EFFECT OF GRAVITON CREATION IN ISOTROPICALLY EXPANDING UNIVERSES ★

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Graviton creation in a Robertson-Walker universe is calculated using a renormalized energy-momentum tensor. If the background matter has an equation of state $p = \gamma\rho$, then for $\gamma > \frac{1}{3}$, graviton energy created near the Planck time t_p can alter the expansion to $a \sim t^{1/2}$ in $\lesssim 10^2 t_p$.

Particle creation by the gravitational field in different cosmological backgrounds has been considered by various authors [1–13]. Near the Planck time $t_p \simeq 10^{-43}$ sec it is a powerful mechanism that could have strongly influenced the state of the early universe. Here we investigate the effect of gravitons created by the expansion near t_p in a Friedmann universe containing background matter with equation of state $p_m = \gamma\rho_m$.

Graviton creation. Gravitational wave perturbations $h_i{}^j(x, t)$ in a flat isotropic universe with Robertson-Walker (RW) metric [$\hbar = c = G = 1$]

$$ds^2 = dt^2 - a^2(t)(dx_1{}^2 + dx_2{}^2 + dx_3{}^2) , \qquad (1)$$

obey the Lifshitz equation (here the synchronous condition $h_{0\mu} = 0$ and transverse, traceless conditions are imposed [14]),

$$\ddot{h}_i{}^j(x, t) + 3\frac{\dot{a}}{a}\dot{h}_i{}^j(x, t) - \frac{1}{a^2}\Delta h_i{}^j(x, t) = 0 ,$$

$$(i, j = 1, 2, 3) , \qquad (2)$$

where $\cdot \equiv d/dt$ and Δ is the Laplace-Beltrami operator on a constant t hypersurface, with eigenvalues k and eigenfunctions $G^{(k)}{}_i{}^j(x)$. Each component $h_k(t)$ of the wave $h_i{}^j(x, t) = \int d^3k\, h_k(t) G^{(k)}{}_i{}^j(x)$, obeys the wave equation in τ-time (defined by $dt = V\, d\tau$ with $V = a^3$),

$$h''_k(\tau) + \Omega_k{}^2(\tau) h_k(\tau) = 0 , \qquad (3)$$

where $' \equiv d/d\tau$ and $\Omega_k = V\omega_k$ with $\omega_k = k/a$. This equation is of the same form as that of a massless scalar wave minimally coupled to a flat RW background. The solution to the wave equation (3) can be written in the form (the subscript k on h_k will be dropped hereafter)

$$h(\tau) = \frac{\alpha}{\sqrt{2\Omega}} e_- + \frac{\beta}{\sqrt{2\Omega}} e_+ , \qquad e_\pm = \exp(\pm i \int^\tau \Omega\, d\tau), \qquad (4)$$

where α and β are complex functions of τ. The Wronskian condition guarantees that $|\alpha|^2 - |\beta|^2 = 1$. The quantity $s = \beta\beta^* = \frac{1}{2}(\Omega|h|^2 + \Omega^{-1}|h'|^2)$ gives a measure of the parametric amplification of the wave, corresponding to quantum mechanical particle production. Near the singularity, for small t or low ω, $|\int^\tau \Omega\, d\tau| \ll 1$, the sudden approximation to the wave equation (3) yields [4 (1971)]

$$s = |C_1|^2\Omega + |C_2|^2/\Omega - \tfrac{1}{2} , \qquad (5)$$

where C_1 and C_2 are complex constants satisfying $C_1^* C_2 + C_1 C_2^* = 1/2$.

Regularized energy-momentum tensor. Gravitational wave perturbations on the metric of eq. (1) can be described by two independent scalar waves corresponding to the two polarizations of the gravitational wave, each of which obeys eq. (2), and can be quantized in the same way as a minimally coupled scalar field [15].

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The effective energy-momentum tensor $T_{\mu\nu}$ of these quantized gravitational waves is taken to be the sum of the energy-momentum tensors corresponding to those two minimally coupled scalar fields. In a semi-classical theory in which the background metric is treated classically, the energy density ρ and pressure p of the gravitons are given as quantum expectation values $\rho = \langle T_0{}^0 \rangle$, $p\delta_i{}^j = -\langle T_i{}^j \rangle$. The state vector is chosen such that no quanta corresponding to waves of the form (4) are present. (Complete specification of the state vector still requires a choice of initial conditions on α and β.) Then one finds that [6]

$$\rho = \langle T_0{}^0 \rangle = (16\pi)^{-1}(2\pi)^{-3} \int \mathrm{d}^3k\, \rho_0(k)\,, \tag{6}$$

