CONVECTIVE INSTABILITY IN A COMPRESSIBLE ATMOSPHERE

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(Received 22 February, 1973)

Abstract. The degree of convective instability as expressed by the growth rate ω of linear modes, is calculated for a plane parallel polytropic atmosphere in the presence of radiative damping, without using Boussinesq approximation. A comparison with the results based on the Boussinesq approximation reveals that the use of the Boussinesq approximation leads to an overestimation of the radiative damping. The computation of ω as a function of the horizontal wave number yields a wavelength of maximal instability under a variety of conditions. For reasonable choices of physical parameters appropriate to the solar atmosphere, the fastest growing wavelengths turn out to be in the range 600–1200 km, and their e-folding times are in the range 200–2000 s.

1. Introduction

The problem of convective instability of a fluid layer heated from below has for a long time been the subject of investigation largely in the framework of the Boussinesq approximation. This approach is demonstrably valid (Spiegel and Veronis, 1960) only when the depth of the layer under consideration is much smaller than any scale-height. Clearly, for a possible application of the study of convective instability to stellar atmospheres where the density variations are certainly not inappreciable we must take into account the full effect of compressibility. Historically, the basic theory for the problem was first set up by Lord Rayleigh (1916) who studied an homogeneous, incompressible atmosphere with the density variations taken into account only when they were coupled with the gravitational acceleration through the buoyancy force. It was later shown by Jeffreys (1930) that Rayleigh's formulation could be applied to a compressible medium provided the temperature gradient is replaced by the corresponding superadiabatic gradient and the specific heat at constant volume, c_V is replaced by the one at constant pressure, c_P.

A considerable effort has since been directed to the investigation of polytropic atmospheres (Lamb, 1945; Skumanich, 1955). These are the simplest inhomogeneous atmospheres with a linear variation of the temperature with height for which the growth rates are found to be a monotonic increasing function of the horizontal wave number in the absence of viscosity and heat conduction. It then becomes clear that unless

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Solar Physics **30** (1973) 309–318. *All Rights Reserved*

some damping mechanism was taken into consideration, the large wave numbers would always remain the most unstable. This prompted Böhm and Richter (1959) to include a radiative damping term in the study of a polytropic atmosphere and indeed their calculations yielded a maximum growth rate at somewhat large values of the wave number. But it was pointed out by Spiegel (1964) that the work of Böhm and Richter was based on the Eddington approximation which is valid for the disturbances that are optically thick and that it overestimates the radiative damping at large wave numbers. In fact, the quantitative behavior is found to be significantly changed when the radiative exchange in a polytropic atmosphere is treated more accurately.

Spiegel (1965) has explored several computational procedures for the study of convective instability in a compressible atmosphere by setting up hydrodynamical equations for an ideal gas with constant coefficients of viscosity and thermal conductivity. He was able to show that for a fluid layer of extremely small vertical extent the problem of convective instability is essentially similar to the Boussinesq approximation with the modification suggested by Jeffreys and even though the changes brought about in the flow field by density variations are qualitatively similar to those obtained by Skumanich for large-scale motions in an inviscid, non-conducting atmosphere, it is the small-scale flow pattern which is grossly modified by dissipative mechanisms. The study of idealized polytropic atmospheres with constant viscosity and thermal conductivity was extended by Unno *et al.* (1960) by including the pressure-fluctuation term in the linearized equations. Their variational calculations provided reasonable estimates of the critical Rayleigh number for marginal stability in the case of a variety of density variations over the fluid layer.

