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ON THE SOLAR DIFFERENTIAL ROTATION AND MERIDIONAL CURRENTS

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ABSTRACT

The solar differential rotation, and large-scale meridional currents are investigated with the axially symmetric, time-independent hydrodynamic equations of motion, including anisotropic convective viscosity forces. The ϕ-component of the equations of motion is integrated to give a linear, ordinary differential equation determining the angular velocity distribution. The Reynolds number for the differential rotation in the convection zone is shown to be large, and an approximation based on this fact is used to solve the equations of motion to first order, under the assumption that the polar heating effects are negligible and that the convection zone is barytropic. A good fit to the observed differential rotation is obtained if the anisotropy parameter satisfies $s - 1 \simeq \frac{1}{5}$, and the differential rotation is then approximately independent of the magnitude of the dynamic convective viscosity η. The circulational velocities near the surface at high latitudes are $\sim (s - 1)\rho^{-1}$ grad η. Reasonable agreement with observed values is shown.

I. INTRODUCTION

The problem of constructing a theoretical model of the solar differential rotation and the large-scale circulation currents is one of long standing, and several attempts have been made to find solutions of the hydrodynamic equations of motion which reproduce the fluid motions observed in the Sun.

These motions are on a scale much larger than the granulation size (i.e., much larger than 1000 km) and are of two sorts: First is the differential rotation, the fact that the Sun's angular velocity at the equator is about 20 per cent larger than that near the poles. Second is the over-all meridional circulation (the component of the fluid velocity in planes of constant longitude), which in the photosphere is observed to be of the order of 10^3–10^4 cm/sec, in contrast to an equatorial rotational velocity of about 2×10^5 cm/sec. Systematic studies of sunspot motions, which presumably reflect velocities deep in the convection zone, show meridional velocities of about 200 cm/sec.

The most widely employed approach to the theoretical understanding of these motions has been to assume that they are primarily generated by the Sun's over-all rotation and that the velocity fields concerned do not depend on time and longitude. This latter assumption, while not definitely established observationally, simplifies the equations of motion a great deal. The present paper employs these assumptions throughout. The effects of "supergranulation" are not discussed.

We here take the standpoint, in common with many previous authors, that the differential rotation and the associated circulations may be described by stationary, axially symmetric solutions of the hydrodynamic equations of motion, with convective (turbulent) viscosity forces included. The ordinary molecular and radiative viscosities can be shown to be many orders of magnitude smaller. This immediately limits the validity of the analysis to dimensions larger than the scale of the turbulence, which is about 500 km at the top of the convection zone and about 7×10^4 km at the bottom. Since the total depth of the convection zone is, in the model used here, about 2×10^5 km, we therefore expect our conclusions about motions at the bottom to be very rough.

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Let us sketch briefly the progress made up to now by other authors and indicate the approach of the present paper. For a more comprehensive review the reader is referred to Mestel (1965).

Most of the previous efforts have centered on the "polar heating" hypothesis, which would derive the meridional circulation and the differential rotation from the fact that in a rotating star strict radiative equilibrium is not possible, the temperature on a given equipotential surface being higher near the pole than at the equator. This sets up an unbalanced acceleration force in the meridional planes.

This idea is based on the well-known von Zeipel theorem, and Randers (1942) showed that the resulting differential rotation is of the proper sign if viscous effects are neglected. Schwarzschild (1947), using numerical integration techniques, predicted a differential rotation of the proper order of magnitude. Sweet (1950) and Öpik (1951) calculated the meridional velocities in detail, and recent refinements have been made by Smith (1966) and Mestel (1966). For the Sun, the velocities predicted by the polar heating theory are generally 10^{-10}–10^{-5} cm/sec, and thus are many orders of magnitude slower than those actually observed.