where $\rho_0 = |\dot{h}_k(t)|^2 + \omega_k{}^2(t)|h_k(t)|^2 = V^{-2}\Omega(1+2s)$. As is well-known from flat space quantum theory, this integral is divergent in the high frequency end of the spectrum, and some kind of renormalization is necessary to yield a physically meaningful energy density. We expect our results to be insensitive to the renormalization method. By means of adiabatic regularization [6], one obtains the finite energy density

$$\rho_{\mathrm{reg}} = (128\pi^4)^{-1} \int \mathrm{d}^3k\, [\rho_0(k) - \rho_{\mathrm{div}}(k)]\,, \tag{7}$$

where

$$\rho_{\mathrm{div}}(k) = a^{-4}[k + \dot{a}^2(2k)^{-1} + A(8k^3)^{-1}]$$

and

$$A = a^4[(\ddot{a}/a)^2 - 2(\ddot{a}/a)(\dot{a}/a)^2 - 2(\dddot{a}/a)(\dot{a}/a) + 3(\dot{a}/a)^4]\,.$$

If one restricts oneself to the confines of Einstein's theory with only a linear curvature term in the gravitational Lagrangian, then it will not be compatible to include subtractions, like those involving A, which are higher than second order. (However, if such a term were included, it might give rise to avoidance of the singularity [17], or other effects.)

Reaction on the background. Near the singularity, or at low frequencies, s can be approximated by eq. (5). As our purpose is to investigate the back reaction of created gravitons, we choose C_1 and C_2 such that the quantity $\rho_0(k) - V^{-2}\Omega$ (proportional to the energy density in mode k with the usual vacuum energy subtracted) will vanish at the initial time t_0. Thus, we take $|C_1|^2 = (4\Omega_0)^{-1}$, $|C_2|^2 = \Omega_0/4$, where $\Omega_0 = a_0{}^2 k$, which corresponds to the initial condition $\beta_k = 0$ at t_0. At a given time t, the graviton energy density ρ consists of two parts, a quantum term ρ_{q} and a classical remnant ρ_{c}. The first part ρ_{q} is the contribution to eq. (7) of all modes $\boldsymbol{k}$ satisfying the condition $\omega \leqslant \omega_{\mathrm{m}} = 1/t$. Those are the modes in which quantum effects such as graviton creation and vacuum polarization are significant. The second part ρ_{c} is the contribution of the higher frequency modes, which will be treated classically, and consists of the remnant of the gravitons created in those modes $\boldsymbol{k}$ at earlier times when they were in the quantum regime. The quantum contribution ρ_{q} is found by integrating eq. (7) over all $\boldsymbol{k}$ with $k = |\boldsymbol{k}| < k_{\mathrm{m}} = a/t$, using the approximation of eq. (5). One finds

$$\rho_{\mathrm{q}} = \frac{1}{128\pi^3}\left[\frac{k_{\mathrm{m}}{}^4}{2a^4}\left(\frac{a}{a_0} - \frac{a_0}{a}\right)^2 - \frac{k_{\mathrm{m}}{}^2}{a^2}\left(\frac{\dot{a}}{a}\right)^2\right]$$

$$= \frac{1}{128\pi^3}\left[\frac{2\sinh^2 x}{t^4} - \frac{1}{t^2}\left(\frac{\dot{a}}{a}\right)^2\right], \tag{8}$$

where $x = \ln(a/a_0)$. The expression for ρ_{q} does not vanish at t_0 because of the higher order subtraction involving $(\dot{a}/a)^2$. That term is only significant for an interval of less than 0.1 t_{p} after t_0, and has negligible influence on the time for graviton creation to alter the expansion to a $\sim t^{1/2}$.

The classical remnant of created gravitons behaves like a relativistic fluid. Its energy density ρ_{c} at time t is given by the sum of all previous increments $\delta\rho_{\mathrm{c}}(k_{\mathrm{m}}(t'), t')$ at earlier times $t' < t$, arising from the change of $k_{\mathrm{m}}(t')$ with time, and being red-shifted by a factor $[a(t')/a(t)]^4$. Thus, the energy density of the classical remnant of all previously created gravitons is

$$\rho_{\mathrm{c}}(t) = \int_{t_0}^{t} \left[\frac{a(t')}{a(t)}\right]^4 \left[-\frac{\partial\rho_{\mathrm{q}}(k_{\mathrm{m}}(t'), t')}{\partial k_{\mathrm{m}}}\right]_a \frac{\partial k_{\mathrm{m}}(t')}{\partial t'}\,\mathrm{d}t'\,, \tag{9}$$

where the subscript a on $\partial\rho_{\mathrm{q}}/\partial k_{\mathrm{m}}$ indicates that $a(t')$ is kept fixed during differentiation. One can write $\rho_{\mathrm{c}}(t)$ in the form

$$\rho_{\mathrm{c}}(t) = (16\pi^3)^{-1}(a_0/a(t))^4 F(t)\,, \tag{10}$$

where $F(t)$ can be found by integration of (9) if $a(t')$ is given.