The foregoing investigations of idealized models are indeed the first step towards our understanding of the full theory of turbulent convection. There have been attempts notably by Malkus and Veronis (1958), at more sophisticated levels, to examine the instability for the perturbations of higher order in the case of finite amplitude convection, of which some results have been verified experimentally (Townsend, 1959). One of the most heroic efforts to calculate the growth of disturbances by perturbing a fully-turbulent steady-state model of the solar hydrogen convection zone of Böhm-Vitense (1958) was made by Böhm (1963). His extensive calculations indicated that the growth rates increased approximately linearly with the wave number much faster than the corresponding rates for a polytropic atmosphere, with no apparent cut-off of the unstable modes due to the radiative damping. However, Spiegel has emphasized that the growth rates probably increase even faster at large wave number because of the overestimate of the radiative exchange in higher layers on the diffusion approximation.

The aim of the present work is to explore the influence of a large density variation and that of radiative exchange on the degree of convective instability. The complete stability problem with viscosity and thermal diffusivity investigated in the past, has been confined to the study of marginal stability of the convective modes. The principal thrust of the present investigation is to undertake exact calculations of the growth rate as a function of the horizontal wave number in a polytropic atmosphere. The motivation arises partly from the hope to find a maximum growth rate, in the presence of radiative damping, at not too large a value of the wave number in accordance with the observed spectrum of the solar granulation and partly to bring out the fact that the information about the convective modes yielded by a full treatment of the compressibility effect differs significantly from that obtained using the Boussinesq equations. The present work will hopefully serve as a basis for attempting the more general problem which includes the effect of the variation of the radiative conductivity with depth and the penetration of the convective elements into the bounding layers.

2. Equations of the Problem

2.1. Governing equations

Our main task is to calculate the degree of instability from the linearized hydrodynamic equations in a polytropic atmosphere with a given variation of the physical quantities. We therefore consider a compressible, inviscid, optically thick fluid layer of infinite horizontal extent governed by the following set of hydrodynamical equations:

Momentum:

$$\varrho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\text{grad})\,\mathbf{v}\right) = -\,\text{grad}\,P + \varrho g\,.$$

Continuity:

$$\frac{\partial \varrho}{\partial t} + \text{div}\,(\varrho\mathbf{v}) = 0\,. \tag{1}$$

Energy:

$$\varrho c_p\left(\frac{\partial T}{\partial t} + \mathbf{v}\cdot\text{grad}\;T\right) - \left(\frac{\partial P}{\partial t} + \mathbf{v}\cdot\text{grad}\;P\right) = K\nabla^2 T.$$

State:

$$P = \mathscr{R}\varrho T.$$

Here we have assumed the coefficient of thermal conductivity K, the gas constant $\mathscr{R}$, the specific heat at constant pressure c_P to be constants, thus neglecting any change in the degree of ionization.

2.2. Unperturbed state

We measure z downwards in the direction of the constant gravity and choose the level where the temperature would vanish as $z = -a$, so that the fluid layer is bounded between $z = 0$ at the top and $z = d$ at the bottom. We shall assume the unperturbed state to be in static equilibrium i.e., $\mathbf{v} = 0$. The unperturbed physical field for constant thermal conductivity is then prescribed by a polytropic law thus:

$$
\begin{aligned}
P_e &= P_e(0)\left(1+\frac{z}{a}\right)^m,\\
\varrho_e &= \varrho_e(0)\left(1+\frac{z}{a}\right)^{m-1},\\
T_e &= T_e(0)\left(1+\frac{z}{a}\right).
\end{aligned}
\tag{2}
$$

$P_e(0)$, $\varrho_e(0)$, $T_e(0)$ are evidently the values of the physical variables at the top surface; and m is the polytropic index. Equilibrium in the unperturbed state requires that:

$$m = \frac{ga}{\mathscr{R}T_e(0)}.$$

2.3. Perturbed Equations

We shall neglect any perturbations in the thermal conductivity arising from the motion and linearize the equations by expressing any physical variable as:

$$q = q_e(z) + q_1(z)\exp(\omega t + ik_x x + ik_y y),$$

to get:

$$
\begin{aligned}
&\varrho_e\omega v = -\operatorname{grad} P_1 + \varrho_1 g \mathbf{1}_z,\\
&\omega\varrho_1 + \operatorname{div}(\varrho_e \mathbf{v}) = 0,\\
&\varrho_e c_P(\omega\theta + WT_e') - \omega P_1 + \varrho_e c_p W - WP_e' =\\
&\qquad = K\left(\frac{d^2}{dz^2} - k^2\right)\theta, \qquad P_1 = \mathscr{R}\varrho_e\theta + \mathscr{R}\varrho_1 T_e.
\end{aligned}
\tag{3}
$$

