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ELECTRICAL CONDUCTIVITY AND CONDUCTIVE OPACITY OF A RELATIVISTIC ELECTRON GAS

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ABSTRACT

The electrical conductivity and the conductive opacity are computed for a system of relativistic degenerate electrons in the presence of a system of ions. In the density-temperature range we are interested, in $10^6 \leq \rho \leq 10^{12}$ (g cm^{-3}), $10^5 \leq T \leq 10^9$ ° K, the ion-ion interaction is taken into account by means of the pair-correlation function. The numerical values of the electron conduction opacities are given for different values of the parameter Γ, which characterizes the strength of the ion-ion interaction, and for different values of $2 \leq Z \leq 26$.

I. INTRODUCTION

The problem of the electrical conductivity of an electron gas, after the classical papers by Marshak (1941), Mestel (1950), and Lee (1950), has not been significantly changed until very recently, when Hubbard (1966), using Kubo formulation for transport phenomena, and Hubbard and Lampe (1969), using a two-polynomial Chapman-Enskog method, substantially improved the original calculations by using the Kubo quantum-mechanical expression for the conductivity tensor. One of the improvements consists in deriving a finite (nondivergent) expression for the Coulomb-scattering cross-section through a better treatment of the ion-ion correlation. Hubbard (1966) and Hubbard and Lampe (1969) also take into account electron-electron scattering, which is neglected in the original treatment. Electron-electron scattering becomes important even when a small fraction of electrons are above the Fermi sea. This occurs when the gas is partially degenerate. The Lorentz model of noninteracting electrons has been shown to be valid when the electrons are strongly degenerate in the density range of $\sim 10^{25}$–10^{30} particles cm^{-3}. However, Hubbard (1966) and Hubbard and Lampe (1969) have not considered either relativistic effects or the influence of an external magnetic field of arbitrary strength.

In this paper we will consider a relativistic extension of the previous theories, and we will compute the electrical conductivity for a degenerate relativistic electron gas without a magnetic field. The presence of a magnetic field and the corresponding computation of the longitudinal and transverse conductivities are considered in subsequent publications (Canuto 1970; Canuto and Chiuderi 1970). The ion-ion interaction is taken into account by using the pair-correlation function $g(r)$ which has been tabulated numerically by Brush, Sahlin, and Teller (1966) for thirteen values of the parameter Γ, which is the ratio of the ion-ion Coulomb interaction to their average kinetic energy kT (Salpeter 1961). The many-body correlations are taken into account by introducing a dielectric constant which reduces, in the static case, to a screening radius r_0^{-2} which is a function of the density. The general expression for the conductivity σ is shown to reduce in the limit of high temperatures to the classical expression of Drude theory (Jackson 1962). The numerical values of the conductivity σ and the conductive opacity k_c are given as functions of the temperature for densities ranging between 10^6 and 10^{12} g cm^{-3} and for temperatures between 10^5 ° and 10^9 ° K. This range covers the interior of white dwarfs and the nonneutron envelope of neutron stars.

II. DERIVATION OF THE DECAY PROBABILITY FOR MOTT SCATTERING

We shall start by first considering the Lagrangian describing the interaction of one electron taking place at x with the αth nucleus Ze located at R_α (see Fig. 1). The Lagrangian is

$$\mathfrak{L}(x, R_\alpha) = -ie\psi^*(x)\gamma_\mu\psi(x)A_\mu(x, R_\alpha)\,, \tag{1}$$

where $\psi^* = \psi^\dagger\gamma_4$ and $\psi^\dagger$ is the Hermitian conjugate of ψ. In the case of Coulomb interaction, equation (1) reduces to

$$\mathfrak{L}(x, R_\alpha) = -e^2Z\psi^\dagger(x)\frac{1}{|x - R_\alpha|}\psi(x)\,. \tag{2}$$

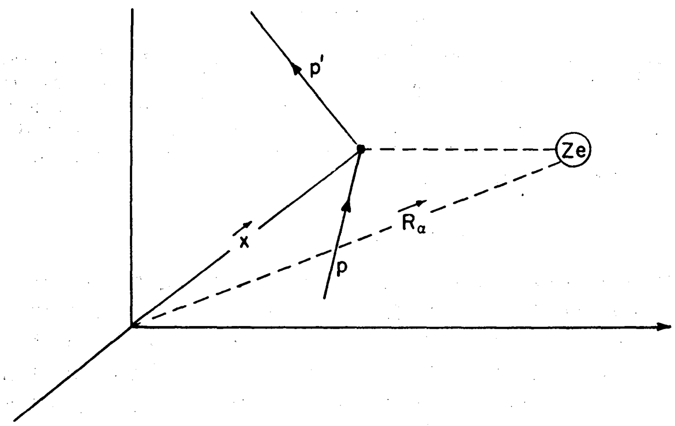

FIG. 1.—Feyman diagram for elastic scattering of an electron by a heavy ion, Ze

The spinors $\psi(x)$ are solutions of the free Dirac equation

$$(i\gamma_\mu\partial_\mu + mc/\hbar)\psi = 0\,,$$

and they are given by (Ω = normalization volume, $s = \pm 1$ spin index)

$$\psi = \Omega^{-1/2}e^{ip\cdot r/\hbar}u^{(s)}(p)\,, \quad u^{(s)}(p) = \left(\frac{E + mc^2}{2E}\right)^{1/2}\begin{pmatrix} 1 \\ \dfrac{\sigma\cdot pc}{E + mc^2}\end{pmatrix}, \tag{3}$$
$$u^\dagger u = 1\,, \quad u^* u = E/mc^2\,,$$

