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PULSATIONAL STABILITIES OF A STAR IN THERMAL IMBALANCE: COMPARISON BETWEEN THE METHODS

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ABSTRACT

The stability coefficients for quasi-adiabatic pulsations for a model in thermal imbalance are evaluated using the dynamical energy (DE) approach, the total (kinetic plus potential) energy (TE) approach, and the small amplitude (SA) approaches. From a comparison among the methods, it is found that there can exist two *distinct* stability coefficients under conditions of thermal imbalance as pointed out by Demaret. It is shown that both the TE approaches lead to one stability coefficient, while both the SA approaches lead to another coefficient. The coefficient obtained through the energy approaches is identified as the one which determines the stability of the velocity amplitudes.

For a prenova model with a thin hydrogen-burning shell in thermal imbalance, several radial modes are found to be unstable both for radial displacements and for velocity amplitudes. However, a new kind of pulsational instability also appears, viz., while the radial displacements are unstable, the velocity amplitudes may be stabilized through the thermal imbalance terms.

Subject headings: instabilities — stars: interiors — stars: pulsation

I. INTRODUCTION

At the present time, there is no *unique* generally accepted definition of vibrational stability coefficient for a star in thermal imbalance. Based upon a general discussion outlined in Ledoux (1963, 1965), Cox, Hansen, and Davey (1973, henceforth CHD) and Cox, Davey, and Aizenman (1974, henceforth CDA) developed the small-amplitude approach appropriate to the thermal imbalance situation and obtained integral expressions for the stability coefficient. Using asymptotic methods, Aizenman and Cox (1974) derived solutions to the pulsation equations for some simple stellar models and showed that the asymptotic methods lead to the same stability coefficient as above for these particular cases. In a separate treatment, Demaret (1974, 1975) obtained an asymptotic solution for the radial displacements considering the effects of thermal imbalance as a perturbation. This method also leads to a stability coefficient which essentially agrees with the one from the small-amplitude approach of CHD and CDA. Both methods recognize that the stability coefficient using the relative radius amplitudes is different from the coefficient using the absolute radial displacements.

Alternately, the stability coefficient may be determined using the rate of change of the variation of the total pulsation energy averaged over a period. In all different variations of this method (Kato and Unno 1967; Simon, 1970, 1971; Simon and Sastri 1972; Sastri and Simon 1973; Axel and Perkins 1971; Davey and Cox 1974) the vibrational stability coefficient is found to be different from the one using the small amplitude methods. However, Davey and Cox (1974, henceforth DC) showed that the desired agreement may be achieved if the kinetic energy *only* (instead of total energy) is used in the analysis. But Demaret (1976) strongly criticized their analysis as incorrect and showed that a proper treatment would not lead to the agreement that Davey and Cox claimed. He further suggested that there exist two separate and distinct stability coefficients for a star in thermal imbalance and that they are connected by a simple relation. A similar suggestion was advanced by Simon (1974, 1977), who pointed out that the energy approach would lead to the determination of the stability coefficient for the velocity amplitudes, while the one using the absolute radial displacements would lead to the coefficient for radial displacements and that these two are distinct. He further suggested that for complete stability of a star, it is necessary that both the absolute radial displacements and the velocity amplitudes be stable.

In a previous paper (Sastri and Simon 1973, henceforth SS), we reported the multimodal pulsational instability of an actual prenova model using the total dynamical energy approach and showed that, for some modes, the star may become stable as the result of the damping effects of thermal imbalance terms. Cox (1974) evaluated the stability coefficient for absolute radial amplitudes for a model of a helium-shell-burning star. Besides these, all the other applications and comparison between the methods have been to special cases of homologously contracting stars or to homologous oscillations of polytropes in homologous contraction.

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The purpose of this paper, then, is twofold. Primarily, it is to apply the different methods to an actual model and to get a feel for the importance of thermal imbalance in determining the stability. Apart from quantitative differences, the two stability coefficients (using the small-amplitude and the total-energy methods) may be qualitatively different. In particular, for the same model and for the same normal mode, the two analyses might lead to stability if one method is used, and to instability if the other is used. A situation of this kind occurs for the fourth mode for the present model. Second, but more important from our point of view, it is to see if the two total energy approaches (viz., SS and Demaret or SS and DC) agree when applied to an actual model. The model in question is one of an evolutionary sequence of 1 $M_\odot$ nova models supplied by Dr. Sparks. Some of the peculiarities of such a model and the modifications required to make the computations possible have been discussed in SS. However, the present model is from a different evolutionary sequence, and the polytropic index for the interior fitted model is 1.647.

In the following, the results of the pulsational analysis of the model using the integral expressions in CDA, those given in Demaret (1976), and the expressions in SS will be presented. Specifically, we wish to compare, in § II*b*, the stability coefficients as given in CDA and those in Demaret (1976) and show that they agree exactly. In § II*c*, the relationship between the stability coefficients with absolute and relative radial displacements will be studied. Section II*d* will deal with the evaluation of stability coefficients using the energy approaches (SS versus Demaret, and SS versus DC). Taking into account the inadequacies in the model treatment (see SS for details) and those inherent in the equilibrium models themselves, we believe that the treatments are consistent and that the two different treatments lead to different stability coefficients.