Here P_1, ϱ_1, and θ are the perturbations of the pressure, density and temperatures respectively, W is the z-component of $\mathbf{v}$. The subscript e denotes the equilibrium quantities and $\mathbf{1}_z$ is the unit vector along the z-axis. In order to complete the specification of the problem we must prescribe the boundary conditions at the top and bottom surfaces, which we take to be free-surface conditions, i.e.

$$
\begin{aligned}
W &= 0, \quad \theta = 0 \quad \text{at} \quad z = 0\\
W &= 0, \quad \theta = 0 \quad \text{at} \quad z = d.
\end{aligned}
\tag{4}
$$

It is convenient at this stage to introduce the following definitions:

$$
\begin{aligned}
k^2 &= k_x^2 + k_y^2,\\
\alpha &= ka,\\
\omega_0^2 &= \frac{P_e(0)}{\varrho_e(0)\,a^2},\\
G_g &= \frac{ga^{-1}}{\omega_0^2},
\end{aligned}
\tag{5}
$$

$$G_K = \frac{Ka^{-2}}{\mathscr{R}\varrho_e(0)\,\omega_0},$$
$$D \equiv \frac{d}{d\zeta}, \tag{5}$$
$$\zeta = \frac{z}{a}.$$

Further, when we express distances in units of a, timescales in units of ω_0^{-1}, and pressures, temperatures and densities in units of $P_e(0)$, $T_e(0)$ and $\varrho_e(0)$ etc., all in a consistent way, the relevant equations reduce to the following dimensionless form:

$$\begin{aligned}
\varrho_e\omega u &= -\, i\alpha_x P_1\,,\\
\varrho_e\omega v &= -\, i\alpha_y P_1\,,\\
\varrho_e\omega W &= -\, DP_1 + G_g\varrho_1\,,\\
\left(\frac{\gamma}{\gamma-1}\right)\varrho_e(\omega\theta + WT_e') &- (\omega P_1 + WP_e') = G_K(D^2 - \alpha^2)\,\theta\,,
\end{aligned} \tag{6}$$

and

$$P_1 = \varrho_e\theta + \varrho_1 T_e\,,$$

where $\gamma = c_P/c_V$.

After a certain amount of manipulation these equations can be reduced to a set of two coupled second-order linear differential equations for the vertical velocity component W and the perturbed temperature θ:

$$G_K D^2\theta = \left(\frac{\gamma}{\gamma-1}\right)\varrho_e(\omega\theta + WT_e') - \omega P_1 - WP_e' + \alpha^2\theta\,, \tag{7}$$

and

$$P_{2W}D^2W = \varrho_e\omega W + D_1P_1 + G_g\,\frac{(P_1 - \varrho_e\theta)}{T_e}\,, \tag{8}$$

where

$$P_{2W} = \frac{\varrho_e T_e\omega}{\alpha^2 + \omega^2},$$

$$D_1P_1 = -\,\frac{\varrho_e''\omega - 2\varrho_e' DW + \left(\dfrac{\varrho_e'}{T_e} - \dfrac{\varrho_e T_e'}{T_e^2}\right)\omega\theta + \omega\,\dfrac{\varrho_e}{T_e}\,D\theta}{(\alpha^2 + \omega^2)/T_e\omega}\,.$$

The system of Equations (7) and (8) thus take the general form:

$$\begin{aligned}
D^2W &= A_0W + A_1DW + B_0\theta + B_1D\theta\,,\\
D^2\theta &= C_0W + C_1DW + D_0\theta + D_1D\theta\,,
\end{aligned} \tag{9}$$

where A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1, are all well determined functions of z and of the unknown eigenvalue ω. The foregoing system of equations together with the free-free boundary conditions allows for the determination of the degree of convective instability as measured by the growth rate ω if k_x and k_y are prescribed.