where $E = \epsilon\, mc^2 = (p^2c^2 + m^2c^4)^{1/2}$ is the relativistic expression for the particle energy. If the Fourier transform of the potential, i.e.,

$$r^{-1} = 4\pi(2\pi)^{-3}\int q^{-2}e^{iq\cdot r}d^3q\,, \tag{4}$$

is used, then the Hamiltonian of the system becomes

$$\mathfrak{H} = 4\pi e^2Z(2\pi)^{-3}\Omega^{-1}u^{*(s')}(p')\gamma_4u^{(s)}(p)\sum_{\alpha=1}^{N_i}\int d^3qq^{-2}e^{iq\cdot R_\alpha}\int d^3re^{i(p-p'+\hbar q)\cdot r/\hbar}\,, \tag{5}$$

where N_i = number of ions.

Integrating first over r and then using the δ-function to perform the integration over q, we obtain

$$\mathfrak{H} = 4\pi e^2Z\Omega^{-1}u^{*(s')}(p')\gamma_4u(p)\,|(p' - p)\hbar^{-1}|^{-2}\sum_{\alpha=1}^{N_i}e^{iq\cdot R_\alpha}\,, \tag{6}$$

where q stands for $(p' - p)\hbar^{-1}$ and the summation over a runs over the position R_a of all the ions. As stated earlier, equation (6) can be generalized to include the many-body effect by the following replacement:

$$|p' - p|^2\hbar^{-2} \rightarrow \eta^l |p' - p|^2\hbar^{-2} \equiv |p' - p|^2\hbar^{-2} + r_D^{-2} \quad (7)$$

with (Silin 1960)

$$r_D^{-2} = \lim_{k\to 0} \lim_{\omega/k\to 0} k^2(\eta^l - 1) , \quad (8)$$

where η^l is the longitudinal dielectric constant. A more rigorous expression for r_D^{-2} is given by Fradkin (1967) as

$$r_D^{-2} = \Pi_{44}(0) ,$$

where $\Pi_{\mu\nu}(k)$ is the polarization operator, defined by ($p_\mu = p,ip_0$)

$$\Pi_{\mu\nu}(k) = ie^2(2\pi)^{-4}\int Tr\gamma_\mu G(p + k,p_0 + k_0)\gamma_\nu G(p,p_0)d^3p dp_0 ; \quad (9)$$

it relates an external field A_ν with the induced current j_μ through the relation

$$j_\mu = \Pi_{\mu\nu}A_\nu , \quad (10)$$

where $G(p)$ is the Fourier transform of the electron Green function. The electron Green function and the tensor $\Pi_{\mu\nu}$ have been evaluated by Tsytovich (1961) for a relativistic electron gas at any temperature and density. The dielectric constant is accordingly computed, and the following expression is derived for r_D^{-2}:

$$r_D^{-2} = -8\pi e^2(2\pi\hbar)^{-3}\int d^3p \frac{\partial}{\partial E} f(p) , \quad (11)$$

where $f(p)$ is the Fermi distribution. For a degenerate electron gas, r_D^{-2} is easily seen to become [$\alpha = \frac{1}{137}$]

$$r_D^{-2} = \lambda\!\!\!^{-}{}_c^{-2} \frac{2\alpha}{\pi} \mu(\mu^2 - 1)^{1/2} \equiv 2r_0^{-2}\lambda\!\!\!^{-}{}_c^{-2} \quad (\lambda\!\!\!^{-}{}_c = \hbar/mc) , \quad (12)$$

where μ is the chemical potential plus the rest mass in units of mc^2. From equation (6) we now obtain

$$|\mathfrak{H}|^2 = (4\pi e^2 Z)^2 \lambda\!\!\!^{-}{}_c^4 \Omega^{-2} |u^{*(s')}(P')\gamma_4 u(P)|^2[(P - P')^2 + 2r_0^{-2}]^{-2} \sum_{a=1}^{N_i} \sum_{a'=1}^{N_i} e^{iQ(R_a - R_{a'})} \quad (13)$$

where $P = p/mc$, $Q = \lambda\!\!\!^{-}{}_c^{-1}(P - P')$. We will now carry out the double summation in equation (13). If we first average equation (13) on the canonical distribution function

$$\exp(-U/kT) , \quad (14)$$

where U is the interaction energy among the ions, we easily obtain

$$\left\langle \sum_{a=1}^{N_i} \sum_{a'=1}^{N_i} e^{iQ\cdot(R_a - R_{a'})} \right\rangle = N_i\left[1 + \frac{N_i}{\Omega} g(Q)\right] , \quad (15)$$

where $g(Q)$ is the Fourier transform of the pair-correlation function $g(r)$ (Brush, Sahlin, and Teller 1966):

$$g(Q) = \int e^{iQ\cdot r} g(r) d^3r , \quad (16)$$

$$g(r_1 r_2) = \Omega^2 N(N - 1)N^{-2}[\int dr_1 \ldots dr_N e^{-U/kT}]^{-1} \int dr_3 \ldots dr_N e^{-U/kT} . \quad (17)$$