For a Robertson-Walker background containing a fluid with equation of state $p_{\mathrm{m}} = \gamma\rho_{\mathrm{m}}$ $(0 \leqslant \gamma \leqslant 1)$, which is not interacting with the created gravitons, the conservation law $T^{\mu\nu}{}_{;\nu} = 0$ yields

$$\rho_m = \rho_{0m}(V/V_0)^{-(1+\gamma)} . \tag{11}$$

To find the effect of graviton creation, one adds the graviton energy density $\rho_g = \rho_q + \rho_c$ to the matter source ρ_m, and solves the Einstein equation

$$3(\dot{a}/a)^2 = 8\pi(\rho_m + \rho_g) . \tag{12}$$

With $\rho_c(t)$ given as a functional of $a(t)$, the above equation becomes an integro-differential equation. However, the integrand of eq. (9) will become small after a short time interval, so that $\rho_c a^4$ becomes constant after that time. Therefore, one only needs $a(t)$ at early times ($t \lesssim 1$) to find $\rho_c(t)$. Because $\rho_m + \rho_g \sim \rho_m$ for $t \lesssim 1$, we approximate $a(t)$ for $t \lesssim 1$ as the solution of the Einstein equation with ρ_g neglected relative to ρ_m; namely $a = a_0(t/t_0)^\alpha$, with $\alpha = 2[3(\gamma+1)]^{-1}$. [In this case $\rho_{0m} = 3\alpha^2/(8\pi t_0{}^2)$.] Thus, substituting ρ_q from eq. (8) and $a = a_0(t/t_0)^\alpha$ in eq. (9), one finds that

$$F(x) = \frac{1}{8t_0{}^4}\left[\frac{e^{2(3-2/\alpha)x} - 1}{3\alpha - 2} + \frac{e^{2(1-2/\alpha)x} - 1}{\alpha - 2} - \frac{e^{4(1-1/\alpha)x} - 1}{\alpha - 1}\left(1 + \frac{\alpha^2}{2}\right)\right] , \tag{13}$$

where $x = \ln(a/a_0)$. For the case $\alpha = 2/3$, the first term is replaced by $2x/\alpha$. If the $(\dot{a}/a)^2$ subtraction were omitted from ρ_q, then the factor of $(1 + \alpha^2/2)$ would be replaced by 1 in eq. (13). The function $F(x)$ becomes constant after a short time (except for $\alpha = 2/3$, when it goes as $\ln(t/t_0)$). With ρ_m, ρ_g and ρ_c given by eqs. (11), (8), (10) and (13), respectively, Einstein's equation becomes

$$\dot{x}^2 = \left(\frac{\alpha^2}{t_0{}^2} e^{-2x/\alpha} + \frac{\sinh^2 x}{24\pi^2 t^4} + \frac{1}{6\pi^2} e^{-4x} F(x)\right) \Big/ \left(1 + \frac{1}{48\pi^2 t^2}\right) . \tag{14}$$

We have solved this equation for $x(t)$ numerically for different values of γ. The initial time t_0 was taken as the time when ρ_m is about equal to the Planck energy $c^7\hbar^{-1}G^{-2} \sim 10^{114}$ erg/cm^3, which gives $t_0 \approx \alpha(3/8\pi)^{1/2}$. The behaviour of $a(t)$ is shown in fig. 1 for several values of γ.

The graviton energy density depends on the time t_0 when particle creation begins. In the present models, as t_0 is between 0.1 and 0.2, the effect of the second order subtraction term in eq. (8) is largely insignificant. The densities ρ_c and ρ_q build up to a maximum almost instantaneously, and then decrease as a result of the expansion and the rapid fall off in production rate [ρ_q is initially negative (not shown in figures) but