II. PULSATIONAL STABILITY

a) *Preliminaries*

In carrying out the pulsational analysis for the model, the input parameters, like the power law in the nuclear energy generation and the part of nuclear energy, ϵ_β, emitted via β-decays (see SS for further details), are kept identical in all the methods to make the necessary comparison between the methods. Pulsational periods and the relative radial and density amplitudes have been determined using the quasi-adiabatic approximation (Davey 1974). The nonadiabatic effects have been ignored in the analysis, since these were shown by CDA to be not significant except for low-mass, highly luminous red supergiants.

In the analysis using the total energy through the method adopted by SS, the thermodynamic quantities do not directly appear in the integrals. But the integral expressions of CDA and those of Demaret (through $\dot{\omega}/\omega$, where ω represents the pulsation frequency and dot denotes differentiation with respect to time) involve pressure, temperature, and density explicitly. For the model available to us, these quantities are known to three significant figures only. From the way these quantities appear in the integral expressions, we estimate that the expressions could lead to results differing up to 10%. For this reason, differences of this magnitude in the computed integrals are intrinsic to the model.

b) *Absolute Radial Amplitudes: Demaret versus CDA*

Combining the linearized momentum, energy and mass conservation equations, CDA obtain a third order differential equation for the absolute radial amplitudes and attempt a solution to this equation of the form

$$\delta r(m, t) = \eta(m, t) \exp(-\kappa_{\mathrm{II}} t) \exp(i\sigma_{\mathrm{II}} t), \tag{1}$$

where σ_{II} refers to the pulsation frequency. The suffix II in σ_{II} and $-\kappa_{\mathrm{II}}$ is used by CDA to denote quantities obtained by using absolute radial amplitudes, while the suffix I as in σ_{I} and $-\kappa_{\mathrm{I}}$ is used for similar quantities obtained through using relative radial amplitudes.

The stability coefficient $-\kappa_{\mathrm{II}}$, is given by

$$-\kappa_{\mathrm{II}} = \frac{1}{2J\Sigma^2}\left[-\tfrac{1}{2}I_1 - 2I_2 + I_3 + I_4 - \tfrac{3}{2}I_5 + C + D\right], \tag{2}$$

where

$$I_1 = 12\int \frac{Gm}{r_0}\frac{\dot{r}_0}{r_0}\xi^2 dm, \qquad I_2 = -\int \frac{\Gamma_{10}P_0}{\rho_0}\left(\frac{\delta\rho}{\rho_0}\right)\left[\frac{\dot{r}_0}{r_0}\left(\frac{\delta\rho}{\rho_0}\right) - 16\pi^2 r_0{}^5\rho_0{}^2\frac{\partial}{\partial m}(r_0\xi)\frac{\partial}{\partial m}\left(\frac{\dot{r}_0}{r_0}\right)\right]dm,$$

$$I_3 = \int \frac{\Gamma_{10}P_0}{\rho_0}\left(\frac{\delta\rho}{\rho_0}\right)\frac{\dot{\rho}_0}{\rho_0}\left(\frac{\delta P}{P_0}\right)dm, \qquad I_4 = \int \frac{2P_0}{\rho_0}\left(\frac{\delta P}{P_0}\right)\left[\frac{\dot{r}_0}{r_0}\left(\frac{\delta\rho}{\rho_0}\right) - 4\pi r_0{}^3\rho_0\xi\frac{\partial}{\partial m}\left(\frac{\dot{r}_0}{r_0}\right)\right]dm,$$

$$I_5 = \int \left(\frac{\delta\rho}{\rho_0}\right)^2\frac{\partial}{\partial t}\left(\frac{\Gamma_1 P_0}{\rho_0}\right)dm, \qquad C = \int (\Gamma_3 - 1)_0\left(\frac{\delta\rho}{\rho_0}\right)\delta\left(\epsilon - \frac{\partial L}{\partial m}\right)dm,$$

$$D = \int (\Gamma_3 - 1)_0\left(\epsilon - \frac{\partial L}{\partial m}\right)\left(\frac{\delta\rho}{\rho_0}\right)^2 dm, \qquad J\Sigma^2 = 2\int\left[\frac{\Gamma_{10}P_0}{2\rho_0}\left(\frac{\delta\rho}{\rho_0}\right)^2 - \frac{2Gm}{r_0}\xi^2\right]dm.$$

Demaret, on the other hand, treats the effects of thermal imbalance as a perturbation on the equilibrium model and tries a solution which is a superposition of the normal modes, viz.,

$$\delta r(r_0, t) = \sum_\nu A_\nu \eta_\nu(r_0, t) \exp\left(-\int_0^t \sigma_\nu''(t')dt'\right) \sin\left(\int_0^t \omega_\nu(t')dt'\right). \tag{3}$$

It must be noted that σ_ν and σ_{II} refer to two different entities and should not be confused. In the situation when the different normal modes are uncoupled, we can establish a correspondence between the two expressions and may identify that

$$-\kappa_{\mathrm{II}} = -\int_0^{\mathrm{PERIOD}} \sigma_\nu''(t')dt'/\mathrm{PERIOD} = -\sigma_\nu'' \quad (t_{\mathrm{ff}}/t_s \ll 1) \tag{4}$$

and

$$\sigma_{\mathrm{II}} = \int_0^{\mathrm{PERIOD}} \omega_\nu(t')dt'/\mathrm{PERIOD}\,, \tag{5}$$

where t_{ff} is the free-fall time of the order of a pulsation period, while t_s is the secular time of the order of the Kelvin time or nuclear time.