The function $g(r)$ is dimensionless, hence the dependence on r is of the form $g(r/a)$, where a is a parameter with the dimension of length. Carrying out the Fourier transform in equation (16), we obtain

$$\left\langle \sum_{a=1}^{N_i} \sum_{a'=1}^{N_i} \exp\left[iQ(R_a - R_{a'})\right] \right\rangle = N_i[1 + 3\int g(x)\,(\xi x)^{-1} \sin\,(\xi x) x^2 dx] \tag{18}$$

$$\equiv N_i \phi(\xi)\,, \tag{19}$$

where

$$\xi = (3/4\pi)^{1/3}(\Omega/N_i)^{1/3}\lambda_c^{-1}|P - P'|\,. \tag{20}$$

The length a has been taken to be equal to the ion-sphere radius $(3\Omega/4\pi N_i)^{1/3}$. Let ρ be the matter density of the mixture of electrons and ions; from the theory of ionized stellar material we have (Chandrasekhar 1957, p. 254)

$$n_i \equiv N_i/\Omega = (\rho/m_{\rm H})\Sigma_k x_k A_k^{-1} = (\rho/m_{\rm H})\mu_i^{-1}\,, \tag{21}$$

where x_k is the number of grams of particles of type k per gram of mixture

$$x_k = n_k m_k (\Sigma_k n_k m_k)^{-1} \tag{22}$$

and μ_i is the molecular weight of the ions (the average number of proton masses per ion particle). With only one type of ions, $x \to 1$, $\mu_i = A$. For the electron gas we have, equivalently,

$$n_e \equiv N_e/\Omega = (\rho/m_{\rm H})\Sigma_k x_k A_k^{-1}(A_k n^*_k - 1) \equiv (\rho/m_{\rm H})\mu_e^{-1}\,, \tag{23}$$

where

$$A_k n^*_k = 1 + \nu_e(k) \tag{24}$$

is what Chandrasekhar has called the "mean ionization per unit atomic weight" (Cox 1968). The quantity $\nu_e(k)$ is the number of free electrons contributed by the particle of type k; for complete ionization $\nu = Z$. When we have only one type of ion and complete ionization, from equation (23) we obtain

$$\mu_e^{-1} = Z/A\,. \tag{25}$$

Let

$$\rho_6 = 10^{-6}\rho/\mu_e \tag{26}$$

be the matter density in units of $10^6\ \mu_e$ g cm^{-3}. We then have

$$n_i = \rho_6(10^6/m_{\rm H})(\mu_e/\mu_i)\,, \qquad n_e = \rho_6(10^6/m_{\rm H})\,.$$

The expression for ξ (eq. 20) thus becomes

$$\xi = 2.69\rho_6^{-1/3}\mu_i^{1/3}\mu_e^{-1/3}[\tfrac{1}{2}(\epsilon^2 + \epsilon'^2) - 1 - (\epsilon^2 - 1)^{1/2}(\epsilon'^2 - 1)^{1/2}\cos\theta]^{1/2}\,. \tag{27}$$

Substituting equation (19) into equation (13), we finally obtain

$$|\mathfrak{H}|^2 = (4\pi Z e^2)^2 \Omega^{-1} \lambda_c^4 n_i [(P - P')^2 + 2r_0^{-2}]^{-2} |u^{*(s')}(P')\gamma_4 u^{(s)}(P)|^2 \phi(\epsilon,\epsilon',\theta)\,. \tag{28}$$

We next average over the initial and sum over the final spin states. We obtain ($g = 2s + 1 = 2$)

$$\begin{aligned} g^{-1}\Sigma_s\Sigma_{s'}|u^{*(s')}(P')\gamma_4 u^{(s)}(P)|^2 &= (4gm^2c^4)^{-1} Tr[\gamma_4(mc^2 - i\gamma Pc)\gamma_4(mc^2 - i\gamma P'c)] \\ &= g^{-1}\epsilon\epsilon^{-1}(1 + \epsilon\epsilon' + PP'\cos\theta)\,. \end{aligned} \tag{29}$$

Finally, the transition probability for the process defined as

$$W = (2\pi/\hbar)|\mathfrak{H}|^2\delta(E - E') , \tag{30}$$

after equations (28) and (29) are substituted, takes the following form:

$$\begin{aligned} W = {} & W_0 n_i g^{-1}\epsilon^{-2}[1 + \epsilon\epsilon' + (\epsilon^2 - 1)^{1/2}(\epsilon'^2 - 1)^{1/2}\cos\theta] \\ & \times [\epsilon^2 + \epsilon'^2 - 2 - 2(\epsilon^2 - 1)^{1/2}(\epsilon'^2 - 1)^{1/2}\cos\theta + 2r_0^{-2}]^{-2}\delta(\epsilon - \epsilon')\phi(\epsilon,\epsilon',\theta) , \end{aligned} \tag{31}$$

where

$$W_0 = 4(2\pi)^3\alpha^3 Z^2 \lambda_c^6(mc^2/\hbar)\Omega^{-1} .$$

In this paper, only the elastic scattering of electrons by ions is considered. The problem of the inelastic scattering of relativistic electrons by lattice vibrations, i.e., phonon scattering, is now being considered (Solinger 1969).