A newer idea is that the anisotropic convective viscosity of the convection zone must be introduced. Wasiutynski (1946) seems to have been the first to consider anisotropic viscosities in solar physics, and Biermann (1951) has described the physical causes and effects of this anisotropic turbulence in stellar and planetary atmospheres. Kippenhahn (1960, 1963) has calculated a differential rotation and system of circulational currents for the Sun, using Wasiutynski's anisotropic viscosity and neglecting the effects of polar heating. Kippenhahn (1963) uses an iterative approximation based on the assumption that the viscous forces are much larger than the inertial ones; i.e., that the Reynolds number for the differential rotation is small. However, the first-order differential rotation that he calculates is much larger than his zeroth-order rotational velocity, and thus the approximation scheme does not converge. We show in § III that in the solar convection zone the viscous forces are actually very small, and hence that the approximation corresponding to a *large* Reynolds number is the valid one.

Also, Sakurai (1966) has investigated the motions generated by the anisotropy of the viscosity, using an expansion in powers of the ratio of the convection zone depth to the solar radius. We discuss his results in § III.

Another school of thought maintains that the differential rotation is really an average over time and longitude of wind fields which are strongly dependent on these variables. The reader is referred to the papers of Ward (1965) and H. H. Plaskett (1966) for considerations of this sort.

The results of the present paper are as follows: In § II we integrate the ϕ-component of the exact equation of motion, obtaining a linear, ordinary differential equation (in general, a singular one) for the rotation law in the convection zone. If the polar heating is neglected, the magnitude of the differential rotation is shown to depend only "weakly" on the magnitude of the viscosity. In § III we apply the results of § II to the solar convection zone, neglecting polar heating and restricting our attention to the anisotropic convective viscosity. The large Reynolds number approximation is introduced, and the equations of motion are integrated to first order, full account being taken of the effects of variable density and viscosity. The approximation is justified in detail by means of a model of the convection zone by Baker and Temesvary (1966) and is shown to be valid everywhere, except for a thin layer at the surface.

Section IV is devoted to a rough power-series analysis of the approximate equations, valid near the rotational axis. The anisotropy parameter can be adjusted to give a differential rotation of the proper sign and magnitude, and values for the meridional velocities then follow, which are shown to be roughly independent of the order of magnitude of the angular velocity and which agree with observations.

II. GENERAL THEORY

We exhibit the hydrodynamic equations of motion with $\partial/\partial t = \partial/\partial\phi = 0$. Both cylindrical coordinates (R,z,ϕ) and spherical coordinates (r,θ,ϕ) are employed throughout, with $R = r \sin\theta$ and $z = r\cos\theta$; ϕ is the longitude angle, and the polar axis ($R = \theta = 0$) coincides with the axis of rotation. In cylindrical coordinates the equations of motion are

$$v_R \frac{\partial v_R}{\partial R} + v_z \frac{\partial v_R}{\partial z} - \frac{v_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} - \frac{\partial \psi}{\partial R} + \frac{1}{\rho} f_R[\eta, s, v_R, v_z], \tag{1}$$

$$v_R \frac{\partial v_z}{\partial R} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\partial \psi}{\partial z} + \frac{1}{\rho} f_z[\eta, s, v_R, v_z], \tag{2}$$

and

$$v_R \frac{\partial v_\phi}{\partial R} + v_z \frac{\partial v_\phi}{\partial z} + \frac{v_R v_\phi}{R} = \frac{1}{\rho} f_\phi[\eta, s, v_\phi], \tag{3}$$

where (v_R,v_z) is the meridional circulation velocity, p is the pressure, and ψ is the gravitational potential; f is the anisotropic convective viscosity force, η is the scalar part of the dynamic viscosity, and s is the anisotropy parameter. We set[1] $\eta = \frac{1}{3}\rho v_c \ell$ by analogy with the kinetic theory of gases, where ρ is the density, v_c is the root-mean-square convective velocity, and ℓ is the mean free path of the convective elements. We put ℓ equal to the mixing length, which is roughly the same as the pressure scale height (Böhm-Vitense 1958); f is a linear functional of η: $f(\alpha\eta) = \alpha f(\eta)$, for constant α.