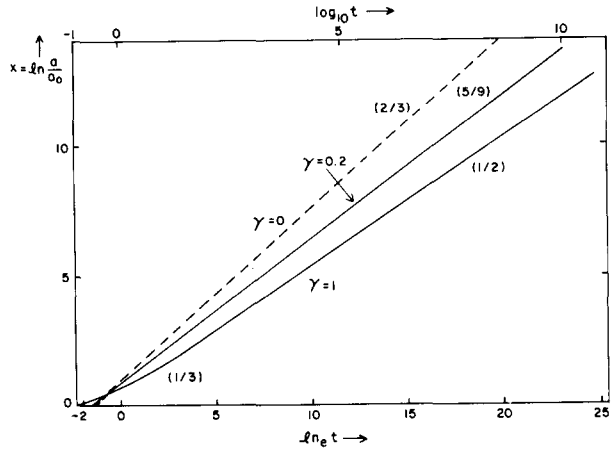

Fig. 1. Relative radius $x = \ln(a/a_0)$ versus time $\ln t$ for a RW universe with graviton creation. The $\gamma = 0.2$ solution starts at $t_0 = 0.192$ and remain Friedmannian all along with a slope α [$\equiv 2/(3(\gamma+1))$] = 5/9 (shown in parentheses) while the $\gamma = 1$ solution starts at $t_0 = 0.115$ with $\alpha = 1/3$ but turns into a $\alpha = 1/2$ solution in $t_0 \simeq 2$ (unit of time is the Planck time $t_p = 5.4 \times 10^{-44}$ sec).

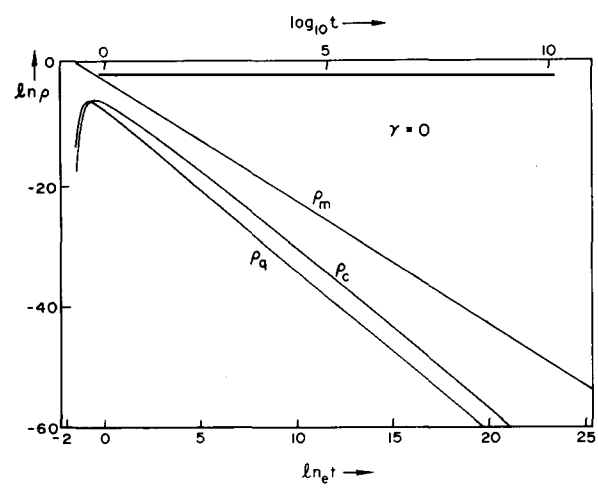

Fig. 2. $\ln \rho$ versus $\ln t$ for $\gamma = 0$. ρ_m dominates for all t (unit of energy density is the Planck density $\rho_p = 5 \times 10^{114}$ erg/cm^3).

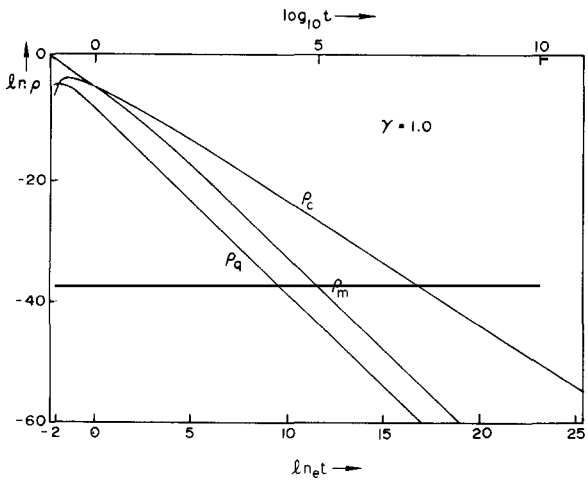

Fig. 3. $\ln p$ versus $\ln t$ for $\gamma = 1$. ρ_c overtakes ρ_q almost immediately, and becomes equal to ρ_m at $t_e \simeq 2$.

becomes positive almost immediately.] The graviton energy comes mainly from the classical remnant after a short time $t \lesssim 1$. For $\gamma \leqslant 1/3$ (see fig. 2), ρ_m dominates and $a \sim t^\alpha$ for the entire time (for $\gamma = 1/3$ there is no graviton creation in the exact solution). For $\gamma > 1/3$ (see fig. 3), ρ_m dominates ($a \sim t^\alpha$) until a time t_e when $\rho_g = \rho_m$, after which $\rho_g (\sim \rho_c)$ becomes dominant and $a \sim t^{1/2}$. In general, for non-interacting fluids $\rho_m \sim a^{-3(1+\gamma)}$ and $\rho_g \sim a^{-4}$, so that ρ_g must eventually dominate for $\gamma > 1/3$, as noted previously by Grishchuk in ref. [7 (1976)]. We find from our numerical solution that for $\gamma = 1$, $t_e \sim 2$, for $\gamma = 0.7$, $t_e \sim 20$, for $\gamma = 0.4$, $t_e \sim 60$, and for $\gamma < 1/3$, $t_e = \infty$ (see fig. 1). For the values of $\gamma > 1/3$ considered, the expansion becomes radiation dominated by $t \lesssim 10^2$. Our calculations show that in these models graviton creation can have a significant effect on the expansion of the universe.

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