III. EXPRESSION FOR THE ELECTRICAL CONDUCTIVITY

The electrical conductivity will now be calculated from the Boltzmann equation. To this end we first compute the relaxation time, which is defined as (Argyres and Adams 1956; Chester and Thellung 1959[1])

$$\tau^{-1}(\epsilon) = \Sigma_f[1 - (P'_x/P_x)]W(\epsilon,\epsilon',\theta) , \quad \Sigma_f = \Omega(2\pi\lambda_c)^{-3}\int d^3P , \tag{32}$$

where we have taken the rate of change of momentum in the x-direction. Since the system is isotropic, the relaxation time will not depend on the direction of momentum chosen. If the system of coordinates is defined such that

$$\begin{aligned} P'_x &= P[\cos\omega\cos\theta + \sin\omega\sin\theta\cos\phi] , \\ P_x &= P\cos\omega , \qquad d\Omega' = \sin\theta d\theta d\phi , \end{aligned} \tag{33}$$

then equation (32) becomes

$$\tau^{-1}(\epsilon) = \tau_1^{-1}\int_0^\pi H(\theta,\epsilon)\sin\theta d\theta \equiv \tau_1^{-1}F(\epsilon) , \tag{34}$$

where

$$\begin{aligned} H(\theta, \epsilon) = {} & \epsilon(\epsilon^2 - 1)^{1/2}\epsilon^{-2}(1 - \cos\theta)[(\epsilon^2 - 1)(1 - \cos\theta) + r_0^{-2}]^{-2} \\ & \times [1 + \epsilon^2 + (\epsilon^2 - 1)\cos\theta]\phi(\epsilon, \theta) , \end{aligned} \tag{35}$$

$$\tau_1^{-1} = \pi\alpha^2 Z^2\lambda_c^3(mc^2/\hbar)n_i . \tag{36}$$

In the case $r_0^{-2} \to 0$, $\phi \to 1$, which implies no screening and a random distribution of ions, the quantity $H(\epsilon,\theta)$ has an angular dependence like $(1 - \cos\theta)^{-1}$, and therefore the integral in equation (34) diverges at $\theta = 0$. The divergence in the nonrelativistic case and $r_0^{-2} \to 0$, $\phi \to 1$, is removed by introducing a cutoff angle θ_0. In this case we then have (Mestel 1950, eq. [31])

$$\tau = 2m^2v^3(2\pi e^4 Z^2 \ln\Lambda)^{-1}n_i^{-1} , \tag{37}$$

$$\Lambda = 2(1 - \cos\theta_0)^{-1} . \tag{38}$$

The angle θ_0 is the minimum scattering angle caused by screening. In our case screening is automatically included. Once the relaxation time is obtained, it is a simple matter to

[1] The latter reference contains a very detailed, critical analysis of the applicability of the Boltzmann equation.

evaluate the conductivity. We first observe that the definition of the current in the x-direction is given by

$$J_x = -g\Omega(2\pi\hbar)^{-3}\int p^2 dp \int d\Omega f(p) j_x \,, \tag{39}$$

where

$$j_x = e\int d^3x \psi^* \gamma_x \psi \,.$$

After evaluation, we obtain

$$j_x = ecp_x E^{-1}\Omega^{-1} = ec\Omega^{-1}\epsilon^{-1}P_x \,. \tag{40}$$

The distribution function f is solution of the Boltzmann equation with the electric field E in the x-direction treated as a first-order perturbation. The result is (Argyres and Adams 1956; Kelly 1969)

$$f = f_0 + eE\tau\epsilon^{-1}(mc)^{-1}\frac{\partial}{\partial P_x} f_0 \,, \tag{41}$$

where f_0 is the distribution function for thermal equilibrium without an electric field. Substituting equations (40) and (41) into equation (39), we obtain

$$\frac{J}{E} = \sigma_0 = -ge^2(2\pi\lambda\!\!\!^{-}_c)^{-3}m^{-1}\int_0^\infty P^2 dP \int_0^\pi \sin\Theta d\Theta \int_0^{2\pi} d\phi\tau(\epsilon)P_x\epsilon^{-2}\frac{\partial f_0}{\partial P_x} \,. \tag{42}$$

Using polar coordinates for the vector P, we can perform the angular integration over Θ, obtaining

$$\sigma_0 = -(4\pi/3)ge^2\tau_1 m^{-1}(2\pi\lambda\!\!\!^{-}_c)^{-3}\int_1^\infty (\epsilon^2 - 1)^{3/2}F^{-1}(\epsilon)\epsilon^{-2}(\partial f_0/\partial\epsilon)d\epsilon \,. \tag{43}$$

For the case of complete degeneracy the derivative can be approximated by $-\delta(\epsilon - \mu)$, where μ is the chemical potential of the electron rest mass (in units of mc^2). For a relativistic degenerate gas we also know that

$$\mu^2 - 1 = \rho_6^{2/3} \,, \tag{44}$$

where, as usual, ρ is the mass density. The final form for σ_0 is given by

$$\sigma_0 = \sigma^*_0 G(\rho_6^{2/3}) \,, \tag{45}$$

$$\sigma^*_0 = \tfrac{8}{3}[(2\pi)^3\lambda\!\!\!^{-}_c{}^3 Z n_i]^{-1}[\alpha Z(\hbar/mc^2)]^{-1} \,, \tag{46}$$

or

$$\sigma^*_0 = 0.33Z^{-1}\rho_6^{-1}10^{23}\ (\mathrm{sec}^{-1}) \,. \tag{47}$$

The function $G(\rho)$ is given by ($\alpha \equiv \frac{1}{137}$)

$$G(\rho) = \rho(1+\rho)^{-1/2}\Big\{\int_0^\pi d\theta \sin\theta(1-\cos\theta)[2+\rho(1+\cos\theta)] \times [\rho(1-\cos\theta) + (2\alpha/\pi)\rho^{1/2}(1+\rho)^{1/2}]^{-2}\phi(\rho,\theta)\Big\}^{-1} \,, \tag{48}$$

where

$$\phi(\rho,\theta) = 1 + 3\int_0^\infty (x\xi)^{-1}\sin(x\xi)g(x)dx \,, \tag{49}$$

$$\xi = 2.697\mu_i^{1/3}\mu_e^{-1/3}(1-\cos\theta)^{1/2} \,. \tag{50}$$

Although ρ does not enter explicitly in ϕ, nonetheless it enters in an implicit form in $g(x)$. As we will discuss later, $g(x)$ is given only for certain values of the parameter Γ which depends on ρ.