It is important to note that we have written f_R and f_z as not containing v_ϕ. This is true for Wasiutynski's form of the viscous stress tensor, as used by Kippenhahn (1960, 1963). Elsässer (1966) has derived this form from kinetic theory, but found it necessary to assume that the convective velocity moments did not depend on position. We know, however, that v_c is definitely a function of r, and hence extra terms will appear in the stress tensor. But one can use Elsässer's formalism to show that in our axially symmetric case these terms do not contain v_ϕ. We mention also that axial symmetry, together with certain symmetry properties of the second velocity moments, implies that f_ϕ contains no extra terms at all, and hence Wasiutynski's form for f_ϕ may be used.

We must also employ the equation of continuity, which reads in our case

$$\frac{1}{R}\frac{\partial}{\partial R}(R\rho v_R) + \frac{\partial}{\partial z}(\rho v_z) = 0. \tag{4}$$

Since this equation has become two-dimensional, it can be satisfied identically by means of a stream function S, such that

$$v_R = -\frac{1}{R\rho}\frac{\partial S}{\partial z}, \qquad v_z = +\frac{1}{R\rho}\frac{\partial S}{\partial R}. \tag{5}$$

Note that this does *not* imply incompressible flow. The lines $S(R,z) =$ constant are then the stream lines of the meridional flow.

We must now say a word about the boundary conditions to be employed in integrating equations (1)–(4). Our interest is primarily in the convection zone, where $\eta \neq 0$, so that one should generally match the currents there to the meridional currents generated by the polar heating in the radiative layers. However, in the solar case, these latter currents are very small, as we have already seen, and it will be reasonable to assume later on for the Sun that the circulation is inclosed entirely in the convection zone.

[1] See Elsässer (1966) The factor $\frac{1}{3}$ should clearly be used.

The continuity of the viscous stress tensor across the boundaries of the convection zone must also be insured. But since we wish to include the effects of a variable viscosity, which we show to be very important, we may satisfy this by letting $\eta \rightarrow 0$ at the boundaries. This must be the case anyway as is shown in § III by direct calculation of η from the convection zone model. This sort of analysis is to be contrasted with the rather artificial boundary conditions on v_ϕ which one must use when assuming η constant.

Let us now proceed to find the integral of equation (3). We do this in a very general way without making any assumption as to the relevance of the polar heating hypothesis. Using equation (5) and defining $Rv_\phi \equiv R^2\Omega \equiv M$ (the angular momentum/mass), we write equation (3) as

$$\rho\left(v_R\frac{\partial v_\phi}{\partial R}+v_z\frac{\partial v_\phi}{\partial z}+\frac{v_R v_\phi}{R}\right)=\frac{1}{R^2}\left(\frac{\partial S}{\partial R}\frac{\partial M}{\partial z}-\frac{\partial S}{\partial z}\frac{\partial M}{\partial R}\right)=f_\phi . \tag{6}$$

Note that the density, not assumed to be constant, has dropped out of the equation.

We also have, quite generally,

$$f_\phi[\eta, s, v_\phi]=g_1\frac{\partial^2 M}{\partial R^2}+g_2\frac{\partial^2 M}{\partial z^2}+g_3\frac{\partial M}{\partial R}+g_4\frac{\partial M}{\partial z}+g_5 M ,$$

where g_1 and g_2 are proportional to η and g_3, g_4, and g_5 contain η and grad η.

