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STUDY OF ATMOSPHERIC TRANSPORT OVER AREA SOURCES BY AN INTEGRAL METHOD

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Abstract—An approximate solution of the steady state two-dimensional diffusion equation, obtained by an integral method, is presented. The solution can be used to predict dispersion of pollutants in the atmosphere over an area–source such as an urban area. The method is applied to the calculation of sulfur dioxide concentration distribution over a 2-h period in Nashville, Tennessee. Calculated values are found to agree well with observations in terms of both the correlation coefficient and the magnitude of relative error.

1. INTRODUCTION

The problem of advective and diffusive transport of emissions from area sources is of interest because of its direct bearing on predicting dispersal of contaminants in urban areas. Currently used models of pollution dispersion range from those aiming to numerically solve the three-dimensional diffusion equation to those containing only one parameter. Recently one of us has compared the merits of the different models with the conclusion that further development of easy to apply but reliable methods is relevant (Hameed, 1974a).

In this paper we describe a simple solution of the steady state, two-dimensional diffusion equation which describes advection in the direction of the wind and diffusion in the vertical direction. The solution is obtained by the integral method originally used in boundary-layer theory (Schlichting, 1968) and in the study of heat transfer problems (Goodman, 1964). The basic idea of this method is to replace the problem of solving the partial differential equation by that of seeking a solution of an ordinary differential equation. This equation is obtained from the original equation by integration over the depth of the contaminated layer such that the total pollutant flux in the layer is conserved. The solution satisfies the correct boundary conditions at the ground and at the top of the layer, together with other compatibility relations. In spite of its simplicity, the integral method has been found to give solutions which usually agree well with exact solutions whenever such a comparison is possible (Goodman, 1964). This encourages us in using it to study the mathematically analogous problem of pollution dispersal over area sources. In Section 2 we give a description of the integral method. Its adaptation to the air pollution problem is discussed in Section 3, while in Section 4 we give an application of the method to the calculation of dispersal of sulfur dioxide in a 2-h, steady-state, period in the atmosphere of Nashville, Tennessee.

2. THE INTEGRAL METHOD

Consider the semi-infinite volume bounded by the planes $z = 0$ and $x = x_0$. If wind is in the x-direction then, under steady state conditions, concentration $X(x, z)$ is given by the diffusion equation

$$u(z)\frac{\partial X}{\partial x} = \frac{\partial}{\partial z} K(z) \frac{\partial X}{\partial z} \tag{1}$$

where u is the mean wind velocity and K is the turbulent diffusion coefficient. Diffusion in the x- and y-directions is neglected. It is assumed that the pollutants are emitted as a steady flux at the surface $z = 0$, i.e.

$$K(z)\frac{\partial X}{\partial z} = -Q(x); \quad z = 0. \tag{2}$$

It is also assumed that, at a given x, the pollutants rise to a height δ, i.e.

$$X(x, z) = 0; \quad z = \delta(x). \tag{3}$$

and that there is no flux outside the contamination layer

$$\frac{\partial X}{\partial z} = 0; \quad z \geq \delta(x). \tag{4}$$

From physical considerations it is further assumed that the top boundary of the layer varies smoothly in the x-direction, i.e. $\partial X/\partial x = 0$ at $z = \delta$. This, together with equations (1 and 4), gives the "smoothing condition",

$$\frac{\partial^2 X}{\partial z^2} = 0, \quad z = \delta(x). \tag{5}$$

The wind velocity and the diffusion coefficient may be written in the empirical form:

$$u(z) = u_1 \left(\frac{z}{z_1}\right)^m; \quad K(z) = K_1 \left(\frac{z}{z_1}\right)^n + K_0.$$

Note that $K(z = 0) = K_0$. The diffusion coefficient should be non-zero at the point of emission for vertical diffusion to be possible.

In the integral method one approximates the z-dependence of the solution for X by a simple function such as a polynomial. Value of the solution at the ground, $X(z = 0)$, in which we are primarily interested, turns out to be insensitive to this approximation (Goodman, 1964). Boundary conditions (3–5) are satisfied if we take the polynomial to be of third order, and write the solution to equation (1) as

$$X(x, z) = c(x)\left(1 - \frac{z}{\delta}\right)^3. \tag{6}$$

Condition (2), then, gives

$$c(x) = \frac{\delta(x)\, Q(x)}{3K_0}. \tag{7}$$

Thus, the only unknown in the solution is $\delta(x)$, the depth of the polluted layer. An expression for $\delta(x)$ is obtained by integrating equation (1) from 0 to δ:

$$\int_0^{\delta} u(z)\frac{\partial X}{\partial x}\,dz = K(z)\frac{\partial X}{\partial z}\Big|_{z=0}^{z=\delta} = Q(x) \tag{8}$$

using equations (4 and 2). Integration over x then gives

$$\int_{x_0}^{x} Q(x')\,dx' = \int_0^{\delta} u(z)X(x,z)\,dz = \frac{1}{3}\frac{u_1}{z_1^m K_0}B(m+1,4)Q(x)\,\delta(x)^{m+2} \tag{9}$$

where $B(m,n) = [\Gamma(m)\Gamma(n)]/[\Gamma(m+n)]$ *is the* β-function. It is assumed that $c(x_0) = 0$, i.e. $\delta(x_0) = 0$.