The conductive-opacity coefficient $k_c{}^0$ is defined by

$$k_c{}^0 = (4ac/3\rho)T^3\lambda_0{}^{-1}\,, \quad ac = \pi^2k^4/15c^2\hbar^3 \tag{51}$$

where $k = 1.38024 \times 10^{-16}$ erg deg^{-1} is the Boltzmann constant. The parameter λ_0 is the thermal-conductivity coefficient, which can be related to σ_0 through the Widemann-Franz law which is valid in the case of high degeneracy (Hubbard 1966)

$$\lambda_0 = (\pi^2/3)(k/e)^2\sigma_0 T\,. \tag{52}$$

Using equations (48) and (45), we now obtain

$$k_c = k^*{}_c{}^0 G^{-1}(\rho_6{}^{2/3})\,, \tag{53}$$

$$k^*{}_c{}^0 = (1/5)(2\pi)^3 m_{\rm H}{}^{-1}\alpha^2\phi^2\lambda_c{}^2Z^2\mu_i{}^{-1}\,, \tag{54}$$

where

$$\alpha = \tfrac{1}{137}\,, \quad m_{\rm H} = 1.6594 \times 10^{-24}\ {\rm g}\,, \quad \phi \equiv kT/mc^2\,.$$

In the case of complete ionization, inserting the value of the constants, we obtain

$$k^*{}_c{}^0 = 6.753T_6{}^2(Z^2/A)10^{-8}\ {\rm cm}^2\ {\rm g}^{-1}\,, \tag{55}$$

where[2] $T_6 = T \times 10^{-6}$.

IV. THE PAIR-CORRELATION FUNCTION

The most important ingredient in equation (48) is the function $g(x)$, which depends on the value of the parameter

$$\Gamma = [(Ze)^2/kT][4\pi/3)(N_i/\Omega)]^{1/3} = \Gamma_0\rho_6{}^{1/3}T_6{}^{-1}\,, \qquad \Gamma_0 = 22.76Z^{5/3}\,, \tag{56}$$

which is a measure of the ion-ion correlation. For completely randomly distributed ions (high temperature), the function ϕ reduces to 1. For low values of Γ ($\ll$1) the classical Debye-Hückel theory can be used, giving for $g(x)$ an analytic expression

$$g_{\rm DH}(x) = \exp\,[-(\Gamma/x)\,\exp\,(-x[3\Gamma]^{1/2})]\,. \tag{57}$$

When we decrease the temperature, Γ increases, and the Debye-Hückel theory breaks down. Recently, Brush, Sahlin, and Teller (1966), using a Monte Carlo Method, have computed $g(x)$ for values of Γ up to 100, but the use of their table is not straightforward and some care has to be used. First of all, the function $g(x) - 1$ has to be normalized to $-\frac{1}{3}$, i.e.,

$$I \equiv \int_0^\infty [g(x) - 1]x^2dx = -\tfrac{1}{3}\,. \tag{58}$$

This can be easily shown by using equation (18). The Monte Carlo results are given only for $x \leq b$ with $b = 3.5$. The normalization condition is never satisfied, and it gets worse for higher values of Γ. It is therefore necessary to continue $g(x)$ beyond the Monte Carlo data with a plausible analytic form. We have used a form proposed by Hubbard (1966)

$$g(x) = 1 - A[\exp\,(-Bx)]x^{-1}\cos\,(Cx + D)\,, \tag{59}$$

[2] Dr. B. Paczynski has communicated to the author that equation (53) is fairly well fitted by a simple expression of the form $k_c \simeq T^2\rho^{-1}$.

where A, B, C, and D have been determined in the following way:[3] C and D were found by using the last two points of the Monte Carlo data for which $g(x) = 1$ and $dg/dx \gtrless 0$, respectively. A and B are then determined by the least-squares method. In Table A we quote the values of A, B, C, and D and those of the normalizing integral I for $\Gamma > 1$. For $\Gamma \leq 1$ we used the Debye-Hückel formula. For $\Gamma = 2.5$ no extension was required. A detailed discussion concerning this point is found in Hubbard's paper (1966).