3. The Method of Solution

Together with the four homogeneous boundary conditions

$$W = 0, \quad \theta = 0 \quad \text{at} \quad \zeta = 0 \tag{10}$$

and

$$W = 0, \quad \theta = 0 \quad \text{at} \quad \zeta = d/a, \tag{11}$$

Equations (9) define a characteristic value problem for ω, when all other parameters are specified.

Let (W, θ) be the unknown solution of (9), satisfying (10) and (11) and Ω be the corresponding unknown *proper* value of ω. We can write:

$$W = C_{\mathrm{I}} W_{\mathrm{I}} + C_{\mathrm{II}} W_{\mathrm{II}} \tag{12a}$$

and

$$\theta = C_{\mathrm{I}} \theta_{\mathrm{I}} + C_{\mathrm{II}} \theta_{\mathrm{II}} \tag{12b}$$

where $(W_{\mathrm{I}}, \theta_{\mathrm{I}})$ is the solution (say I) of (9) satisfying the modified boundary conditions:

$$(W = 0, \quad \theta = 0, \quad DW = 1, \quad D\theta = 0 \quad \text{at} \quad z = 0), \tag{13}$$

and $(W_{\mathrm{II}}, \theta_{\mathrm{II}})$ is the solution (say II) of (9) satisfying another set of modified boundary conditions:

$$(W = 0, \quad \theta = 0, \quad DW = 0, \quad D\theta = 1 \quad \text{at} \quad z = 0). \tag{14}$$

The solutions I and II are linearly independent, and each of them satisfies (10), so that the solution (W, θ) given by (12) also satisfies (10). The conditions (11) however, are not satisfied by (W, θ) in (12), except by appropriate choice of C_{I} and C_{II}.

For satisfying (11), C_{I} and C_{II} must satisfy

$$\left.\begin{aligned} C_{\mathrm{I}} W_{\mathrm{I}}\left(\frac{d}{a}\right) + C_{\mathrm{II}} W_{\mathrm{II}}\left(\frac{d}{a}\right) &= 0 \\ \text{and} \quad C_{\mathrm{I}} \theta_{\mathrm{I}}\left(\frac{d}{a}\right) + C_{\mathrm{II}} \theta_{\mathrm{II}}\left(\frac{d}{a}\right) &= 0 \end{aligned}\right\}, \tag{15}$$

This is possible, nontrivially, only when:

$$W_{\mathrm{I}}\left(\frac{d}{a}\right) \theta_{\mathrm{II}}\left(\frac{d}{a}\right) - W_{\mathrm{II}}\left(\frac{d}{a}\right) \theta_{\mathrm{I}}\left(\frac{d}{a}\right) = 0. \tag{16}$$

This last equation defines the 'proper' value of ω. To find the proper value of ω for a specified set of values of α and other parameters, the following numerical method was employed.

A function $F(\kappa)$ is defined as the left-hand side of Equation (16), where $W_{\mathrm{I}}(d/a)$, $W_{\mathrm{II}}(d/a)$, $\theta_{\mathrm{I}}(d/a)$ and $\theta_{\mathrm{II}}(d/a)$ are obtained through numerical determinations of solutions I and II, by substituting κ for ω in the expressions for coefficients A_0, A_1 etc., in (9). The numerical determination of the solutions I and II is simple and straightforward, since the boundary conditions (13) and (14) are one-point boundary conditions.

A simple numerical iterative method of finding the zeroes of a given function is then used to determine the root of the equation $F(\kappa)=0$. This root is obviously the required 'proper' value of ω.