We now assume the source strength to be independent of x. It is convenient to first obtain the solution for constant source strength and to extend it later to the type of variable sources appropriate to urban areas, as shown in Section 3. Putting $Q(x) = Q_0$ in equation (9) we get:

$$\delta(x) = \left[\frac{3z_1^m K_0}{u_1 B(m+1,4)}(x - x_0)\right]^{1/m+2}. \tag{10}$$

Substitution for δ in equation (7) gives us the concentration at the ground:

$$c(x) = \frac{1}{3}\frac{Q_0}{K_0}\left[\frac{3z_1^m K_0}{u_1 B(m+1,4)}\right]^{1/m+2}(x - x_0)^{1/m+2}. \tag{11}$$

An estimate of the error in the solution given by equation (11) may be obtained by considering the case when the velocity u is independent of z, i.e. $m = 0$. Equation (11) in this case reduces to

$$c(x) = \frac{2}{\sqrt{3}}\frac{Q_0}{(u_1 K_0)^{1/2}}(x - x_0)^{1/2}, \tag{12}$$

while the exact solution is (Carslaw and Jaeger, 1959):

$$c(x) = \frac{2}{\sqrt{\pi}}\frac{Q_0}{(u_1 K_0)^{1/2}}(x - x_0)^{1/2}. \tag{13}$$

The error in the approximate solution (12) is less than 2 per cent.

A weakness of the integral method may be noted by observing that the z-dependence of the diffusion coefficient is not incorporated in the solution, as is evident from equation (8). Thus only K_0, the value of the coefficient at the point of emission of flux, appears in the solution, equation (11).

3. DISPERSION OF URBAN AIR POLLUTION

The solution found in equation (11) is for a source which extends to all $x > x_0$. If the source extends up to a finite distance $x = x_1$ and is zero for $x > x_1$ then the solution for $x > x_1$ is obviously obtained by adding to the solution in equation (11) a similar solution with source strength $(-Q_0)$ for $x > x_1$. Hence the solution for the "single cell problem" for $x > x_1$ is

$$c(x) = \frac{1}{3}\frac{Q_0}{K_0}\left[\frac{3z_1^m K_0}{u_1 B(m+1,4)}\right]^{1/m+2}[(x-x_0)^{1/m+2} - (x-x_1)^{1/m+2}]. \tag{14}$$

The solution for a source of finite width is thus obtained by the superimposition of two complementary semi-infinite sources, for each of which we have the solution in equation (11). This can be extended to the problem of dispersion of pollutants over an urban area where the emission inventory is usually in the form of succeeding area–sources of assigned strengths, as shown in Fig. 1. The solution for $x_1 < x < x_2$ is obtained, as before, by

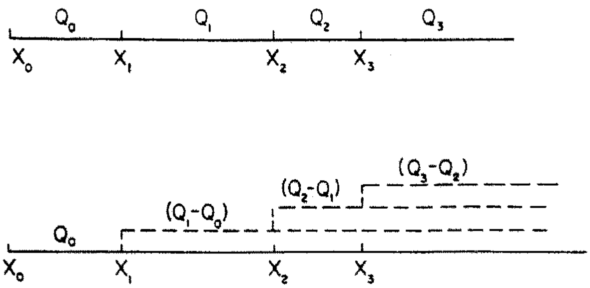

Fig. 1. A typical emission inventory is in the form of succeeding area–sources of assigned strengths as shown in the upper figure. To obtain the value of concentration at a given point we superimpose a number of sources of semi-infinite extent with strengths shown in the lower figure.

adding to the expression in equation (11), a similar expression for source strength $(Q_1 - Q_0)$ for $x > x_1$. Proceeding in this manner to larger values of x one obtains the appropriate source strengths to be assigned to the semi-infinite sources which have to be superimposed on each other in order to obtain the solution for the given source distribution. Source strengths to be adopted for the semi-infinite problems are shown in the lower part of the figure. Thus, to obtain the solution for a given x we superimpose a number of solutions like equation (11) and obtain, with reference to the figure,

$$X(x,z) = \sum_{l=0}^{s} X_l(x,z); \quad x_s < x < x_{s+1} \tag{15}$$

where each X_l satisfies equation (1) and the boundary condition

$$K(z)\left.\frac{\partial X_l}{\partial z}\right|_{z=0} = -(Q_l - Q_{l-1}). \tag{16}$$