The stability coefficient $-\sigma''$ (henceforth, we drop the suffix ν) is given by

$$-\sigma'' = \frac{1}{2\omega^2}\int_0^M \left(\frac{\delta T}{T_0}\right)\left[\delta\left(\epsilon - \frac{\partial L}{\partial m}\right)\right]dm - \frac{1}{2\omega^2}\int_0^M \left(\frac{\delta T}{T_0}\right)^2\left(\epsilon - \frac{\partial L}{\partial m}\right)dm - \frac{1}{2}\frac{\dot\omega}{\omega}\,,$$

where the integrals are normalized to $J = 1$. Thus in the notation of CDA, the above can be written as

$$-\sigma'' = \frac{1}{2J\Sigma^2}\int_0^M \left(\frac{\delta T}{T_0}\right)\left[\delta\left(\epsilon - \frac{\partial L}{\partial m}\right)\right]dm - \frac{1}{2J\Sigma^2}\int_0^M \left(\frac{\delta T}{T_0}\right)^2\left(\epsilon - \frac{\partial L}{\partial m}\right)dm - \frac{1}{2}\frac{\dot\omega}{\omega}\,. \tag{6}$$

In our computations, all the pulsational variables are treated as real since the nonadiabatic effects are being ignored. In evaluating (6), $\dot\omega/\omega$ is obtained using equation (72) of CDA, viz.,

$$\frac{1}{\omega}\frac{\partial\omega}{\partial t} = \frac{1}{2J\Sigma^2}[I_1 + 2I_2 + I_5]\,. \tag{7}$$

Several of the integrals in (2) and (7) involve time derivatives of the thermodynamic quantities and thus require considerable reduction. This reduction is performed using the conservation-of-mass equation, viz.,

$$\rho_0 = \frac{1}{4\pi r_0^2 r_0'}\,, \tag{8}$$

and the linearized momentum equation

$$\frac{1}{\rho_0}\frac{\partial P_0}{\partial t} = \frac{\Gamma_{10}P_0}{\rho_0}\frac{\dot\rho_0}{\rho_0} + (\Gamma_3 - 1)_0\left(\epsilon - \frac{\partial L}{\partial m}\right). \tag{9}$$

The final expressions, then, are

$$-\kappa_{\mathrm{II}} = \frac{1}{2J\Sigma^2}[A1 + A2 + C + (1 - \tfrac{3}{2}\Gamma_{10})A3]\,, \tag{10}$$

$$\frac{1}{\omega}\frac{\partial\omega}{\partial t} = \frac{1}{2J\Sigma^2}[A4 - 2A2 + \Gamma_{10}A3]\,, \tag{11}$$

where

$$A1 = \int \frac{\dot r_0}{r_0}\left\{-\frac{6Gm}{r_0}\xi^2 + \frac{\Gamma_{10}P_0}{\rho_0}\left(\frac{\delta\rho}{\rho_0}\right)\left[-2\xi + (\Gamma_1 - 1)_0\frac{\delta\rho}{\rho_0}\right]\right\}dm\,,$$

$$A2 = \int \frac{\dot r_0'}{r_0'}\left\{\frac{\Gamma_{10}P_0}{\rho_0}\left(\frac{\delta\rho}{\rho_0}\right)\left[2\xi + \left(\frac{\Gamma_{10}+1}{2}\right)\frac{\delta\rho}{\rho_0}\right]\right\}dm\,,$$

$$A3 = \int (\Gamma_3 - 1)_0\left(\frac{\delta\rho}{\rho_0}\right)^2\left(\epsilon - \frac{\partial L}{\partial m}\right)dm\,,$$

$$A4 = \int \frac{\dot r_0}{r_0}\left\{\frac{12Gm}{r_0}\xi^2 - \frac{2\Gamma_{10}P_0}{\rho_0}\left(\frac{\delta\rho}{\rho_0}\right)\left[(\Gamma_1 - 1)_0\left(\frac{\delta\rho}{\rho_0}\right) - 2\xi\right]\right\}dm\,.$$

TABLE 1

THERMAL IMBALANCE CONTRIBUTIONS AND STABILITY COEFFICIENTS
(absolute radial amplitudes)

i	Demaret TI*	CDA TI	Demaret $-\sigma''$	CDA $-\kappa_{II}$
0	-0.4141(-14)	-0.4141(-14)	3.6485(-14)	+3.6498(-14)
1	-0.3864(-13)	-0.3863(-13)	6.6067(-13)	+6.6208(-13)
2	-0.1212(-12)	-0.1212(-14)	3.1958(-12)	+3.2078(-12)
3	-0.3995(-13)	-0.3996(-13)	2.6719(-12)	+2.6814(-12)
4	+0.2587(-13)	+0.2587(-13)	5.3758(-14)	+5.376(-14)
5	-0.5686(-13)	-0.5687(-13)	1.3811(-12)	+1.3812(-12)
6	-0.1870(-13)	-0.1872(-13)	1.9245(-12)	+1.9276(-12)

* TI denotes thermal imbalance contributions only.