V. NUMERICAL RESULTS

The Monte Carlo data and the extended analytic expression for $g(x)$ were used to compute the function G (see eqs. [45] and [53]), which is given in Tables 1 and 2 for different values of Z ($2 \leq Z \leq 26$) and ρ_6. For a given value of the density ρ_6, the atomic number, and the parameter Γ, the temperature is easily computed from equation (56). In Tables 1 and 2 the seven values of log G, corresponding to each couple of values of log ρ_6 and Z, correspond to $\Gamma = 0$, 2.5, 10, 20, 40, 75, and 100, respectively. $\Gamma = 0$ means completely randomly distributed ions, i.e., $\phi \rightarrow 1$. The density ρ ranges from 10^6 to 10^{12}, while the temperature T ranges from 10^5 ° to 10^8 ° K. To compare our results with the

TABLE A

VALUES OF A, B, C, AND D

Γ	A	B	C	D	I
5....	0.583242	1.26555	5.23598	1.57081	−0.333233
10....	0.772943	0.909053	3.69599	2.67959	−0.332123
14....	0.517428	0.608818	3.69599	3.04920	−0.342655
20....	0.882497	0.735184	3.77006	2.95252	−0.325321
30....	6.23226	1.13223	3.92699	2.74890	−0.315717
40....	2.53091	0.702997	3.92699	2.74890	−0.347549
50....	0.679080	0.241315	4.05367	2.38153	−0.391829
75....	1.72393	0.361834	4.18879	1.98968	−0.320898
100....	3.32609	0.475877	4.53214	0.849743	−0.303438

one obtained by Mestel (1950) for the nonrelativistic region, we have computed the ratio KHH/K_c^0, where KHH is the Haselgrove and Hoyle (1959) analytic formula which fits the Mestel conductivities and K_c^0 is the one computed in this paper. The ratio has been evaluated at $\rho = 10^6$ g cm^{-3} for different values of Γ and Z (see Table 2). As we can see from Table 2, the ratio is less than 1 for $\Gamma \leq 40$ and it is slightly greater than 1 for higher Γ. A ratio $KHH/K_c^0 < 1$ for $\rho = 10^6$ has also been found by Hubbard and Lampe (1969). It can also easily be checked that at $\rho = 10^6$ g cm^{-3} the opacity K_c^0 computed here is very close to the one given by Hubbard and Lampe (1969) in their Tables 2 and 3.

Finally, the validity of the results has to be mentioned. As pointed out by Salpeter (1961), the ratio Z/A can be significantly modified because of free-electron capture (also called inverse β-decay in astrophysics). Let E_Z be the β-decay energy of the electron where the nucleus $(Z - 1,A)$ decays to the nucleus (Z,A). The electron Fermi energy of the plasma is simply

$$E_F/mc^2 = (1 + \rho_6^{2/3})^{1/2} - 1 .$$

If $E_F > E_Z$, the nucleus (Z,A) can decay back to the nucleus $(Z - 1,A)$ via inverse β-decay. We have computed E_F and E_Z for the nuclei in Table 1. The results are given in Table 3. The second column of this table is taken from the 1964 Atomic Mass Table (Mattauch, Thiele, and Wapstra 1965). Apart from ^{52}Fe and ^{40}Ca, ^{4}He is stable up to

[3] This procedure has been suggested to us by W. B. Hubbard.

TABLE 1

VALUES OF THE QUANTITIES log G (see Eq. [48]) AS A FUNCTION OF THE DENSITY $\rho_6(=10^{-6}\rho)$