4. Results and Discussion

The coupled system of Equation (9) is solved numerically by the method described in Section III, and Tables I, II, III, IV summarize the dimensionless growth rates (expressed in units of ω_0; cf. Equations (5)) as a function of the (dimensionless) horizontal wave number $\alpha=ka$ for a variety of values of the radiative exchange parameter G_K, and for given values of the polytropic index m and the ration of specific heats y. In all computations the value of d/a is tentatively taken as unity. The tables give an over-all dependence of ω on α, G_K, m and it is quite clear that for every choice of the parameters G_K, y and m, (allowed to vary in the range applicable to the solar atmosphere) there exists a horizontal wave number $\alpha=\alpha_{\max}$, for which the growth rate ω attains a maximum.

It is instructive and illuminating to compute the growth rate of instability as a function of the horizontal wave number also in the framework of the Boussinesq approximation. Following Danielsen (1961), and neglecting viscosity and electrical resistivity, the equation for the fundamental growth rate expressed in units of ω_0 for free-free boundary conditions comes out to be:

$$(\alpha^2 + 1)\,\omega^2 + \frac{G_K\pi^2}{(1 - \mathscr{R}/c_p)}\,(\alpha^2 + 1)^2\,\omega - G_g\alpha^2 = 0\,.$$

TABLE I

The dimensionless growth rate ω as a function of the non-dimensional horizontal wave number $\alpha=ka$ for series of G_K with $m=2.5$, $\gamma=1.5$. The numbers in the parentheses are the dimensionless growth rates yielded by using the Boussinesq approximation

G_K \ α	1	2	3	4	5	6	7
0.1	0.210	0.374	0.504	0.507	0.583	0.513	0.472
	(0.141)	(0.346)	(0.421)	(0.398)	(0.343)	(0.285)	(0.236)
0.5	0.056	0.133	0.159	0.160	0.136	0.108	0.056
	(0.028)	(0.069)	(0.084)	(0.080)	(0.068)	(0.057)	(0.047)
1.0	0.028	0.068	0.085	0.080	0.068	0.056	0.046
	(0.014)	(0.034)	(0.057)	(0.040)	(0.034)	(0.028)	(0.023)
1.5	0.018	0.046	0.057	0.052	0.048	0.038	0.032
	(0.009)	(0.023)	(0.038)	(0.026)	(0.024)	(0.019)	(0.016)
2.0	0.014	0.034	0.042	0.040	0.034	0.028	0.019
	(0.007)	(0.017)	(0.022)	(0.020)	(0.017)	(0.014)	(0.012)

TABLE II

The dimensionless growth rate ω as a function of the non-dimensional horizontal wave number $\alpha = ka$ for a series of G_K with $m = 1$, $\gamma = 1.5$. The numbers in the parentheses are the dimensionless growth rates yielded by using the Boussinesq approximation.

G_K \ α	1	2	3	4	5	6	7
0.1	0.082	0.165	0.214	0.231	0.221	0.205	0.178
	(0.056)	(0.138)	(0.168)	(0.159)	(0.137)	(0.114)	(0.094)
0.5	0.022	0.053	0.066	0.062	0.055	0.042	0.023
	(0.011)	(0.028)	(0.034)	(0.032)	(0.027)	(0.023)	(0.019)
1.0	0.011	0.027	0.034	0.032	0.027	0.023	0.019
	(0.006)	(0.014)	(0.023)	(0.016)	(0.014)	(0.011)	(0.009)
1.5	0.007	0.018	0.023	0.021	0.018	0.015	0.013
	(0.004)	(0.009)	(0.015)	(0.011)	(0.009)	(0.007)	(0.007)
2.0	0.005	0.014	0.017	0.016	0.014	0.012	0.008
	(0.003)	(0.007)	(0.011)	(0.008)	(0.007)	(0.006)	(0.005)