For the present model, the values of r_0, $\dot{r}_0$ are available from the model and the values of r_0', $\dot{r}_0'$ (primes denote differentiation with respect to mass) are evaluated numerically from those quantities.

The results of the computation are shown in Table 1. The first two columns refer to the thermal imbalance terms according to CDA and Demaret, respectively; the last two columns refer to the stability coefficients for the same. Results for the fundamental mode and the first six overtones are tabulated. The perfect agreement is not surprising, since it is shown both by CDA and by Demaret that the expressions are equivalent and reduce from one to the other under assumptions which are valid for our model. This, however, provides a check that the integrals are evaluated correctly.

c) Absolute versus Relative Radial Amplitudes

Davey and Cox (1974) pointed out that, for a star in thermal imbalance, the stability coefficients for the absolute and for the relative radial amplitudes ($-\kappa_{II}$, $-\kappa_I$, respectively) are different and that the physically meaningful quantity for pulsational stability is the one for absolute radial amplitudes. However, the two coefficients are related by

$$-\kappa_{II} = -\kappa_I + I_0/J \tag{12}$$

where

$$I_0 = \int_0^M \frac{\dot{r}_0}{r_0} r_0^2 \xi^2 dm$$

and

$$J = \int_0^M |r_0\xi|^2 dm \quad \text{with } \xi = \partial r/r_0 \, .$$

Demaret (1975, 1976) also obtained a similar relationship between the coefficients.

We have computed $-\kappa_I$ and I_0/J for the model to see if the difference between the coefficients is significant and to determine how it compares with the other thermal imbalance contributions. The expression for $-\kappa_I$ is given in CDA, viz.,

$$-\kappa_I = \frac{1}{2J\Sigma^2}\left[\frac{I_1}{6} - 2M_2 + I_3 + I_4 - \tfrac{3}{2}I_5 + C + D\right], \tag{13}$$

where

$$M_2 = \int_0^M 16\pi^2\Gamma_{10}P_0\rho_0 r_0^6 \left(\frac{\partial\xi}{\partial m}\right)\left(\frac{\delta\rho}{\rho_0}\right)\frac{\partial}{\partial m}\left(\frac{\dot{r}_0}{r_0}\right)dm \, .$$

The other integrals are defined in § II*b*. After performing the necessary reductions, we obtain

$$-\kappa_I = \frac{1}{2J\Sigma^2}\left[A4 + A5 + C + (1 - \tfrac{3}{2}\Gamma_1)A3\right], \tag{14}$$

where

$$A4 = \int_0^M \frac{\dot{r}_0}{r_0}\left\{\frac{2Gm}{r_0}\xi^2 + \frac{\Gamma_{10}P_0}{\rho_0}\left(\frac{\delta\rho}{\rho_0}\right)\left[(\Gamma_3 - 3)_0\frac{\delta\rho}{\rho_0} - 4\xi\right]\right\}dm\,,$$

$$A5 = \int_0^M \frac{\dot{r}_0'}{r_0'}\left\{\frac{\Gamma_{10}P_0}{\rho_0}\left(\frac{\delta\rho}{\rho_0}\right)\left[\frac{(\Gamma_3 + 1)_0}{2}\left(\frac{\delta\rho}{\rho_0}\right) + 4\xi\right]\right\}dm\,.$$

Table 2 presents the results for these calculations. Columns designated (1) and (2) show the thermal imbalance contributions to $-\kappa_{\text{II}}$ using equations (2) and (12) respectively, while columns designated (3) and (4) give the corresponding total stability coefficients. It might also be mentioned that the stability coefficients for the absolute radial and relative amplitudes differ significantly for the fourth mode. For the same mode, the term I_0/J dominates not only the thermal imbalance terms but also the terms from nuclear burning. The differences between the thermal imbalance terms themselves for the different modes are not significant, keeping in mind the discussion presented in § II*a*.

d) *Energy Approaches: Demaret versus SS and DC versus SS*

A final comparison is made between the energy approaches as outlined in Demaret (1976) and Sastri and Simon (1973). The stability coefficient using this method according to Demaret is

$$-\sigma_E'' = -\sigma'' + \frac{1}{\omega}\frac{\partial\omega}{\partial t}\,. \tag{15}$$

According to Davey and Cox (1974), their equation (12) gives the rate of charge of total pulsational energy. When the quasi-adiabatic approximation is invoked and the initial model is assumed to be in hydrostatic equilibrium, the stability coefficient, $-\kappa_E$, is given by

$$-\kappa_E = -\kappa_{\text{II}} + \frac{1}{\omega}\frac{\partial\omega}{\partial t}\,,$$

which is essentially the same as (15).