Log ρ_6 / Z	0.0	0.6	1.4	2	2.6	3.4	4	5	5.6	6.2
2	-1.380	-0.543	0.410	1.055	1.674	2.483	3.086	4.087	4.687	5.287
	-0.892	-0.007	0.988	1.648	2.274	3.085	3.689	4.691	5.291	5.891
	-0.740	0.164	1.181	1.849	2.479	3.293	3.897	4.899	5.499	6.099
	-0.694	0.219	1.246	1.919	2.551	3.367	3.971	4.973	5.573	6.173
	-0.586	0.367	1.446	2.144	2.788	3.610	4.216	5.219	5.819	6.419
	-0.644	0.291	1.847	2.034	2.672	3.491	4.096	5.099	5.699	6.299
	-0.646	0.292	1.853	2.041	2.680	3.500	4.105	5.107	5.708	6.308
6	-1.380	-0.543	0.410	1.055	1.674	2.483	3.086	4.087	4.687	5.287
	-1.052	-0.179	0.824	1.460	2.083	2.894	3.497	4.499	5.099	5.699
	-0.960	-0.076	0.918	1.578	2.204	3.016	3.619	4.621	5.221	5.821
	-0.931	-0.044	0.954	1.615	2.241	3.054	3.657	4.659	5.259	5.859
	-0.881	0.015	1.023	1.688	2.315	3.129	3.732	4.734	5.334	5.935
	-0.891	-0.001	1.001	1.668	2.290	3.108	3.706	4.708	5.308	5.908
	-0.883	0.007	1.009	1.671	2.298	3.111	3.714	4.716	5.316	5.916
8	-1.380	-0.543	0.410	1.055	1.674	2.483	3.086	4.087	4.687	5.287
	1.085	-0.216	0.765	1.420	2.043	2.854	3.457	4.458	5.058	5.659
	-1.002	-0.123	0.867	1.525	2.150	2.962	3.565	4.566	5.167	5.767
	-0.975	-0.093	0.900	1.559	2.184	2.996	3.599	4.601	5.201	5.801
	-0.933	-0.044	0.956	1.618	2.245	3.058	3.661	4.662	5.263	5.863
	-0.939	-0.054	0.942	1.602	2.228	3.040	3.644	4.645	5.245	5.846
	-0.932	-0.047	0.950	1.610	2.236	3.049	3.652	4.653	5.254	5.854
12	-1.380	-0.543	0.410	1.055	1.674	2.483	3.086	4.087	4.687	5.287
	-1.127	-0.262	0.716	1.869	1.992	2.803	3.405	4.407	5.007	5.607
	-1.056	-0.182	0.803	1.459	2.088	2.895	3.497	4.499	5.099	5.699
	-1.032	-0.156	0.882	1.489	2.113	2.925	3.527	4.529	5.129	5.729
	-1.000	-0.118	0.875	1.534	2.160	2.972	3.575	4.576	5.177	5.777
	-1.004	-0.124	0.867	1.525	2.150	2.962	3.565	4.567	5.167	5.767
	-0.999	-0.118	0.874	1.532	2.157	2.969	3.572	4.574	5.174	5.774
14	-1.380	-0.543	0.410	1.055	1.674	2.483	3.086	4.087	4.687	5.287
	-1.142	-0.278	0.699	1.351	1.974	2.785	3.387	4.389	4.989	5.589
	-1.074	-0.202	0.781	1.437	2.060	2.871	3.474	4.476	5.076	5.676
	-1.052	-0.177	0.808	1.465	2.088	2.900	3.503	4.504	5.105	5.705
	-1.023	-0.148	0.848	1.506	2.131	2.942	3.545	4.547	5.147	5.747
	-1.027	-0.149	0.841	1.498	2.123	2.935	3.538	4.539	5.139	5.739
	-1.021	-0.142	0.847	1.505	2.130	2.942	3.545	4.546	5.147	5.747
20	-1.380	-0.543	0.410	1.055	1.674	2.483	3.086	4.087	4.687	5.287
	-1.174	-0.312	0.662	1.314	1.936	2.746	3.349	4.350	4.950	5.550
	-1.114	-0.246	0.734	1.388	2.011	2.822	3.425	4.426	5.027	5.627
	-1.094	-0.224	0.758	1.413	2.036	2.847	3.450	4.452	5.052	5.652
	-1.071	-0.196	0.790	1.446	2.070	2.881	3.484	4.486	5.086	5.686
	-1.072	-0.198	0.786	1.442	2.066	2.877	3.480	4.481	5.082	5.682
	-1.067	-0.198	0.792	1.448	2.072	2.883	3.486	4.488	5.088	5.688
26	-1.380	-0.543	0.410	1.055	1.674	2.483	3.086	4.087	4.687	5.287
	-1.195	-0.335	0.637	1.288	1.910	2.720	3.323	4.324	4.925	5.525
	-1.141	-0.275	0.703	1.356	1.978	2.789	3.392	4.398	4.994	5.594
	-1.123	-0.255	0.725	1.379	2.002	2.812	3.415	4.417	5.017	5.617
	-1.102	-0.231	0.752	1.407	2.080	2.841	3.444	4.446	5.046	5.646
	-1.102	-0.232	0.750	1.404	2.028	2.839	3.442	4.443	5.043	5.643
	-1.098	-0.227	0.755	1.410	2.088	2.845	3.447	4.449	5.049	5,649

TABLE 2

COMPARISON OF THE CONDUCTIVE OPACITIES *KHH* AND $K_c{}^0$ FOR DIFFERENT VALUES OF Z AND Γ AT $\rho_6=1$

Z	Log KHH	LogK_c^0	KHH/K_c^0
2	-0.33985E 01	-0.24624E 01	0.91194
	-0.42985E 01	-0.47104E 01	2.58206
	-0.47485E 01	-0.53564E 01	4.05478
	-0.51985E 01	-0.60644E 01	7.34458
	-0.56185E 01	-0.65664E 01	8.87088
	-0.57985E 01	-0.68044E 01	10.13833
6	-0.17215E 01	-0.11213E 01	0.25107
	-0.26364E 01	-0.24333E 01	0.62656
	-0.30864E 01	-0.30623E 01	0.94616
	-0.35364E 01	-0.37123E 01	1.49957
	-0.39414E 01	-0.42423E 01	1.99971
	-0.41364E 01	-0.45103E 01	2.36574
8	-0.12969E 01	-0.56338E 00	0.18469
	-0.21964E 01	-0.18464E 01	0.44663
	-0.26464E 01	-0.24734E 01	0.67137
	-0.30964E 01	-0.31154E 01	1.04464
	-0.35014E 01	-0.36494E 01	1.40594
	-0.36964E 01	-0.39164E 01	1.65946
12	-0.67325E 00	-0.25471E 00	0.11804
	-0.15854E 01	-0.10363E 01	0.28241
	-0.20353E 01	-0.16603E 01	0.42165
	-0.24853E 01	-0.22923E 01	0.64116
	-0.28903E 01	-0.28283E 01	0.86690
	-0.30853E 01	0.30933E 01	1.01851
14	-0.44389E 00	0.55666E 00	0.09987
	-0.13385E 01	-0.71134E 00	0.23593
	-0.17884E 01	-0.13333E 01	0.35070
	-0.22534E 01	-0.19823E 01	0.53575
	-0.26584E 01	-0.25183E 01	0.72438
	-0.28384E 01	-0.27643E 01	0.84327
20	-0.82401E 01	0.12636E 01	0.06589
	-0.80921E 00	-0.16441E 01	0.16115
	-0.12586E 01	-0.63644E 00	0.23869
	-0.17085E 01	-0.12594E 01	0.35559
	-0.21135E 01	-0.17984E 01	0.48413
	-0.23085E 01	-0.20634E 01	0.56881
26	0.43709E 00	0.17785E 01	0.04556
	-0.41174E 00	0.50450E 00	0.12127
	-0.85992E 00	-0.11350E 00	0.17930
	-0.13096E 01	-0.73450E 00	0.26601
	-0.17145E 01	-0.12745E 01	0.36304
	-0.19095E 01	-0.15385E 01	0.42556

