Exact and Approximate Solutions for Multiple Scattering by Cloudy and Hazy Planetary Atmospheres

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ABSTRACT

Solutions are obtained for the problem of multiple scattering by a plane parallel atmosphere with anisotropic phase functions typical of cloud and haze particles. The resulting albedos, angular distributions of intensities, and planetary magnitudes are compared to solutions obtained with approximate analytic phase functions and, in the case of the cloud phase function, to the solution obtained with the forward diffraction peak omitted from the phase function.

It is shown that the cloud phase function with the truncated peak yields results practically identical to those obtained with the complete cloud phase function, not only for albedos and magnitudes, but also for the angular distribution; the approximation introduces errors of several per cent in the angular distribution for direct backscattering (the region of the glory), for emergent angles near grazing regardless of the incident angle, and, of course, a larger error occurs for total scattering angles near 0°. However, the errors are unimportant for many applications, and hence a large reduction in computer time is possible. This is particularly useful, for example, in making practical the computations needed for interpreting the phase curve, limb darkening and spectral reflectivity of Venus.

It is shown that the Henyey-Greenstein phase function, based on the asymmetry factor $\langle \cos\theta \rangle$, yields spherical and plane albedos and planetary magnitudes (for optically thick atmospheres) close to those obtained with the cloud and haze phase functions. The Kagiwada-Kalaba phase function, based on the ratio of forward to backward scattering, gives significantly less satisfactory results for the same quantities. Neither of the two analytic phase functions can accurately duplicate the true angular distribution of scattering by thin clouds; however, the results are better with thick layers, especially for hazes. The results indicate that the Henyey-Greenstein phase function may be useful for problems such as line formation in planetary atmospheres.

1. Introduction

The scattering of light in planetary atmospheres is in most cases characterized by a significantly anisotropic phase function (scattering diagram) since the presence of even a small number of haze or cloud particles results in a forward elongation in the single scattering. As a consequence of this anisotropy it is usually necessary to employ numerical methods to solve problems of diffuse reflection (multiple scattering); however, serious difficulties occur in the numerical approach if the phase function contains a narrow diffraction spike or if solutions are required at hundreds of wavelengths, as in the case of line formation. Hence, it is of interest to examine whether approximate phase functions can yield substantially the same multiple scattering results as the exact phase functions.

In the scattering of light by clouds, if the cloud particles are significantly larger than the wavelength then Fraunhofer diffraction around the particles gives rise to a sharp forward peak in the phase function. It has sometimes been assumed, without numerical verification, that radiation scattered into the narrow diffraction spike may be treated as being unscattered, i.e., the spike may be truncated from the phase function and the interaction optical thickness of the cloud reduced correspondingly. That assumption greatly reduces the computational difficulties and it is thus of interest to determine the angles, if any, for which the approximation is accurate. [Potter (1969) is also making computations to test the truncated peak approximation and we are indebted to him for a preprint of his results. Potter includes graphs of the transmission function and makes computations only for the azimuth independent case, but where comparable our results are in substantial agreement.]

In other scattering problems it is sometimes known that the phase function is anisotropic but its exact shape, which depends on the particle concentration, shape, size distribution, refractive index and absorbtivity, is unknown. In such cases it may be advantageous to employ a family of analytic phase functions which are a function of a single parameter, representing the degree of anisotropy of the phase function. It is not expected that such analytic phase functions could accurately match the multiple scattering results if the phase functions to be matched were of any imaginable shape; however, the scattering by cloud and haze particles (both of which we assume to be water) has a certain regularity, the degree of forward scattering

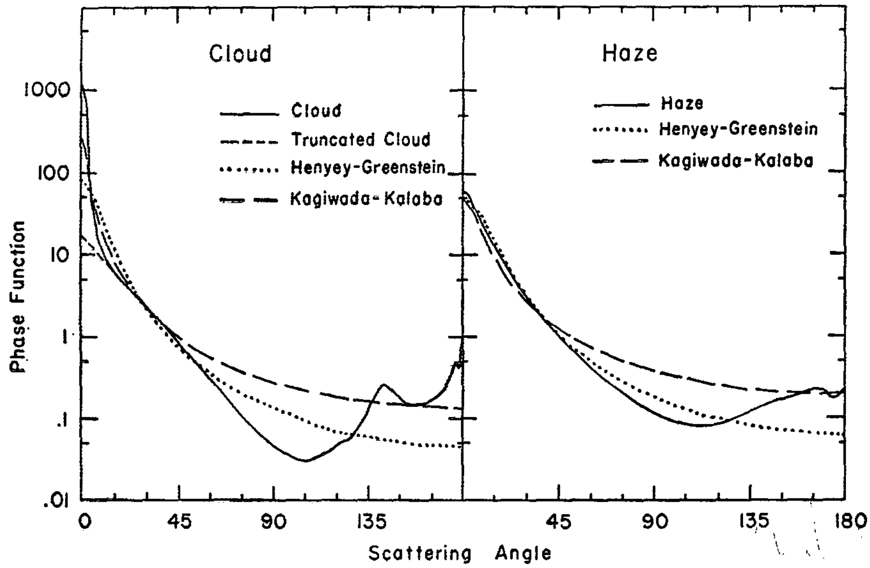

FIG. 1. Phase functions for scattering by cloud particles (left) and haze particles (right). The exact phase functions are from Mie theory for spherical water particles; the Henyey-Greenstein functions are from (3) with $g=0.844$ (left) and 0.794 (right); the Kagiwada-Kalaba functions are from (4) and (5) with $b=1.00105$ (left) and 1.00819 (right).