The corresponding expression from Sastri and Simon is

$$\begin{aligned} -\kappa_s = \frac{1}{2J\Sigma^2}\Bigg\{&\int_0^M (\Gamma_3 - 1)_0\left(\frac{\delta\rho}{\rho_0}\right)\left[\delta\epsilon - \frac{\partial(\delta L)}{\partial m}\right]dm \\ &- \int_0^M (\Gamma_3 - 1)_0^2\left(\epsilon - \frac{\partial L}{\partial m}\right)\left(\frac{\delta\rho}{\rho_0}\right)^2 dm + \int_0^M \frac{\rho_0^2}{2T_0}\left(\frac{\partial^3 U}{\partial S\partial\rho^2}\right)_0\left(\epsilon - \frac{\partial L}{\partial m}\right)\left(\frac{\delta\rho}{\rho_0}\right)^2 dm \\ &+ \int_0^M 2(\Gamma_3 - 1)_0\left(\epsilon - \frac{\partial L}{\partial m}\right)\rho_{*0}\,dm\Bigg\}\,, \end{aligned} \tag{16}$$

TABLE 2*

THERMAL IMBALANCE CONTRIBUTIONS AND STABILITY COEFFICIENTS (relative radial and absolute radial amplitudes)

$-\kappa_{\text{I}}$(TI)	I_0/J	$-\kappa_{\text{I}}$(TI) + I_0/J (1)	$-\kappa_{\text{II}}$(TI) (2)	$-\kappa_{\text{I}}$	$-\kappa_{\text{I}} + I_0/J$ (3)	$-\kappa_{\text{II}}$ (4)
-0.4730(-14)	+0.0865(-14)	-0.3865(-14)	-0.4141(-14)	3.5909(-14)	3.6774(-14)	3.6498(-14)
-0.4927(-13)	+0.1268(-13)	-0.3659(-13)	-0.3864(-13)	6.5145(-13)	6.6413(-13)	6.6208(-13)
-0.1663(-12)	+0.0510(-12)	-0.1152(-12)	-0.1212(-12)	3.1628(-12)	3.2138(-12)	3.2078(-12)
-0.9290(-13)	+0.5745(-13)	-0.3545(-13)	-0.3996(-13)	26.2840(-13)	26.859(-13)	26.814(-13)
-0.0773(-13)	+0.3539(-13)	+0.2766(-13)	+0.2587(-13)	0.2016(-13)	0.5554(-13)	0.5376(-13)
-1.0620(-13)	+0.5245(-13)	-0.5375(-13)	-0.5687(-13)	13.319(-13)	13.844(-13)	13.812(-13)
-0.7482(-13)	+0.5906(-13)	-0.1583(-13)	-0.1872(-13)	18.686(-13)	19.276(-13)	19.247(-13)

*TI denote the thermal imbalance contributions only.

where ρ_{*0} is the second-order density variation given by

$$\rho_{*0} = 3x_*^2 + \tfrac{1}{2}r_0^2 x_*'^2 + 2r_0 x_* x_*' - 3u_* - r_0 u_*' \tag{17}$$

(where primes here denote differentiation with respect to r_0 and $x_* = \partial r/r_0$). It may be mentioned that the latter approach does not require a knowledge of the time derivatives of the spatial and thermodynamic quantities, while it includes the intrinsically second-order terms, u_*, in the analysis. That relation (15) is true for homologous oscillations of polytropes in homologous contraction and for homologous contraction in stars, has been shown by Davey and Cox (1974) and also by Demaret (1976). Our calculations for the polytrope $n = 1.5$ for the overtones have also led to the same result. For the present model, which does not fall under the above special categories, the calculations have been performed using the methods discussed in SS and the results are shown in Tables 3 and 4. Again keeping in mind the discussion as to the possible accuracy that can be expected, the results substantiate the agreement between the coefficients obtained with the two energy approaches.

III. DISCUSSION

As pointed out by Demaret (1976) and as is obvious from the present calculations for a prenova model, there do exist two distinct stability coefficients for a star in thermal imbalance. The question of which one of these is significant for an evolving model is beyond the purview of the present investigation. We can, however, speculate on what the second stability coefficient (viz., the one obtained through the energy methods) might mean in physical terms. For convenience of interpretation, we rewrite equations (6) and (15) as

$$-\sigma'' = -\frac{\dot{\omega}}{2\omega} + \frac{1}{2J\Sigma^2}\left\{\int_0^M \left(\frac{\delta T}{T_0}\right)\left[\delta\left(\epsilon - \frac{\partial L}{dm}\right)\right]dm - \int_0^M \left(\frac{\delta T}{T_0}\right)^2\left(\epsilon - \frac{\partial L}{\partial m}\right)dm\right\}, \tag{18}$$

$$-\sigma_E'' = +\frac{\dot{\omega}}{2\omega} + \frac{1}{2J\Sigma^2}\left\{\int_0^M \left(\frac{\partial T}{T_0}\right)\left[\delta\left(\epsilon - \frac{\partial L}{\partial m}\right)\right]dm - \int_0^M \left(\frac{\delta T}{T_0}\right)^2\left(\epsilon - \frac{\partial L}{\partial m}\right)dm\right\}. \tag{19}$$

