{U'V'}}{\partial y}\right|, \qquad (11a)$$

$$\left|\overline{\frac{\alpha'}{\sigma}\frac{\partial V'}{\partial p}}\right| \ll \left|\frac{\partial \overline{U'V'}}{\partial y}\right|. \qquad (11b)$$

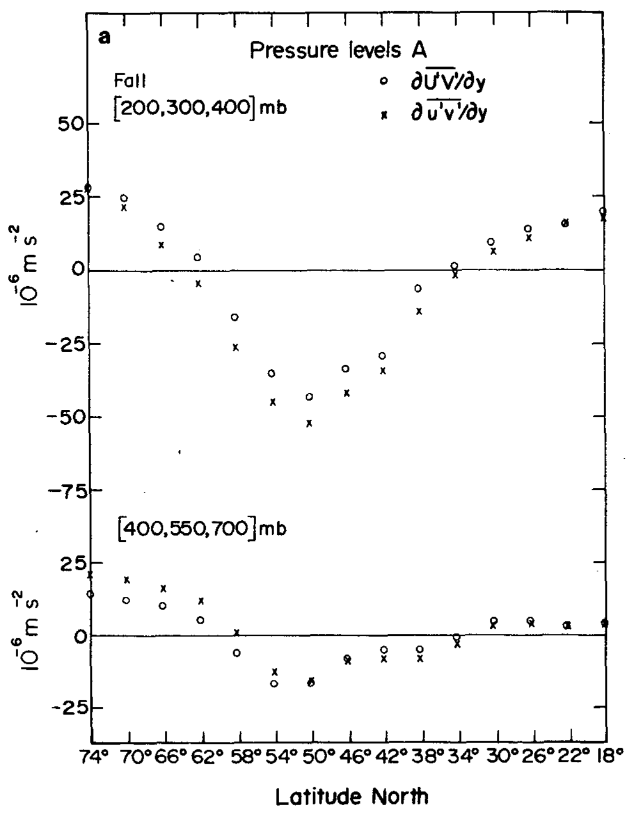

FIG. 4a. As in Fig. 1a, but for fall.

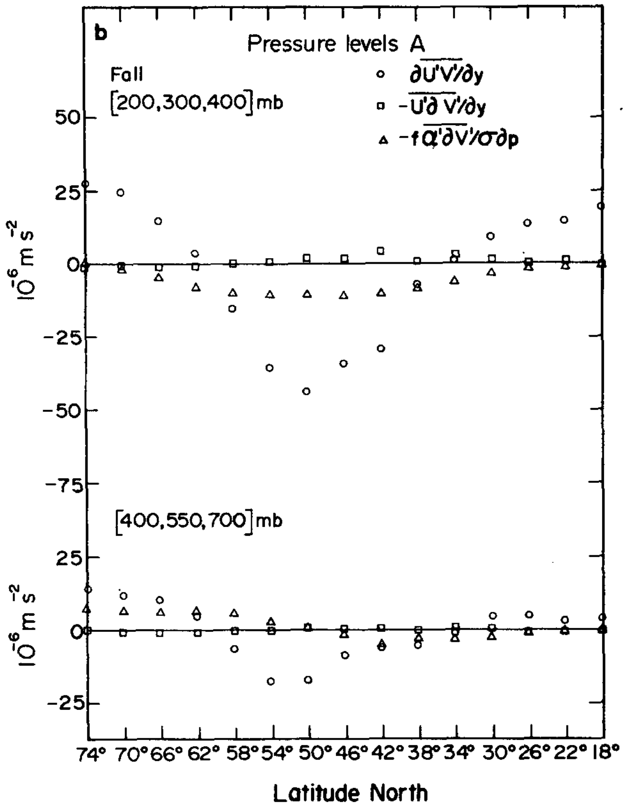

FIG. 4b. As in Fig. 1b, but for fall.

Inequality (11a) is automatically satisfied when (11b) is satisfied because (as Figs. 1b–4b show) $|\overline{\alpha'\partial V'/\sigma\partial p}| \geqslant |\overline{U'\partial V'/\partial y}|$ almost everywhere during all seasons. Thus we will use $|\overline{\alpha'\partial V'/\sigma\partial p}|/|\partial\overline{U'V'}/\partial y|$ as a measure of the degree of quasi-geostrophy. We will define the relative error as the ratio

$$|\overline{\alpha'\partial V'/\sigma\partial p}|_{\max}/|\partial\overline{U'V'}/\partial y|_{\max}$$

and the absolute error as $|\overline{\alpha'\partial V'/\sigma\partial p}|_{\max}$. The subscript max refers to the maximum values of these quantities for a given tropospheric level and season.

The relative and absolute errors for pressure levels A are listed as a function of tropospheric level and season in Table 2. As can be seen, the degree of quasi-geostrophy of the observed momentum flux convergence, as measured by the relative error, ranges from 25–41% in the upper troposphere and 26–60% in the mid-troposphere. In the upper troposphere the relative error for winter, spring and fall is about 30% less than for summer. Although the relative error is similar for winter, spring and fall, the absolute error for winter is 60% greater than for spring or fall. In the mid-troposphere the absolute error changes little from season to season but the relative error changes by over 100%. A comparison of mid and upper tropospheric relative errors indicates that the relative errors in the

TABLE 2. Relative and absolute errors as a function of season for pressure levels A, B and pressure levels A with constant σ.