generally increasing with increasing particle size, and hence it is conceivable that a well chosen analytic phase function may be useful. Moreover, in many applications an integrated intensity, such as the spherical albedo, plane albedo, or planetary magnitude at a given phase angle, is desired. In these cases the effect of secondary features in the phase function (such as the glory or cloudbow) should be at least partially averaged out, thus increasing the possibility that the use of a smooth analytic phase function may yield adequate results.

It is the purpose of this paper to determine the accuracy of the multiple scattering solutions obtained by omitting the diffraction peak from cloud phase functions; a second purpose is to compare exact multiple scattering solutions for phase functions typical of both clouds and hazes to solutions obtained with analytic phase functions—in particular, the functions proposed by Henyey and Greenstein (1941) and by Kagiwada and Kalaba (1967). For all the phase functions the calculations are made for both optically thin and optically thick atmospheres.

Irvine (1968) has previously made exact computations for an azimuthally symmetric radiation field and compared them to intensities computed with the small angle method of Romanova and to fluxes computed with the Eddington and two-stream approximations. Irvine found the small angle method to be accurate but the approximations for the flux resulted in serious errors in some cases.

2. Phase functions

The phase function which we have taken as being typical of clouds is shown in Fig. 1. This was kindly computed by H. Cheyney from Mie scattering theory for the "cloud model" size distribution (Deirmendjian, 1964) of spherical water droplets; this size distribution has its maximum at the particle diameter $d=8\,\mu$ and it has a mean extinction diameter $\sim 10\,\mu$. Here, as elsewhere in this paper, polarization is neglected, i.e., the theoretical phase function is obtained as the average over the two perpendicular polarizations. The computations were for the wavelength $\lambda=8189$ Å, but, since the absorbtivity of water is negligible in the range 3000 Å$\lesssim\lambda\lesssim$9000 Å and the real refractive index is only mildly wavelength dependent, the results are approximately valid for other wavelengths in that range with the particle sizes modified such that the size parameter $x=\pi d/\lambda$ is the same as indicated above.

The asymmetry parameter $\langle\cos\theta\rangle$, which is defined as the weighted mean over the sphere of the cosine of the scattering angle with the phase function $P(\cos\theta)$ as the weighting function, i.e.,

$$\langle\cos\theta\rangle\equiv\tfrac{1}{2}\int_{-1}^{1} P(\cos\theta)\,\cos\theta d(\cos\theta), \qquad (1)$$

has the value 0.844 for the cloud phase function in Fig. 1. The asymptotic value as $x\rightarrow\infty$ is ~ 0.87 using 1.33 for the real refractive index (van de Hulst, 1957, p. 226). The significance of the asymmetry parameter, which will be employed in our approximate calculations, has been discussed by van de Hulst (1957) and by Irvine (1963, 1965a) who computes that quantity for single particles for various refractive indices and particle size parameters.

The phase function for haze particles, also shown in Fig. 1, was computed by Deirmendjian (1964) at

$\lambda = 7000$ Å for his haze model M, which has its maximum concentration at diameter $0.1\,\mu$. It is typical of haze particles for visual wavelengths; there is no sharp diffraction peak since the particle diameters are less than the wavelength and the back peak is less pronounced than for cloud particles. For this phase function $\langle\cos\theta\rangle = 0.794$.

The first approximate phase function employed in our multiple scattering computations was that obtained by truncating the forward diffraction peak; the truncated phase function was obtained by taking the slope of the logarithm of the phase function as being constant for $\theta < 20°$, with the slope equal to that of the untruncated phase function at 20° (Fig. 1). For the particular cloud phase function in Fig. 1 the fraction of light contained in the forward peak,

$$f = \int_{4\pi}\int (P - P')\frac{d\omega}{4\pi}, \tag{2}$$

is 0.434, where P and P' are the cloud and truncated cloud phase functions, respectively; hence, computations with P for an optical thickness τ should be compared to computations with P' for a thickness 0.566τ. (For nonconservative scattering ω_0 as well as τ must be scaled; see, for example, Potter, 1969).

The analytic phase function which has been used most extensively in the literature was introduced by Henyey and Greenstein (1941); it is defined by

$$P(\cos\theta) = \frac{1 - g^2}{(1 + g^2 - 2g\cos\theta)^{\frac{3}{2}}}. \tag{3}$$

The merits of this phase function have been discussed by van de Hulst and Irvine (1963) and van de Hulst and Grossman (1968). For the Henyey-Greenstein function, $\langle\cos\theta\rangle = g$, and this parameter may vary from 0 for isotropic scattering to 1 for an infinitely narrow forward beam. [Values of g in the range $(-1,0)$ could be used to represent predominately backward scattering or a combination of two Henyey-Greenstein phase functions, one with positive g and one with negative, could be used to simulate atmospheric phase functions (Irvine, 1965b), but the latter case would require the use of three parameters.] To correspond to the cloud and haze phase functions the values $g = 0.844$ and $g = 0.794$, respectively, were used in (3); the resulting Henyey-Greenstein phase functions are shown in Fig. 1.