The only difference between the expressions on the right-hand side of equations (18) and (19) is the sign of $\dot{\omega}/2\omega$. In general, $\dot{\omega}/2\omega$ is a small quantity compared to the term within braces, $\{\}$. If, at some stage of evolution with thermal imbalance or for some normal mode, the two terms become comparable, then two situations arise: (i) If $|\dot{\omega}/2\omega| < |\{\}|$, then $-\sigma''$ and $-\sigma_E''$ are of the same sign and the model is stable or unstable depending on that sign. (ii) If $|\dot{\omega}/2\omega| > |\{\}|$, then $-\sigma''$, $-\sigma_E''$ are of different signs; a positive $\dot{\omega}/2\omega$ would make $-\sigma''$ negative, thus leading to the stability of absolute radial amplitudes. This reduction in the radial amplitudes leads to a decrease in the variation of the potential energy of the star. Since the star is evolving rapidly at this stage, no readjustment of the structure can take place; instead, the readjustment takes place in the form of an increase in the variation of the kinetic energy over a period; this in turn leads to an increase in the velocity amplitudes. Now, if we look at equation (19), we see that in this particular situation $-\sigma_E''$ is positive and large. Thus it is possible to identify the second coefficient with the stability of the velocity amplitudes. (That the energy approach of Demaret [1976] leads, in the case of thermal imbalance, to the determination of stability of velocity amplitudes is shown in Simon [1977].) A similar argument to the above can be used in case $\dot{\omega}/2\omega$ is negative with the difference that the readjustment would be in the opposite direction. In the absence of thermal imbalance, however, $\dot{\omega}/\omega$ is zero and both the stability coefficients merge into one.

TABLE 3

STABILITY COEFFICIENTS USING THE ENERGY APPROACHES

i	$-\sigma''$	$\frac{1}{\omega_0}\frac{\partial\omega_0}{\partial t}$	$-\sigma_E''$	$-\kappa_s$
0	0.36485(-13)	+0.26396(-14)	0.39125(-13)	+0.36121(-13)
1	0.66067(-12)	-0.21180(-13)	0.63949(-12)	+0.58404(-12)
2	0.31958(-11)	-0.23621(-12)	0.29236(-11)	+0.28116(-11)
3	0.26719(-11)	-0.33325(-12)	0.23387(-11)	+0.22754(-11)
4	0.53758(-13)	-0.71684(-13)	-0.17926(-13)	-0.23475(-13)
5	0.13811(-11)	-0.91585(-13)	0.12895(-11)	+0.12195(-11)
6	0.19245(-11)	-0.28434(-12)	0.16402(-11)	+0.15650(-11)

TABLE 4

THERMAL IMBALANCE CONTRIBUTIONS USING THE ENERGY APPROACHES

i	$-\sigma''$ (TI only) Demaret	$\frac{1}{\omega}\frac{\partial \omega}{\partial t}$	$-\sigma_E''$ (TI only) Demaret	$-\kappa_S$ (TI only) SS
0	-0.4141(-14)	+0.2640(-14)	-0.1501(-14)	-0.4282(-14)
1	-0.3864(-13)	-0.2118(-13)	-0.5982(-13)	-0.8095(-13)
2	-0.1212(-12)	-0.2362(-12)	-0.3574(-12)	-0.4217(-12)
3	-0.04(-12)	-0.3333(-12)	-0.3733(-12)	-0.3962(-12)
4	+0.2587(-13)	-0.7168(-13)	-0.4581(-13)	-0.5001(-13)
5	-0.5686(-13)	-0.916(-13)	-0.1486(-12)	-0.2175(-12)
6	-0.0187(-12)	-0.2843(-12)	-0.3030(-12)	-0.3246(-12)

From the above arguments we see that both the small-amplitude approach and the total-energy approach lead to physically meaningful quantities, namely, the stability coefficients for the absolute radial amplitudes and for the velocity amplitudes, respectively.

A few comments should be made regarding the treatment of the model and the application of the methods to this model. The 1 $M_\odot$ prenova model available to us is one of an evolutionary sequence and consists essentially of the outer hydrogen-burning shell with only 44 mass zones. As mentioned in SS, we have fitted a suitable polytrope to the interior and performed quadratic interpolation in the model itself to obtain a finer grid. As happens with the averaging and interpolation techniques, the initial static model is somewhat disturbed by this procedure. This departure from equilibrium structure is strongest in the interior zones, and these are the zones which contribute largely to thermal imbalance terms. For the fundamental mode, the radial amplitudes are large in this region and the differences do appear in the computed integrals—in particular, in the thermal imbalance contributions (see Table 4, last two columns). For the overtones, on the other hand, the radial amplitudes fall off and the differences in the integrals are substantially smaller.