Now let us suppose that we have been able to find coordinates $u = u(R,z)$, $w = w(R,z)$, such that $\partial M/\partial u = 0$, and hence so that the lines w = constant are lines of constant M. We show in § III that in the high Reynolds number case $u = z$ and $w = R$ to first order. Then it is easy to show that

$$\frac{\partial S}{\partial R}\frac{\partial M}{\partial z}-\frac{\partial S}{\partial z}\frac{\partial M}{\partial R}=J\left(\frac{u,w}{R,z}\right)\frac{\partial S}{\partial u}\frac{dM}{dw} ,$$

and equation (6) becomes, with $D \equiv d/dw$,

$$\frac{\partial S}{\partial u}DM=f_1(u,w)D^2M+f_2(u,w)DM+f_3(u,w)M . \tag{7}$$

Now since M does not depend on u, we may integrate equation (7) between any two limit curves $u = u_1(w)$ and $u = u_2(w)$, getting

$$[S(u_2,w)-S(u_1,w)]DM = h_1(w)D^2M+h_2(w)DM+h_3(w)M , \tag{8}$$

which is our linear, homogeneous, ordinary differential equation determining the differential rotation. The boundary conditions Ω finite and $\partial\Omega/\partial R = 0$ at $R = 0$ will generally suffice to determine M completely, except for a multiplicative constant, once $S(u_j,w)$ has been specified from boundary conditions.

Now let us suppose that we wish to neglect the polar heating currents entirely, which are vanishingly small in the Sun. Then it is most useful to choose $u_2(w)$ to be the upper boundary of the convection zone, and $u_1(w)$ to be the lower boundary; and thus u_1 and u_2 are stream lines, so that there be no fluid flow across them. Therefore, $S(u_j,w)$ = constant. Further, axial symmetry requires that the polar axis $R = 0$ must also be a stream line (i.e., $S(u,0)$ = constant); and since the polar axis connects u_1 and u_2, the continuity of S requires $S(u_1,w) = S(u_2,w)$. Therefore the left-hand side of equation (8) vanishes, and the resulting equation for M is then invariant under the substitution $\eta \rightarrow \alpha\eta$, where α is a constant.

This shows what we mean by the statement that the differential rotation depends only weakly on the magnitude of η: Once the topology of the lines w = constant has been determined, then the distribution of M from line to line is independent of the substitution $\eta \rightarrow \alpha\eta$. However, the determination of the w-lines themselves will depend in a non-linear fashion on η. In the next section we show that for large Reynolds numbers (small η) the differential rotation is completely independent of the magnitude of η.

Having thus determined the differential rotation, we may then substitute $M(w)$ back into equation (7) to determine $v_w \propto \partial S/\partial u$. We see that the meridional velocities are directly proportional to η (again, in this restricted sense), and hence the more viscous the convection zone, the faster must be the angular momentum transport to compensate for the losses generated by the convective viscosity.

III. APPROXIMATION FOR THE SOLAR CONVECTION ZONE

We now develop an approximation for large Reynolds number, which we show to be valid throughout the solar convection zone, except for a surface layer. This approximation may be treated formally by expanding the dependent variables in powers of a dimensionless parameter λ which is of the order of $\eta(\Omega R_0^2\rho)^{-1}$, an inverse Reynolds number, where $\Omega = 3 \times 10^{-6}$ sec^{-1} is a typical value for the solar angular velocity, and R_0 is the solar radius. Since $\eta = \frac{1}{3}\rho v_c \ell$, we get $\lambda \sim (v_c/R_0\Omega)(\ell/R_0)$. Now the scale height ℓ is much smaller than R_0, and is greatest at the bottom of the convection zone, where $\ell \simeq R_0/10$, whereas $v_c \ll R_0\Omega \simeq 2 \times 10^5$ cm/sec at the bottom, but $v_c \sim R_0\Omega$ at the top. Therefore $\lambda < \frac{1}{10}$ everywhere. This criterion will be more carefully scrutinized later on.