TABLE III

The dimensionless growth rate ω as a function of the non-dimensional horizontal wave number $\alpha = ka$ for a series of G_K with $m = 2.5$, $\gamma = 1.15$. The numbers in the parentheses are the dimensionless growth rates yielded by using the Boussinesq approximation

G_K \ α	1	2	3	4	5	6	7
0.1	0.297	0.545	0.701	0.797	0.844	0.859	0.602
	(0.190)	(0.468)	(0.569)	(0.538)	(0.462)	(0.385)	(0.318)
0.5	0.207	0.422	0.531	0.574	0.568	0.536	0.483
	(0.038)	(0.094)	(0.114)	(0.107)	(0.091)	(0.077)	(0.063)
1.0	0.141	0.313	0.395	0.406	0.383	0.359	0.289
	(0.019)	(0.047)	(0.057)	(0.054)	(0.046)	(0.038)	(0.032)
1.5	0.104	0.244	0.306	0.305	0.277	0.259	0.203
	(0.013)	(0.031)	(0.038)	(0.038)	(0.031)	(0.025)	(0.021)
2.0	0.082	0.195	0.244	0.242	0.215	0.199	0.156
	(0.009)	(0.017)	(0.028)	(0.027)	(0.023)	(0.019)	(0.016)

TABLE IV

The dimensionless growth rate ω as a function of the non-dimensional horizontal wave number $\alpha = ka$ for a series of G_K with $m = 1$, $\gamma = 1.15$. The numbers in the parentheses are the dimensionless growth rates yielded by using the Boussinesq approximation

G_K \ α	1	2	3	4	5	6	7
0.1	0.184	0.351	0.445	0.492	0.508	0.497	0.475
	(0.076)	(0.187)	(0.227)	(0.215)	(0.185)	(0.154)	(0.127)
0.5	0.086	0.195	0.242	0.243	0.234	0.190	0.160
	(0.015)	(0.037)	(0.045)	(0.043)	(0.371)	(0.031)	(0.025)
1.0	0.048	0.114	0.142	0.139	0.121	0.102	0.080
	(0.007)	(0.019)	(0.023)	(0.021)	(0.018)	(0.015)	(0.013)
1.5	0.033	0.080	0.097	0.094	0.082	0.082	0.057
	(0.006)	(0.012)	(0.015)	(0.014)	(0.012)	(0.011)	(0.008)
2.0	0.025	0.060	0.074	0.071	0.061	0.052	0.043
	(0.004)	(0.009)	(0.011)	(0.010)	(0.009)	(0.007)	(0.006)

This quadratic equation in ω was solved for the same sets of values of the parameters G_g, G_K, α and γ as before, and the values of the positive root of the quadratic are given in the parenthesis in Tables I–IV.

The Boussinesq approximation (B.A.) discards the stabilizing effects of the zero order density stratification. Therefore, in the absence of the radiative damping, the use of the B.A. may be expected to yield growth rates higher than those obtained without using it. However, from Tables I–IV one finds that the growth rates ($\omega_{\mathrm{B.A.}}$) obtained by using the B.A. are always *smaller* than those (ω) obtained without B.A. Hence we conclude that the B.A. leads to an overestimation of the stabilization by the radiative damping. This conclusion is also supported by the fact that the ratio $\omega_{\mathrm{B.A.}}/\omega$ decreases with increasing G_K, when other parameters are held constant. For small G_K, the overestimation of the radiative damping in the B.A. leads to *smaller* estimates of the fastest growing wavenumber $\alpha_{\max}$. For example, the B.A. reduces $\alpha_{\max}$ from 5 to 3, for $G_K \approx 0.1$ in Table 1.

It has been realized for quite some time that the Boussinesq equations, even though they may embody the essential features of the theory of convection, lack several characteristics that are typical of stellar convective transport of energy. We have shown here that the resulting growth rates are quantitatively modified, in a significant manner, with the introduction of compressibility and thermal conductivity.