Secondly, the absolute radial displacements in the linear analysis are given by equation (57) of Demaret (1974) for isentropic oscillations and equation (51) of Demaret (1975) for nonisentropic oscillations. In the former case, the radial amplitudes are amplified by the factor $1/\sqrt{\omega}$ and in both cases, the eigenfunctions are modified by some factor which, according to Demaret, represents "a kind of coupling between the different adiabatic pulsation modes." For the present model, several modes are unstable, and some of the unstable modes suffer from strong resonances. In Table 5 the values of ω_i for different modes and their ratios with the fundamental eigenvalue ω_0 are given. (Note that these ω's are the normalized eigenfrequencies and are different from the other ω's used in the paper.) The disagreement (for modes 0, 1, 5; Table 4) between the computed thermal imbalance terms is larger than expected precisely for these modes. Thus these differences are not alarming in view of the close resonances between the unstable modes. However, in view of the fact that, even in the linear analysis, the absolute radial

TABLE 5

NORMALIZED EIGENVALUES FOR DIFFERENT NORMAL MODES AND THE RATIOS WITH THE FUNDAMENTAL

i	ω_i^2	ω_i	ω_i/ω_o
0	3.5241	1.8773	1.00
1	14.674	3.8307	2.04
2	28.533	5.3416	2.84
3	42.867	6.5473	3.49
4	63.377	7.9610	4.24
5	88.157	9.3892	5.001
6	112.19	10.592	5.65

TABLE 6a

FUNDAMENTAL RADIAL FREQUENCIES FOR EVOLVING NOVA MODELS

Model	Evolution Time (10^{11} seconds)	$\omega^2_{o,t}$
Sp. 100	5.3838758	0.97209
Sp. 102	5.3859782	0.97197
Sp. 104	5.3864754	0.97197
Sp. 110	5.3882635	0.97205
Sp. 150	5.3971824	0.97195

displacements given by equation (51) of Demaret (1975) are modified by the resonances from other modes and since the total energy treatment of Sastri and Simon (1973) is not affected by the resonances (note that the resonances between modes affect the second order terms w_* as shown in Simon and Sastri [1972] and not u_*), we tend to believe that the treatment applied by SS is probably valid in all situations. Nevertheless, the evaluation of u_* in the case of multimodal pulsational instability is subject to some uncertainty (see discussion in SS); thus the previous statement should be treated with some caution.

A third point concerns the evaluation of $\dot{\omega}/\omega$. A numerical procedure where the frequencies from two nearby models are used in the form $(\omega_{t_1} - \omega_{t_0})/[\omega_{t_0}(t_1 - t_0)]$ has been adopted by Cox (1974) in estimating the stability coefficients for helium-shell-burning stars. While this may be correct for the models of Gingold and Faulkner (1974) as used in Cox (1974), a similar procedure between two models of the present sequence, viz., models Sp 100 and Sp 110, has led to inconsistent results. The results are shown in Tables 6a and 6b. Apart from the quantitative differences which could be due to the extremely small values of $(\omega_{t_1} - \omega_{t_0})$, there is a qualitative difference. In particular, the evaluation using the integral expressions shows that the fundamental period of the model is decreasing, while the overtone periods increase. On the other hand, from the numerical evaluation procedure, we obtain increasing periods for all the modes. This inconsistency may be understood if one notices that the implicit assumption, that ω_t is a smooth or linear function of time at least between the models considered, is not correct. Table 6a lists the fundamental eigenfrequencies for a chosen set of models from the sequence and their evolution times. The time intervals are smaller than the secular time scales defined in § IV of CDA. This irregularity in period variation may be purely due to the local changes in the thermodynamic variables occasioned by the increased nuclear burning in the shell. For example, between models Sp 100 and Sp 104, the nuclear burning in the bottom shell is enhanced sixfold, and the luminosity rose by 25%; but the overall properties like the surface luminosity and surface radius are essentially the same. The variations on a global scale are evident only when the periods for models Sp 100, Sp 110, and Sp 150 are compared. But these variations do not reflect the instantaneous fluctuations leading to continuous readjustment in the interior. Thus a numerical evaluation of $\dot{\omega}/\omega$ between models Sp 100 and Sp 110, or among models Sp 90, Sp 100, and Sp 110, is probably inaccurate. Table 6b brings out the nature of the qualitative and quantitative differences between the values obtained from instantaneous values through the integrals and those obtained through numerical evaluation. Thus some caution is needed while using $\dot{\omega}/\omega$ from evolutionary models in determining the pulsational stability of a given model.