Pressure		Winter Absolute error (10^{-6} m s^{-2})	Winter Relative error	Spring Absolute error (10^{-6} m s^{-2})	Spring Relative error	Summer Absolute error (10^{-6} m s^{-2})	Summer Relative error	Fall Absolute error (10^{-6} m s^{-2})	Fall Relative error	Annual Absolute error (10^{-6} m s^{-2})	Annual Relative error
Pressure levels A	200 300 400	17	0.29	10	0.25	9	0.41	11	0.26	11	0.29
	400 550 700	6	0.26	8	0.38	6	0.60	7	0.41	5	0.29
Pressure levels B	150 200 250	11	0.17	6	0.16	4	0.20	10	0.25	7	0.18
	200 250 300	14	0.20	11	0.24	10	0.40	9	0.19	10	0.24
Pressure levels A with constant σ	200 300 400	17	0.29	6	0.15	7	0.32	8	0.19	7	0.18
	400 550 700	8	0.35	5	0.24	5	0.50	5	0.29	4	0.24

mid troposphere are 50% greater than the relative errors of the upper troposphere during all seasons except winter. A comparison of absolute errors indicates that the absolute errors in the mid troposphere are consistently smaller than those of the upper troposphere during all seasons. In the annual average, the relative error for the mid and upper troposphere is identical even though for spring, summer and fall the mid tropospheric relative error was 50% greater than the upper tropospheric value. This apparent inconsistency can be explained by the large latitudinal shift with season of the term $|\overline{\alpha'\partial V'/\sigma\partial p}|_{\max}$, which was used in the definition of the relative error. The maximum of $|\overline{\alpha'\partial V'/\sigma\partial p}|$ in the mid troposphere occurs at about 42°N latitude during winter and summer and at 62°N during spring and fall, whereas the maximum in $|\partial\overline{U'V'}/\partial y|$ occurs near 50° N during all seasons (Figs. 1b, 2b, 3b and 4b). Thus, in the mid-troposphere, averaging over four seasons decreases $|\overline{\alpha'\partial V'/\sigma\partial p}|_{\max}$ much more than $|\partial\overline{U'V'}/\partial y|_{\max}$.

b. Pressure levels B

The quantities $\partial\overline{U'V'}/\partial y$, $\overline{U'\partial V'/\partial y}$ and $\overline{\alpha'\partial V'/\sigma\partial p}$ were also computed with pressure level B data (graphs not shown). The increased vertical resolution did not change the latitudinal dependence of any of the quantities from their upper tropospheric values calculated from pressure levels A data, although the relative and absolute errors did decrease somewhat (Table 2). The relative error at [200, 250, 300] mb in winter and fall is only two-thirds of the relative error at [200, 300, 400] mb. The relative error for spring and summer, however, remains unchanged. The [150, 200, 250] mb relative error for winter, spring and summer, and the annual mean are even less than the [200, 250, 300] mb values, but this is largely a reflection of the decrease in the absolute error during these periods.

c. Sensitivity to horizontal resolution and vertical interpolation

To determine the sensitivity of the momentum flux convergence to the horizontal resolution, the NMC data was interpolated to an 8°×10° latitude-longitude grid (for pressure levels A). The results are identical to those described in Section 4a and are not shown here.

The sensitivity of the computed fluxes to quadratic interpolation at 550 mb using 400, 500 and 700 mb data was also investigated. The zonal mean temperature in the middle latitudes with quadratic interpolation was 1–2° higher than with linear interpolation but the change in the monthly average momentum flux convergence was almost imperceptible.

d. Momentum flux convergence in the lower troposphere

Seasonal average values of $\partial\overline{U'V'}/\partial y$, $\overline{U'\partial V'/\partial y}$ and $\overline{\alpha'\partial V'/\sigma\partial p}$ were also computed for the [700, 850, 1000] mb levels. The effect of terrain at a grid point was included by weighting the fluxes above the terrain by the fraction of the layer free of terrain. With this method of computation the quasi-geostrophic momentum flux convergence in mid-latitudes exceeded the observed values by a factor of 10–20 during all seasons. This result is not surprising in view of the fact that diabatic and frictional effects, which dominate the physics of the lower troposphere, are neglected in quasi-geostrophic theory.

5. Summary

With our measure of relative error we have shown that the observed momentum flux convergence averaged seasonally is quasi-geostrophic within 25–40% in the upper troposphere and within 25–60% in the mid-troposphere. The observed momentum flux convergence averaged annually is quasi-geostrophic within 30% in the mid and upper troposphere. The degree of quasi-geostrophy is relatively unaffected by vertical resolution in the upper troposphere and unaffected by horizontal resolution in both the mid and upper troposphere. In the lower troposphere the quasi-geostrophic momentum flux convergence is 10–20 times greater than the magnitude of the observed value.

We conclude that the momentum flux convergence computed from quasi-geostrophic theory is a satisfactory approximation for qualitative studies of the general circulation but less satisfactory for quantitative investigations.

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APPENDIX

List of Symbols

x	eastward distance
y	northward distance
∇	horizontal gradient $[=(\partial/\partial x,\ \partial/\partial y)]$
ϕ	latitude
λ	longitude
r	radius of the earth
p	pressure
t	time
t_m	averaging period (3 months)
U	observed eastward wind speed
V	observed northward wind speed
ψ	streamfunction
u	eastward wind speed $[=-\partial\psi/\partial y]$
v	northward wind speed $[=\partial\psi/\partial x]$
Φ	geopotential
f	Coriolis parameter
f_0	Coriolis parameter at 45°N
T	temperature
α	specific volume
θ	potential temperature
$\bar{A}$	zonal average of A $\left[=(2\pi)^{-1}\int_0^{2\pi} A\,d\lambda\right]$
A'	departure from zonal average of A $[=A-\bar{A}]$
σ	mean static stability $\left[=\left(\frac{2}{\pi t_m}\right)\int_0^{t_m}\int_0^{\pi/2} -\overline{\frac{\alpha}{\theta}\frac{\partial\theta}{\partial p}} \times \cos\phi\, d\phi\, dt\right]$.

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