Kagiwada and Kalaba (1967) have recently introduced a rational phase function defined by

$$P(\cos\theta) = \frac{k}{b - \cos\theta}, \tag{4}$$

where

$$k = 2\left[\ln\left(\frac{b+1}{b-1}\right)\right]^{-1}. \tag{5}$$

The Kagiwada-Kalaba phase function hence depends on the single parameter b and if this function is to be used to simulate scattering by atmospheric particles some procedure must be established to determine the value of b for a given atmospheric phase function. It is not practical to require either $\langle\cos\theta\rangle$ or the fraction of light scattered into the forward hemisphere (which is typically $\gtrsim 0.95$ for cloud particles) to be the same for the Kagiwada-Kalaba phase function as for the atmospheric phase functions because for clouds as many as 10^3–10^4 terms would be required in the cosine expansion of the resulting Kagiwada-Kalaba function. However, it is possible to use a different parameter discussed by Kagiwada and Kalaba, the ratio of forward to backward scattering, $P(\cos 0°)/P(\cos 180°) \equiv r$, in terms of which b is given by

$$b = \frac{r+1}{r-1}. \tag{6}$$

For the cloud phase function $r \sim 1900$ and hence $b \sim 1.00105$, while for the haze phase function $r \sim 245$ and $b \sim 1.00819$; the corresponding Kagiwada-Kalaba phase functions are shown in Fig. 1.

Other simple analytic phase functions such as $P = 1$ (isotropic), $P = \frac{3}{4}(1 + \cos^2\theta)$ (Rayleigh phase function) and $P = 1 + a\cos\theta$ have been employed in the literature; these, however, cannot be used to simulate the anisotropic scattering of clouds or hazes.

3. Computational procedure

The computing method has been described in detail elsewhere (Hansen, 1969). It is based on a doubling principle first stated by van de Hulst (1963); using that principle the scattering and transmission functions for an atmosphere of optical thickness 2τ can be obtained from the same functions for an atmosphere of thickness τ. In the procedure employed here the doubling principle is applied repeatedly after beginning with a layer of such small thickness ($\tau_0 = 2^{-25}$) that multiple scattering is negligible; in that case the initial scattering and transmission functions are given by analytic expressions proportional to the phase function. The phase functions and the scattering and transmission functions were expanded in cosine series with from as few as 40 terms (for the Henyey-Greenstein phase function with $g = 0.794$) to as many as 180 terms (for the cloud phase function). The number of terms employed and the number of intervals used in the Gauss integrations were varied with the results suggesting that the errors do not exceed $\sim 0.1\%$.

Computations were made of the spherical albedo as a function of optical thickness. The spherical albedo, which is the ratio of the total radiation reflected from a spherical atmosphere to the total radiation incident from a distant source, is given in terms of the scattering

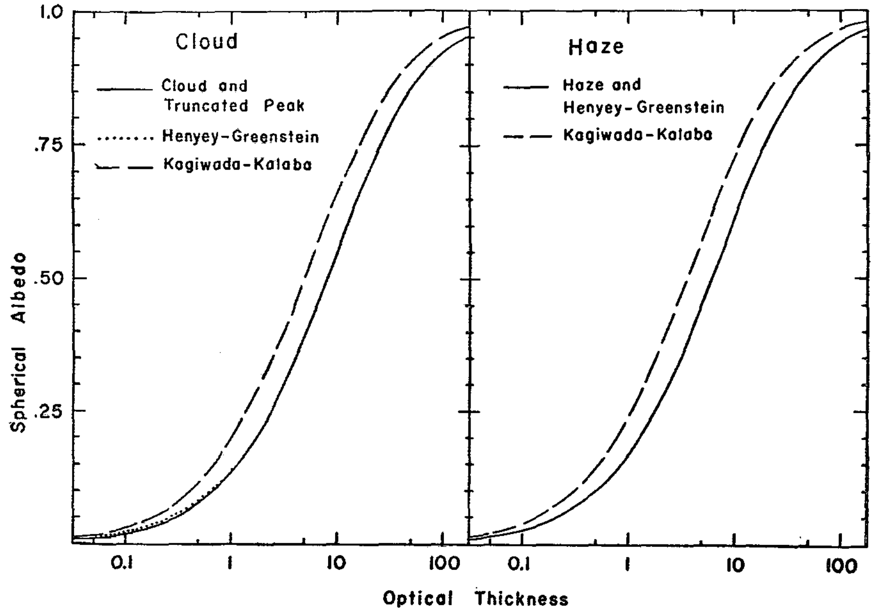

FIG. 2. Spherical albedo of a plane parallel layer with conservative scattering for the phase functions of Fig. 1. The albedos for the cloud and truncated cloud phase functions (left) are indistinguishable, as are the albedos for the haze phase function and the corresponding Henyey-Greenstein function (right). With the truncated peak phase function the optical thickness was reduced by the factor $1-f=0.566$ [see (2)] in the calculations for Figs. 2–9.

function of Chandrasekhar (1950) by

$$A=\int_0^1\int_0^1 S^0(\tau;\,\mu,\mu_0)d\mu d\mu_0, \tag{7}$$

where $S^0(\tau;\mu,\mu_0)$ is the first (azimuth independent) term in the cosine expansion of the complete scattering function $S(\tau;\mu,\phi;\mu_0,\phi_0)$.