Finally in evaluating which method is appropriate for determining stability, we note that both the methods seem to suffer from some drawbacks. The small amplitude approach requires the knowledge of time derivatives, viz., $\dot{P}$, $\dot{T}$, $\dot{\rho}$, etc. At least two different models in time are required to evaluate these quantities. While it may, in some

TABLE 6b

$\dot{\omega}/\omega$ EVALUATED FROM NUMERICAL VALUES AND FROM INTEGRAL EXPRESSIONS

Mode	0	1	2	4
ω_i^2: Sp. 110	0.97205	4.0468	7.8233	17.479
ω_i^2: Sp. 100	0.97209	4.0478	7.8698	17.480
$\dot{\omega}/\omega$ (Num)	-0.469(-13)	-0.282(-12)	-0.674(-11)	-0.652(-13)
$\dot{\omega}/\omega$ (Comp)	+0.2640(-14)	-0.2118(-13)	-0.2362(-12)	-0.7168(-13)

cases, be possible to absorb these quantities into $\dot{\omega}/\omega$ (thus requiring only their periods to be known as in Cox [1974]), this may not always be accurate, as has been pointed out earlier. On the other hand, with the total dynamical energy approach of SS, the stability coefficients for both the radial displacements and velocity amplitudes may be determined, when one combines equations (6), (15), and (16) and identifies that

$$\frac{1}{\omega}\frac{\partial\omega}{\partial t} \approx \frac{1}{2J\Sigma^2}\left[\int_0^M \frac{\rho_0{}^2}{2T_0}\left(\frac{\partial^3 U}{\partial S\partial\rho^2}\right)_0\left(\epsilon - \frac{\partial L}{\partial m}\right)\left(\frac{\delta\rho}{\rho_0}\right)^2 dm + 2\int_0^M (\Gamma_3 - 1)_0\left(\epsilon - \frac{\partial L}{\partial m}\right)\rho_{*0} dm\right]. \tag{20}$$

Then the stability coefficient for absolute radial amplitudes $-\kappa_{s,r}$, would be

$$-\kappa_{s,r} = \frac{1}{2J\Sigma^2}\left\{\int_0^M (\Gamma_3 - 1)_0\left(\frac{\delta\rho}{\rho_0}\right)\left[\delta\epsilon - \frac{\partial}{\partial m}(\delta L)\right]dm - \int_0^M (\Gamma_3 - 1)_0\left(\epsilon - \frac{\partial L}{\partial m}\right)\left(\frac{\delta\rho}{\rho_0}\right)^2 dm \right.$$
$$\left. - \frac{1}{2}\int_0^M \frac{\rho_0{}^2}{2T_0}\left(\frac{\partial^3 U}{\partial S^{\ 2}}\right)_0\left(\epsilon - \frac{\partial L}{\partial m}\right)\left(\frac{\delta\rho}{\rho_0}\right)^2 dm - \int_0^M (\Gamma_3 - 1)_0\left(\epsilon - \frac{\partial L}{\partial m}\right)\rho_{*0} dm\right\},$$

while the coefficient for the velocity amplitudes is given by equation (16). However, this approach, while not requiring the knowledge of two separate models, suffers from the inconvenience of having to solve another second-order differential equation (eq. [8] of Simon 1971) to determine the second-order quantity u_* and its spatial derivatives. Thus which method is appropriate depends essentially on the nature of the models available for analysis.

IV. CONCLUSIONS

The thermal imbalance contributions and the stability coefficients for a thin hydrogen-shell-burning 1 $M_\odot$ nova model have been determined for absolute radial displacements using the expressions given by Demaret and CDA, and it is found that the results show excellent agreement. Next, the stability coefficients for the relative radial and absolute radial amplitudes are obtained. To the order of accuracy expected (see § II*a* for details), the thermal imbalance contributions show good agreement. To nearly the same accuracy, the two total energy approaches (viz., Demaret and SS or DC and SS), using the sum of the kinetic and potential energies and the total dynamical energy respectively, are shown to agree. Arguments are presented to provide physical meaning for the stability coefficient obtained through the energy methods, and it is shown that it represents the stability of the model for velocity amplitudes. Obviously the advantage of not requiring the time derivatives of the radius, pressure, density, etc., in the model is balanced by the necessity of including the second-order terms in the total dynamical energy approach of SS. While the existence of resonances between unstable modes might affect the results in other methods, inclusion of second-order terms which are not sensitive to resonances in the treatment of SS (keeping in mind the note of caution mentioned in the discussion) might make the latter more effective.

Concerning the results for the model studied, the multimodal radial pulsational instability in a hydrogen-shell-burning star, reported earlier by SS, is confirmed. The present model is from an evolutionary sequence different from the one used in SS. Both the absolute radial and the velocity amplitudes are found to be unstable in several modes. There is also a situation, not encountered or studied in the literature before, where the radial amplitudes seem to grow while the velocity amplitudes seem to decline. This corresponds to a different kind of oscillation with large radial amplitudes but small velocity amplitudes. (The author is indebted to Norman Simon for elaborating on this idea, and also for suggesting that the energy approaches might lead to the determination of stability of the velocity amplitudes.) The thermal imbalance terms, in general, are small and have a stabilizing effect. In one particular case, viz., for the fourth mode or the fifth normal mode, the thermal imbalance terms outweigh the energizing terms and lead to stability of the velocity amplitudes. This situation, according to the definitions in Simon (1977), corresponds to the case of displacement (or D) instability.

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