Admittedly we have tackled an idealized problem with the fluid layer confined between rigid boundaries, thus making no allowance for the penetration of convection. We have also not taken into account the dependence of thermal conductivity and specific heat on the state variables, particularly on the temperature. We also expect the variation of the thermal conductivity with height to have a non-negligible effect on the values of the growth rates. The corresponding refinements are intended to be made in a subsequent investigation.

For comparison with the properties of the observed inhomogeneities in the solar atmosphere we must determine appropriate values of the parameters a, d, g, K, m and γ. The idealized Formulae (2) cannot be satisfied exactly and simultaneously by $P(z)$, $\varrho(z)$ and $T(z)$ from any computed model of the mean stratification. Hence the parameters a, d, m can be determined only approximately. The 'mean' values of K and γ also can only be chosen crudely, somewhat arbitrarily. From the Harvard–Smithsonian reference atmosphere (HSRA, Cf. Gingerich *et al.*, 1971), we have $P \approx 1.3 \times 10^5$ dyn cm^{-2}, $\varrho \approx 3.2 \times 10^{-7}$ gm cm^{-3}, and $T \approx 6.4 \times 10^3$ K at the surface $\tau = 1$. This surface is also the boundary of the stable and unstable regions, and is taken as $z = 0$. Assuming a layer-thickness equal to one-density scale height, and using Spruit's model (Spruit, 1972) of the convection zone, we obtain $d \approx H_\varrho \approx 620$ km. Spruit's model is preferred since it fits smoothly with the HSRA. From this model, $P \approx 2.8 \times 10^5$ dyn cm^{-2}, $\varrho \approx 8.0 \times 10^{-7}$ gm cm^{-3} and $T \approx 1.5 \times 10^5$ K at $z = d$. From this data we estimate $a \approx 600$ km, yielding $\omega_0 \sim 10^{-2}$ s^{-1}, and $m = 3$. For estimating m we used ϱ-variation, since P-variation has very steep gradients near $\tau = 1$. Thus, for the present comparison, we choose values from Table I or III. The parameter G_K varies by over three orders of magnitude from ~ 5 at $z = 0$ to $\sim 10^{-3}$ at $z = d$. However, we attach more weight to the largest values of G_K corresponding to the uppermost layers, since here the development of the mode being slowest, the corresponding phase of development will take most of the time required for e-folding. Thus we take $G_K \sim 1$. The mean value of γ is also very difficult to determine; but values 1.15 and 1.5 seem acceptable. From Tables I and III, we notice $\alpha_{max} \sim 3$ to 6 for $G_K \approx 2$ to 0.5, and the corresponding $\omega_{max} \approx 0.05$ to 0.6 times ω_0. This gives the fastest growing modes with wavelengths $\lambda_{max} \sim 600$–1200 km, and with e-folding times $\omega_{max}^{-1} \approx 160$–2000 s, for various values of G_K.

The values of λ_{max} are in good agreement with the observed sizes of the normal granulation cells (Namba and Diemel, 1969). The e-folding times are of the same order as the observed lifespans of normal granules (Bahng and Schwarzschild, 1961).

From the spread of values of ω_{max} and α_{max} in Tables I–IV, it is tempting to conclude that the observed spread in the sizes and the life-spans of normal granules is due to the variations in the thermal fields experienced by each granule during the course of its development.

5. Conclusions

Convective instability in a plane parallel polytropic atmosphere has been investigated in the presence of radiative damping. The results are in qualitative agreement with those of Böhm and Richter (1959). Quantitatively, the results are in good agreement with the observed scales of the normal solar granulation. The investigation points to the fact that the Boussinesq approximation leads to an overestimation of the radiative damping effects.

Acknowledgements

We are grateful to Professor E. A. Spiegel for a very helpful conversation. One of us (S. M. C.) would like to thank Dr R. Jastrow for the hospitality at the Institute for Space Studies.

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