The plane albedo, which is the ratio of the radiation reflected from a plane parallel atmosphere to the incident radiation, was also computed; this quantity is a function of the angle of incidence $\theta_0=\cos^{-1}\mu$ (measured from the surface normal) and is given in terms of the scattering function by

$$a(\mu_0)=\frac{1}{2\mu_0}\int_0^1 S^0(\tau;\,\mu,\mu_0)d\mu. \tag{8}$$

The angular distribution of the scattered light is given in our graphs by the reflection function defined as

$$R(\tau;\,\mu,\phi;\,\mu_0,\phi_0)=\frac{S(\tau;\,\mu,\phi;\,\mu_0,\phi_0)}{4\mu}. \tag{9}$$

This is a convenient quantity to employ since for a Lambert reflector it would be represented by a horizontal line when plotted against either θ or $\phi-\phi_0$.

In observations of planets the planetary disk is often not resolved; hence, the quantity of interest in the magnitude of the planet as a function of the phase angle α, the latter being the angle at the planet between the directions of the earth and sun (divided by 180°, it gives the fraction of the hemisphere turned toward the earth that is in darkness). The magnitude is given by (Horak, 1950)

$$m(\alpha)=C-2.5\log_{10}\int\int S(\tau;\,\mu,\phi;\,\mu_0,\phi_0)d\sigma, \tag{10}$$

where the integration is over the visible hemisphere and C is a constant.

4. Computational results

A comparison of the multiple scattering results for the cloud and truncated cloud phase functions will be presented first. The spherical albedos, shown in Fig. 2, are indistinguishable for those two phase functions on the scale of that diagram. The percentage error introduced by truncating the phase function is greatest for very thin layers since in that case single scattering dominates and hence grazing incident and emergent angles contribute a non-negligible amount to the albedo. At $\tau=0.125$ the error is $\sim 1.0\%$, but for $\tau=1$ it is $\sim 0.3\%$; for thicker layers it is still less.

The local albedos for the same phase functions are compared in Fig. 3. Although differences of several per cent occur for near grazing incident angles ($\mu_0<0.1$), the results are practically indistinguishable for other angles at all optical thicknesses.

Perhaps the most striking result of the computations is the very close agreement of the angular distributions of the diffusely reflected light for the cloud and trun-

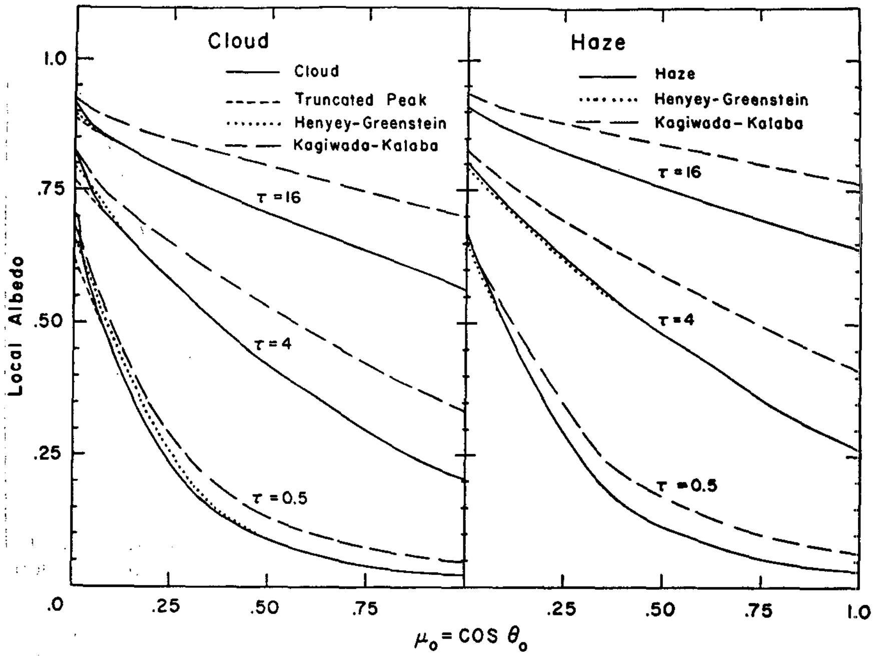

FIG. 3. The albedo of a plane parallel layer as a function of direction for conservative scattering for the phase functions of Fig. 1.

cated cloud phase functions. Examples of the reflectivities at representative incident and emergent angles are shown in Figs. 4–8. For most angles the reflectivities for the two phase functions differ by an amount too small (<1.0%) to be visible in the graphs. As expected, the results begin to deviate for small scattering angles, as shown, for example, in Fig. 6 with $\theta_0 = 85°$ and $\theta \sim 90°$ and in Fig. 8 with $\theta = \theta_0 = 85°$ and $\phi - \phi_0 \sim 0°$. The differences, however, are not very large for total scattering angles

$$\theta \equiv \mu\mu_0 + (1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}} \cos(\phi-\phi_0)$$

larger than 10°; i.e., multiple scattering does not cause the effect of the forward peak to spread out very much.