10^{11} g cm^{-3}, ^{12}C and ^{16}O up to 10^{10} g cm^{-3}, and ^{24}Mg and ^{28}Si up to 10^8 g cm^{-3}. In the case $\phi = 1$, equation (48) would not depend on this process because the variable Z/A enters only in ρ_6, (eq. [26]), and Table 1 depends only on ρ_6. The ratio $A{:}Z/A$ enters in the definition of ξ (eq. [50]). The ion effect as seen from Table 1 increases with Γ; therefore, in each table the values for $\Gamma \approx 75$–100 could be affected by the process of inverse β-decay more than the values for lower Γ.

VI. THE CLASSICAL LIMITS

In this section we will show that in the limit $r_0^{-2} \to 0$, $\phi \to 1$, the formula for the conductivity reduces to the one obtained by simple classical nonrelativistic arguments based on Drude theory (Jackson 1962, p. 459). We first consider that the Fermi-Dirac distribution has to be changed into the Boltzmann distribution

$$f_{\rm FD} = g[1 + Z^{-1} \exp (E/kT)]^{-1} \to Z \exp (-E/kT) , \tag{60}$$

where (Huang 1963)

$$Z = n_e(2\pi\hbar/m)^3(m/2\pi kT)^{3/2} . \tag{61}$$

TABLE 3

ENERGIES AND MATTER DENSITIES OF VARIOUS NUCLEI

Nucleus (1)	E_Z(MeV) (2)	ρ_6 (3)	E_F(MeV) (4)
^{4}He	+25.8	1	0.207
^{12}C	+13.4	10^2	1.87
^{16}O	+10.4		
^{24}Mg	+ 5.51	10^3	4.52
^{28}Si	+ 4.6	10^4	8.623
^{40}Ca	+ 1.31	10^5	22.68
^{52}Fe	− 2.374	10^6	50

We therefore have

$$f_{\rm FD} \to n_e(2\pi\hbar/m)^3 f_{\rm B} , \quad f_{\rm B} = (m/2\pi kT)^{3/2} \exp - (mv^2/2kT) . \tag{62}$$

The nonrelativistic limit of equation (43) gives

$$\sigma_{\rm NR} = -(4\pi/3)e^2\tau_j m^{-1}(2\pi\lambda\!\!\!^{-}_c)^{-3}\int_0^\infty d\rho\rho^3 F_{\rm NR}{}^{-1}(\epsilon)(\partial f_{\rm B}/\partial\rho) . \tag{63}$$

The nonrelativistic limit of $F(\epsilon)$, equation (34), in the limits $\phi \to 1$, $r_0^{-2} \to 0$, is simply

$$F_{\rm NR}(\epsilon) = 2\rho^{-3}\int_0^\pi (1 - \cos\theta)^{-1} \sin\theta d\theta = 2\rho^{-3} \ln \Lambda , \tag{64}$$

where the lower limit has been changed to θ_0 to avoid divergences. We therefore have

$$\sigma_{\rm NR} = -(4\pi/3)e^2\tau_1 m^{-1}(2\pi\lambda\!\!\!^{-}_c)^{-3}c^{-6}(2 \ln \Lambda)^{-1}\int_0^\infty v^6(\partial f_{\rm B}/\partial v)dv . \tag{65}$$

The integration can be first performed by parts, giving

$$-6\int v^5 f_{\rm B} dv , \tag{66}$$

which finally can be evaluated by remembering the general formula (Landau and Lifshitz 1958)

$$\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty v^{2+n}\exp\left(-\frac{mv^2}{2kT}\right)dv = (2\pi^{3/2})^{-1}\left(\frac{2kT}{m}\right)^{n/2}\Gamma[\tfrac{1}{2}(n+3)]\,. \quad (67)$$

We have

$$\sigma_{\rm NR} = (4\pi/3)e^2\tau_1 m^{-1}(2\pi\lambda\!\!\!^{-}_c)^{-3}c^{-6}(2\ln\Lambda)^{-1}6n_e(2\pi\hbar/m)^3\pi^{-3/2}(2kT/m)^{3/2}\,. \quad (68)$$

Substituting the values of τ_1 given in equation (36), we finally obtain

$$\sigma_{\rm NR} = 4(2/\pi)^{3/2}n_e n_i^{-1}Z^{-1}(kT/mc^2)^{3/2}(\alpha Z)^{-1}(mc^2/\hbar)(\ln\Lambda)^{-1}\,. \quad (69)$$

In the case of complete ionization, we have from equations (22) and (24)

$$n_e = (\rho/m_{\rm H})(Z/A)\,,\quad n_i = (\rho/m_{\rm H})A^{-1}\,,\quad n_e n_i^{-1}Z^{-1} = 1\,, \quad (70)$$

and we finally obtain

$$\sigma_{\rm NR} = 4(2/\pi)^{3/2}(kT/mc^2)^{3/2}(\alpha Z)^{-1}(mc^2/\hbar)(\ln\Lambda)^{-1}\,, \quad (71)$$

which coincides with the expression obtained from classical electrodynamics (Jackson 1962, p. 461).

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