For each X_l we obtain an expression, like equation (11), for the concentration at the ground c_l. We then have, for $x_s < x < x_{s+1}$, the ground concentration

$$c(x) = \frac{1}{3K_0}\left[\frac{3z_1^m K_0}{u_1 B(m+1,4)}\right]^{1/m+2}\sum_{l=0}^{s}(Q_l - Q_{l-1})(x - x_{l-1})^{1/m+2}. \tag{17}$$

It is interesting to note that the solution in equation (17) has the same functional dependence on x as the solution obtained by Gifford and Hanna (1970). The constant factor, however, is different. Their assumption that the solution may be represented as a product of two functions is similar to our Ansatz, equation (6), and their use of the continuity condition is equivalent to our equation (9). However, the integral method goes further by imposing boundary conditions (2–5) at the lower and upper limits of the polluted layer.

In the preceding derivation it has been assumed that the emission of the pollutants takes place at $z = 0$. It may be easily shown that if, instead, the emission takes place at a height $z = h$ then equation (17) is still valid for values of x for which $h/\delta \ll 1$, if K_0 is replaced by $K(h)$.

4. DISPERSION OF SULFUR DIOXIDE IN NASHVILLE

A merit of equation (17) is that it is very easy to apply to practical pollution dispersion problems. The area sources in the emission inventory need not have regular geometrical shapes and the wind direction may be arbitrarily oriented. The calculation does not require an electronic computer. To calculate pollutant concentration at a given point one draws a straignt line, along the wind direction, from this point to the upwind edge of the source distribution. Intersections of this line with different source areas can then be marked and the straight line resembles the upper part of Fig. 1. Application of equation (17) to the problem is then straight forward.

Table 1. Comparison of observed and calculated sulfur dioxide concentrations in Nashville. Concentrations are given in pphm (1 pphm = 2.7×10^{-5} g m^{-3} of SO_2)

Observation station No.	Observed conc	Integral method	Numerical model	Multi-cell model
19	5.8	8.7	5.1	4.0
48	13.6	17	6.5	7.3
52	2.9	6.2	0.4	3.2
56	6.9	8.9	0.9	5.9
60	20.9	20	14	6.8
82	13.2	9.7	4.2	3.6
90	4.1	6.8	2.7	2.9
	Correlation coefficient	0.92	0.89	0.70
	Mean relative error	0.37	−0.53	−0.33

We have applied equation (17) to the dispersion of sulfur dioxide in the atmosphere of Nashville, Tennessee over a 2-h period. Source inventory and the meteorological data for the problem have been given by Randerson (1970). The source distribution covers an area of 12 × 13 miles. Observed wind speed and direction were steady for the 2-h period. The vertical profile of wind may be represented by $u(z) = 0.97\, Z^{0.32}$ m s^{-1}. Sulfur dioxide is assumed to be emitted at a height of 15 m above the ground, this being approximate average building height. The turbulent diffusion coefficient at 15 m is estimated to be 3 m^2 s^{-1}.

Measured values of concentration at seven observation stations in the city are compared with the values calculated by equation (17) in Table 1. Results obtained by a numerical solution of the three-dimensional diffusion equation (Randerson, 1970) and by a multi-cell model (Hameed, 1974b) are also given. A comparison of the correlation coefficients and the mean relative errors of the three theoretical calculations shows that the integral method yields satisfactory results.

5. CONCLUSIONS

We have found that the integral method is readily adaptable to the modelling of atmospheric dispersion of pollutants over area sources. The method was applied with the assumption that the distribution of pollutants in the vertical is given by a simple polynomial. The solution thus obtained agrees very well with the exact solution in the case of uniform wind velocity, i.e. $m = 0$, see equations (12 and 13). Application of the solution to the calculation of dispersion of sulfur dioxide in Nashville (with $m = 0.32$) yields results in satisfactory agreement with observations. These results and the simplicity of the solution obtained show that the integral method can be a fruitful technique in studying dispersion problems.

The form of the method presented in this paper ignores the z-dependence of the diffusion coefficient $K(z)$. Also we expect the error in the solution to become significant if the z-distribution of wind velocity is highly non-uniform, i.e. $m \gg 0$. In view of the potential usefulness of the integral method it would be desirable to study the possibility of removing these difficulties and to extend the applicability of the method to cases where the wind velocity and the diffusion coefficient are given by functions more general than those assumed in this paper. Work in these directions is in progress.

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