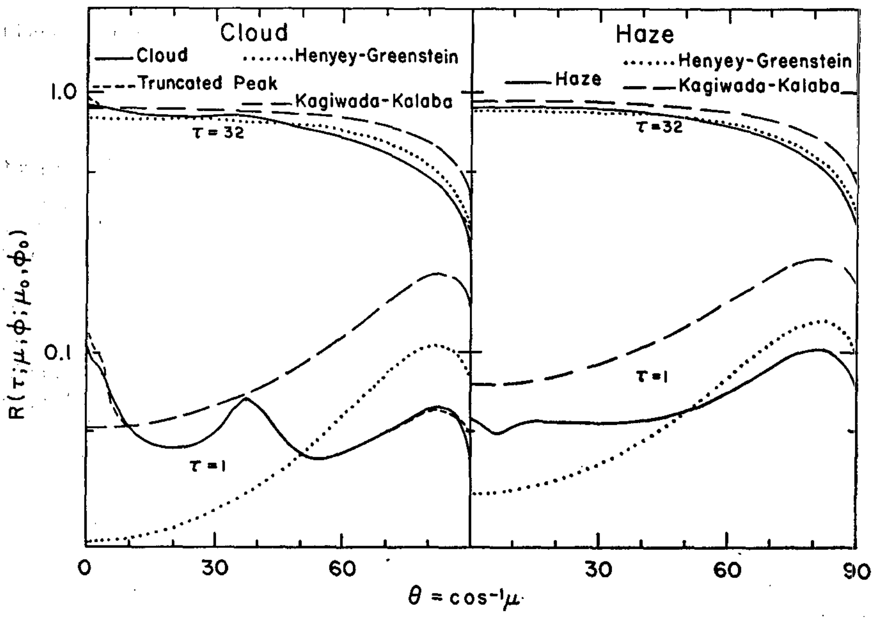

FIG. 4. Reflectivity of a plane parallel layer with conservative scattering for the phase functions of Fig. 1 with $\theta_0 = 0°$. Because of the normal incidence the reflectivity applies for all $\phi - \phi_0$. For most angles the reflectivities with the cloud and truncated cloud phase functions are indistinguishable.

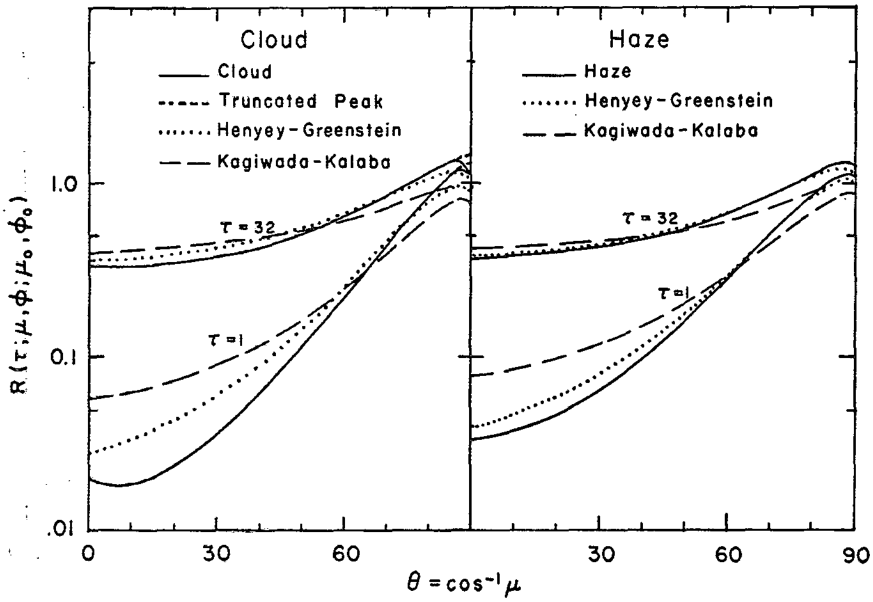

FIG. 5. Same as Fig. 4 except that $\theta_0=60$ and $\phi-\phi_0=0°$.

A second effect of truncating the forward peak, interestingly enough, is to introduce a noticeable error (several per cent at most) for direct *backscattering*. This is illustrated for $\theta_0=0°$ and $\theta\sim0°$ in Fig. 4; the reflectivity for the truncated phase function exceeds that for the true phase function for angles close to direct backscattering. The effect arises from multiply scattered photons whose first scattering, and all subsequent scatterings except the last, are at total scattering angles $\sim0°$, and whose last scattering is at a total scattering angle $\sim180°$. The number of such photons is significant because of the strong forward spike and the back peak (glory) in the phase function. The truncated peak approximation, in effect, replaces photons scattered through a few degrees by photons scattered through 0°; hence, with the true phase function, the back peak in the reflectivity is reduced or spread out more by multiple scattering than it is with the truncated phase function. For the same reason the reflectivity with the true phase function slightly exceeds that with the truncated phase