We solve equations (1)–(4) to first order in λ under the further assumption that the convection zone is barytropic; i.e., that there exists a unique equation of state $p = f(\rho)$. Biermann (1958) and Kippenhahn (1960) have shown that this is valid to great accuracy in the solar convection zone, except for a very thin layer at the surface. This assumption rids us of the last vestiges of the polar heating effect, which depends on $\nabla p \times \nabla \rho \neq 0$, and we can write

$$\rho^{-1}\nabla p + \nabla\psi = \rho^{-1}f'(\rho)\nabla\rho + \nabla\psi = \nabla\left[\int_0^\rho \rho^{-1}f'(\rho)\,d\rho + \psi\right] \equiv \nabla F,$$

where

$$F = \int_0^\rho \rho^{-1}f'(\rho)\,d\rho + \psi.$$

Since η is a first-order quantity, equation (3) implies that v_R and v_z must vanish to zeroth order. Thus we expand $v_R = v_{R1}\lambda + v_{R2}\lambda^2 + \ldots$, and similarly for v_z. Further, $v_\phi = R(\Omega_0 + \Omega_1\lambda + \ldots)$, with the same for F and ρ. Since η is first order, we may immediately write the zeroth- and first-order parts of equations (1) and (2): equation (1) implies $R\Omega_0^2 = \partial F_0/\partial R$ and $2R\Omega_0\Omega_1 = \partial F_1/\partial R$, while equation (2) yields $\partial F_0/\partial z = \partial F_1/\partial z = 0$. Hence F_0 and F_1 are arbitrary functions of R, and therefore Ω_0 and Ω_1 are also functions of R only. As mentioned in § II, the fact that $\eta \rightarrow 0$ at the convection zone boundaries insures the continuity of the viscous stress tensor without the necessity of imposing boundary conditions on Ω.

We thus conclude that, to first order in λ, the lines of constant $M = R^2\Omega$ are lines of constant R, and that we may set $u = z$ and $w = R$.

Let us now turn to equation (6). Since $M = M(R)$ to first order we will obtain $S = S_1\lambda + S_2\lambda^2 + \ldots$ to *second* order by solving

$$-\frac{\partial S}{\partial z}\frac{dM}{dR} = R^2 f_\phi = R^2\left(g_1\frac{d^2M}{dR^2} + g_3\frac{dM}{dR} + g_5 M\right), \qquad (9)$$

which is already in the form of equation (7). From now on, we drop the indices indicating the order of the terms, for we do not wish to go to higher orders.

By assuming axial symmetry, we have met the objections to the Wasiutynski tensor posed by Elsässer (1966), at least as regards f_ϕ (see § II). Note that the Wasiutynski form also neglects distortions of the viscosity tensor from spherical symmetry. However, these distortions are only about 2 parts in 10^5 for the Sun, and may be disregarded. Then f_ϕ is most conveniently expressed in spherical coordinates as

$$f_\phi = \frac{1}{r^3} \sin\theta \frac{\partial}{\partial r}\left[r^4 \eta \frac{\partial \Omega}{\partial r} + 2\eta(1-s) r^3 \Omega \right] + (r \sin^2\theta)^{-1} \frac{\partial}{\partial \theta}\left(s\eta \sin^3\theta \frac{\partial \Omega}{\partial \theta} \right). \quad (10)$$

The meaning of the anisotropy parameter s is as follows: Always, $s \geq 0$; $s = 1$ is complete isotropy, and f_ϕ then reduces to the ordinary molecular viscosity form. The expression $1 > s \geq 0$ means that the turbulent exchange is stronger in the radial (vertical) direction, and $s > 1$ means that it is stronger in the horizontal directions.