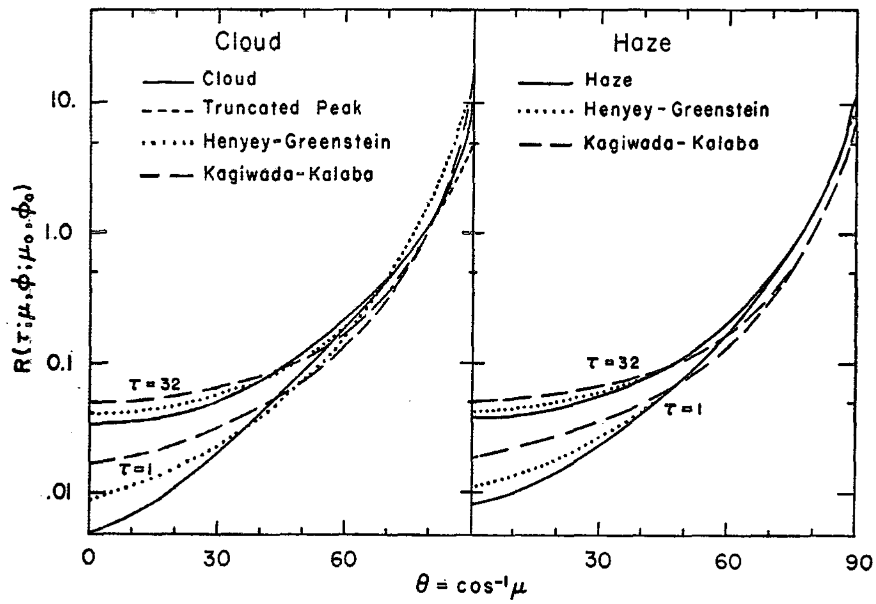

FIG. 6. Same as Fig. 4 except that $\theta_0=85°$ and $\phi-\phi_0=0°$.

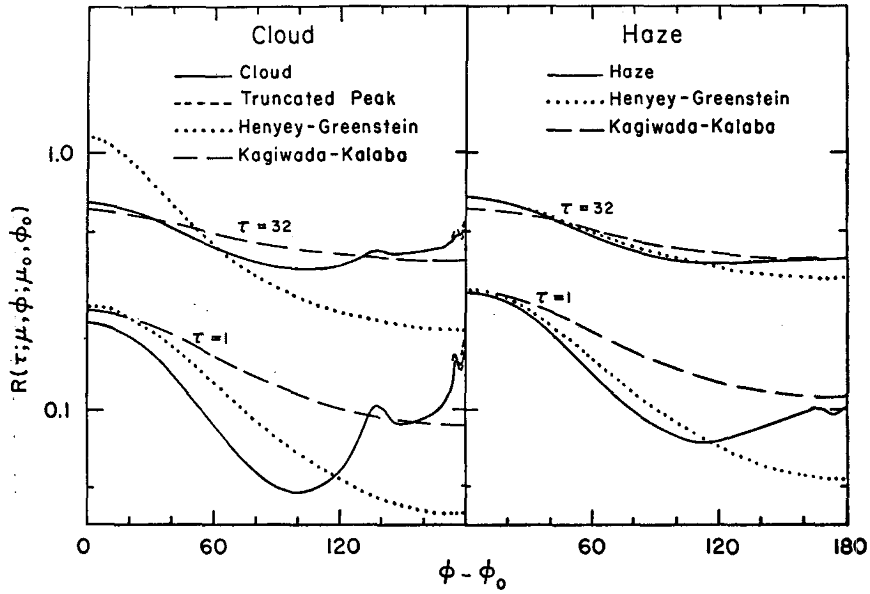

FIG. 7. Reflectivity of a plane parallel layer for conservative scattering for the phase functions of Fig. 1 with $\theta_0=\theta=60°$ (from the normal).

function for scattering angles several degrees from direct backscattering. These effects are also visible in Figs. 7 and 8 for $\phi-\phi_0\sim 180°$.

Still a third noticeable error is introduced by the truncating of the forward spike, again several per cent at most. This occurs for grazing viewing angles $\theta\sim 90°$ (Figs. 4 and 5); it exists regardless of the incident angle, although in Fig. 6 it is not visible since it occurs at the same angle as the first error discussed above. An explanation for the third error is as follows: At a small distance below the top of the atmosphere the diffuse radiation field has some angular distribution, generally anisotropic; photons moving upward, but in a direction nearly parallel to the upper surface, have a greater probability of being scattered (and changing their direction) before emerging than do photons moving more perpendicular to the surface. Hence, a relative limb darkening occurs in the emerging diffuse radiation,