We now examine more carefully the validity of our approximation procedure. Our iterative technique will converge rapidly only if the meridional velocities given by equation (6) turn out to be much smaller than a typical value of $v_\phi \sim R_0\Omega$. We have $dv_\phi/dR \sim v_\phi/R_0 \sim \Omega$, and from observations, $R_0\partial\Omega/\partial r \sim \frac{1}{5}\Omega \sim \partial\Omega/\partial\theta$, etc. But since η runs

TABLE 1

SOLAR CONVECTION ZONE PARAMETERS (BAKER AND TEMESVARY 1966)
[Mixing Length $\ell = 1.5 \times$ (Pressure Scale Height)]

Depth (cm)	ρ (gm/cm^3)	ℓ (cm)	v_c (cm/sec)	$\eta = \frac{1}{3}\rho v_c \ell$	$\rho^{-1}\lvert d\eta/dr\rvert$
3 4(6)	4 0(−7)	2 3(7)	2 4(4)	7 3(4)	3 1(5)
1 3(7)	4 6(−7)	3 3(7)	2 2(5)	1 1(6)	5 7(4)
3 7(7)	8 3(−7)	4 5(7)	1 5(5)	1 9(6)	5 4(4)
1 5(8)	7 2(−6)	8 3(7)	7 6(4)	1 5(7)	2 9(4)
3 6(8)	6 2(−5)	1 5(8)	4 2(4)	1 3(8)	1 6(4)
1 0(9)	8 2(−4)	4 5(8)	2 2(4)	2 6(9)	8 4(3)
3 3(9)	8 3(−3)	1 6(9)	1 1(4)	4 9(10)	4 0(3)
1 1(10)	7 6(−2)	5 5(9)	5 4(3)	7 5(11)	1 6(3)
1 8(10)	1 8(−1)	7 8(9)	2 9(3)	1 4(12)	1 4(3)
1 9(10)	2 1(−1)	8 1(9)	2 0(3)	1 1(12)	. .

through its entire range of values over the convection zone depth ΔR, $d\eta/dr \sim \eta/\Delta R \sim 4\eta/R_0$, and we see that the terms in equation (10) containing $d\eta/dr$ will be more important than the rest. Setting $v_m^2 \equiv v_R^2 + v_z^2$, one can then use equations (6) and (10) to estimate

$$\frac{v_m}{R_0\Omega} \sim \frac{|f_\phi|}{\rho R_0 \Omega^2} \lesssim \frac{|d\eta/dr|}{\rho R_0 \Omega}. \quad (11)$$

Thus we must determine the right-hand side of equation (11) throughout the convection zone. Table 1 has been computed from a model of the convection zone calculated from mixing-length theory by Baker and Temesvary (1966). The mixing-length ℓ has been set equal to $1.5\times$ (pressure scale height). D is the depth from the top of the photosphere, not from the top of the convection zone, which extends from about $D = 2.7 \times 10^6$ cm to $D = 1.9 \times 10^{10}$ cm.

Since $R_0\Omega \simeq 2 \times 10^5$ cm/sec, we see from the last column of Table 1 that $v_m/(R_0\Omega) \ll 1$ is satisfied for $D > 10^8$ cm, whereas for $D < 5 \times 10^7$ cm the agreement is not very good. But 5×10^7 cm $\lesssim \ell$ at this depth, and variations over this small a distance have no relevance for a viscosity theory, which is good only for scales $\gtrsim \ell$. Thus our criterion predicts good convergence of the approximation scheme, except for the uppermost layers, where the predicted velocities are too high. Let us note, however, that near the bottom of the convection zone the mixing length ℓ is of the order of the total depth of the convection zone. Thus any results for the bottom layers will be quite rough. A more serious defect is the fact that at no depth is ℓ less than about $\frac{1}{3}$ this depth, and since the coefficients of equation (8) depend on the application of boundary conditions at the top and bottom, we should not expect the results of any theory employing a convective viscosity to correspond very accurately with observation.

Note also from Table 1 that, at the bottom of the convection zone, η descends to zero from its peak value over a distance considerably smaller than a mixing length. Since the viscous stresses arise from convective elements which have come from distances $\sim\ell$, we cannot contemplate stresses caused by large changes in velocities or viscosity which occur over distances smaller than a mixing length. Thus the last entry in the last column of Table 1 is artificially large.