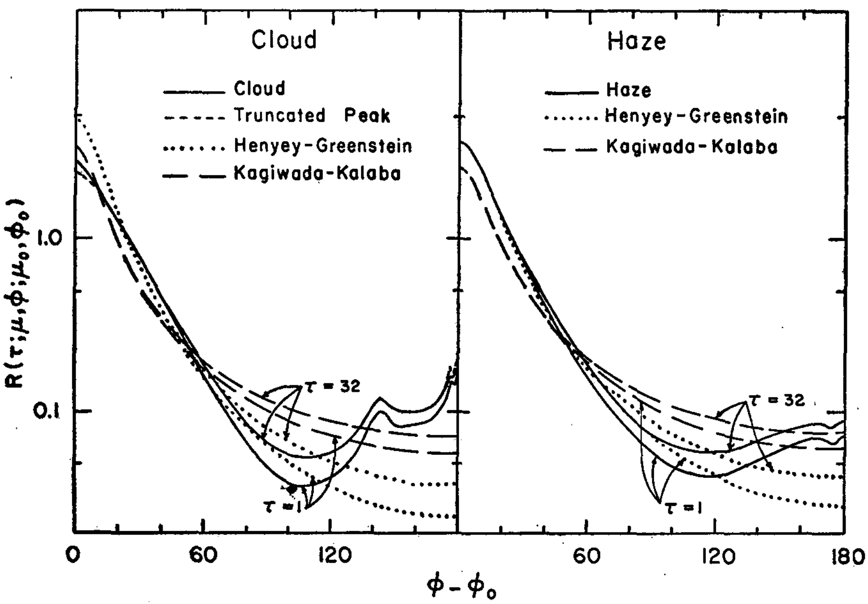

FIG. 8. Same as Fig. 7 except that $\theta=\theta_0=85°$. For $\tau=1$ and $\phi-\phi_0\gtrsim 175°$ the reflectivity for the truncated cloud phase function slightly exceeds that for the cloud phase function, but only the latter is shown due to the crowding of lines at those angles.

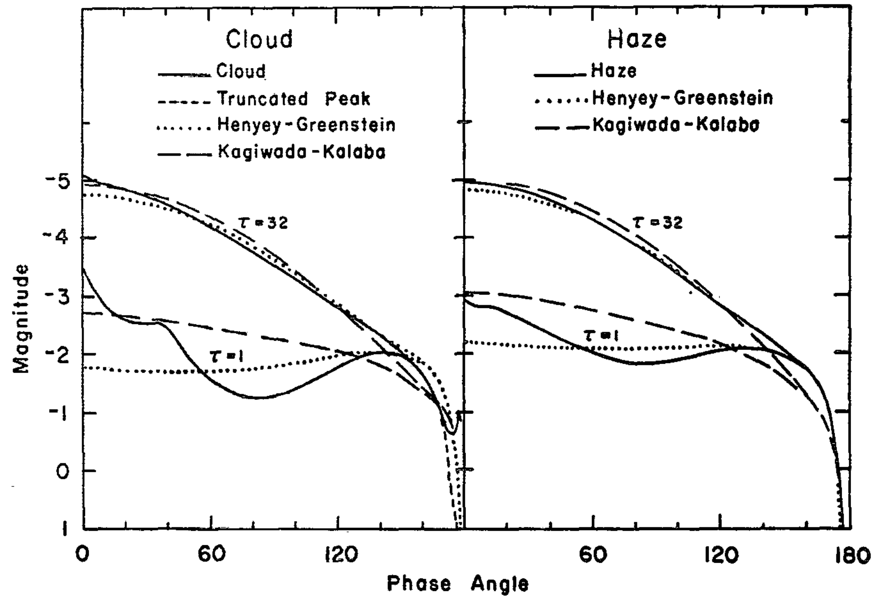

FIG. 9. Planetary magnitude as a function of phase angle (earth-planet-sun angle) for a plane parallel atmosphere with conservative scattering for the phase functions of Fig. 1. The vertical scale applies to the planet Venus; for other planets an appropriate constant must be added. Ground reflection from the planet is neglected.

and the more anisotropic the phase function, the greater is the limb darkening. Therefore, since truncating the peak decreases the scattering anisotropy, the limb darkening is reduced for the truncated phase function. Since the scattering function is symmetric in θ and θ_0, the same error should occur for grazing incident angles; this is verified by the computations. For angles other than those discussed in the above three paragraphs the error introduced by truncating the peak is $\lesssim 1.0\%$.

The results for the planetary magnitude as a function of phase angle are shown in Fig. 9; these indicate that truncating the phase function does not introduce a significant error for phase angles $<170°$. The angular width of the region in which a significant error occurs is not much larger than the width of the spike in the phase function, because single scattering dominates for large phase angles. Calculations were also made with particles half as large (diameters $\sim 4\mu$) as those used for the illustrated phase function; for the smaller particles the truncated peak approximation was good for phase angles $<160°$. For still smaller particles the peak is broad enough that there is no need for it to be truncated.

The results of the computations with the analytic phase functions are also compared in Figs. 2–9 to the computations with the exact cloud and haze phase functions. For both the spherical and plane albedos the Henyey-Greenstein phase function, based on the asymmetry factor $\langle\cos\theta\rangle$, very accurately duplicates the albedos of both the cloud and haze; however, the Kagiwada-Kalaba phase function, based on the ratio of forward to backward scattering, consistently gives albedos that are too large. There are significant errors with the Henyey-Greenstein phase function only for thin layers ($\tau<1$), or, in the case of the local albedo, for incident angles near grazing.

The reflectivities shown in Figs. 4–8 indicate that in general neither of the analytic phase functions can accurately duplicate the angular distribution of scattering by hazes and clouds. However, as would be expected, the results are significantly better for the haze than for the cloud and they are much better for thick layers than for thin ones. Also, in most cases the Henyey-Greenstein phase function yields reflectivities closer to those of the cloud or haze than does the Kagiwada-Kalaba phase function. For thick atmospheres it appears that it would be difficult to distinguish experimentally between the Henyey-Greenstein and cloud phase functions or the Henyey-Greenstein and haze phase functions.

The planetary magnitudes are compared in Fig. 9 which shows clearly that both analytic phase functions lead to poor approximations for thin layers; however, for thick layers the analytic functions provide results which may be sufficiently accurate for some problems.

For scattering by clouds, the truncated peak and analytic phase function approximations may be combined, as suggested by van de Hulst and Irvine (1963); i.e., the light scattered into the forward peak may be treated as unscattered and an analytic function fit to the phase function without the peak. This approach was applied to the cloud with both the Henyey-Greenstein and Kagiwada-Kalaba functions, but the results (not illustrated) did not represent a significant improvement over those obtained by using the analytic phase functions alone.

5. Discussion

The computations presented indicate that if the phase function for scattering by cloud particles is known, then without great computational difficulty it is possible to accurately obtain the angular distribution of scattering by clouds even if there is a strong forward diffraction peak; the error introduced by truncating the forward peak of the phase function is unimportant for most applications (excepting, of course, cases in which the diffraction peak itself is examined, e.g., the study of the aureole in diffuse transmission.) The small error in the reflectivity for viewing or incident angles near grazing ($\theta \sim 90°$) would be very difficult to observe; the error for direct backscattering, which would occur only for phase functions with both a forward spike and back peak (glory), is also a minor effect. A similarly small error would occur for any phase function having a sharp feature in addition to a forward spike; truncating the diffraction peak would cause the corresponding feature in the reflectivity to be sharper than it would be for the true phase function. The largest error caused by truncating the forward spike is that which occurs for total scattering angles near 0°; however, the calculations indicate that the region with significant error is not much wider than the peak itself, a result of the fact that single scattering dominates at grazing angles.

An important application of the truncated peak approximation is to planetary magnitudes. For a particle size distribution with a mean extinction diameter $\sim 10\,\mu$ (for $\lambda \sim 8000$ Å; hence, for size parameters $x \sim 40$), the truncated peak approximation is good for phase angles $\alpha < 170°$. As indicated by Arking and Potter (1968), though, it would be difficult to obtain observational data for phase angles $> 170°$ and the plane parallel atmosphere approximation moreover would become invalid. Therefore, since the truncated peak approximation would be still better for larger particles, it should be very useful for $x > 40$. For size parameters $20 < x < 40$ more care must be exercised since there is still a peak in the phase function and truncation may cause large errors for phase angles $\gtrsim 160°$. For still smaller size parameters the peak is sufficiently broad that the complete phase function may be used without difficulty.

The Henyey-Greenstein phase function, based on the asymmetry factor $\langle \cos\theta \rangle$, accurately simulates the albedos, spherical and plane, of both the cloud and haze; the Kagiwada-Kalaba phase function, at least if it is based on the ratio of forward to backward scattering, is less accurate for the same purpose. This suggests in the case of conservative scattering, for a given τ, that $\langle \cos\theta \rangle$ essentially determines the albedos. Some support for that suggestion follows from calculations we have made (not illustrated here) for several different phase functions having the same value of $\langle \cos\theta \rangle$; the spherical albedos generally agreed within a few per cent and the correspondence improved with increasing optical thickness. Also, Kagiwada *et al.* (1968) showed that the fluxes are nearly equal with isotropic and Rayleigh scattering; both of those phase functions have $\langle \cos\theta \rangle = 0$. The importance of $\langle \cos\theta \rangle$ is apparently due to the fact that both the spherical and plane albedos involve the integral over $\cos\theta$ of the product of the intensity and $\cos\theta$, and in the definition (3) of $\langle \cos\theta \rangle$, P is proportional to the intensity of singly scattered radiation. The importance of $\langle \cos\theta \rangle$ in determining the spherical albedo was recognized by Sagan and Pollack (1967) and incorporated in their two-stream approximation for the flux.

The analytic phase functions cannot, in general, accurately simulate the angular distribution of scattering by haze or clouds. However, for thick atmospheres the Henyey-Greenstein phase function is sufficiently accurate for many purposes; for example, it could be used, as suggested by van de Hulst and Grossman (1968), to test the effect of anisotropic scattering in the problem of line formation in planetary atmospheres. The analytic phase functions cannot, of course, be used in problems in which the effects of specific features in the phase function (such as the cloudbow or glory) are searched for; this is the case, for example, with the photometric magnitude of Venus as a function of phase angle (Arking and Potter, 1968).

Our calculations were made only for conservative scattering but this is the relevant case for scattering by clouds and haze in the visual range; moreover, it is clear that the truncated peak approximation would be at least as accurate in the nonconservative case (Potter, 1969). But, since the analytic phase functions provide their poorest results for single scattering, the multiple scattering solutions obtained with those functions would become less accurate if the single particle albedo were decreased.

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