Sakurai (1966) has also investigated this anisotropic viscosity theory. He uses an interesting iterative approximation based on the smallness of the ratio of the convection zone depth to the solar radius and concludes that his method is self-consistent only if the convective velocity is $\sim 10^4$ cm/sec. Baker and Temesvary (1966) show that this low a velocity is obtained for $D \gtrsim 10^9$ cm (see Table 1). In his numerical analysis Sakurai assumed v_c = constant.

To continue with our analysis of equation (9), we may convert equation (10) into cylindrical coordinates and write, assuming that s is independent of position and that $\partial\eta/\partial\theta = 0$,

$$\begin{aligned} -(R^2\Omega)'\frac{\partial S}{\partial z} &= R^3\Omega''[1+(s-1)\cos^2\theta]\eta \\ &+ 3R^2\Omega'[1+(1-s)(\sin^2\theta-\cos^2\theta)]\eta + 6R\Omega(1-s)\sin^2\theta\,\eta \\ &+ R^2[R\Omega' + 2(1-s)\Omega]\sin\theta\,\frac{d\eta}{dr} \equiv g(R,z)\,, \end{aligned} \tag{12}$$

where the primes indicate d/dR.

As discussed in § II, we wish to integrate this equation neglecting the polar heating currents. Since Ω does not depend on z, the above equation implies $v_R(R,z) = v_R(R,-z)$, and from equation (5), we then obtain $v_z(R,z) = -v_z(R,-z)$. Hence v_z vanishes at at $z = 0$, and S is also constant on $z = 0$. Therefore the arguments of § II imply that we may take the lower boundary of the integration to be $z = z_1(R) = 0$ for $R_i \leq R \leq R_0$, and $z_1(R) = (R_i^2 - R^2)^{1/2}$ for $0 \leq R \leq R_i$, where $R_i = R_0 - \Delta R$ is the inner radius of the convection zone, and the upper boundary is $z_2(R) = (R_0^2 - R^2)^{1/2}$. Thus we may restrict our attention to the region $z \geq 0$.

Since by the arguments in § II, $S(R,z_1) = S(R,z_2)$ (= constant), we immediately integrate equation (12) to find

$$-(R^2\Omega)'\int_{z_1}^{z_2}\frac{\partial S}{\partial z}\,dz = \int_{z_1}^{z_2} g(R,z)\,dz \equiv h_1(R)\Omega'' + h_2(R)\Omega' + h_3(R)\Omega = 0\,. \tag{13}$$

Equation (13) is the approximate form of equation (8), valid to first order. Note that η and $d\eta/dr$ occur linearly, and that the equation is invariant under $\eta \rightarrow \alpha\eta$, where α is a constant. Thus, to first order, the magnitude of the differential rotation does not depend on the magnitude of η. However, the anisotropy parameter plays a crucial role, for $s = 1$ implies that Ω = constant.

It is possible to integrate equation (13) exactly. Using equation (10) in cylindrical coordinates, one can show that f_ϕ may be written as $R^2 f_\phi = \partial F_1/\partial R + \partial F_2/\partial z$. Therefore one may perform an integration by parts with respect to z in equation (13), and since η vanishes on the convection zone boundaries and $\partial\Omega/\partial\theta$ vanishes at $z = 0$, the boundary terms may be thrown away. Equation (13) then becomes a perfect differential in R, and one easily integrates to find

$$\frac{\Omega'}{\Omega}=2R(s-1)\int_{z_1}^{z_2}dz\,\frac{\eta(r)}{r^2}\left[\int_{z_1}^{z_2}dz\,\frac{\eta(r)}{r^2}(R^2+z^2s)\right]^{-1}\equiv F(R),$$

where $r=(R^2+z^2)^{1/2}$. The integration constant has been set equal to zero so that Ω may remain finite at $R=0$.

We may obviously integrate again to get finally

$$\Omega(R)=a_0\exp\left[\int_0^R F(R')\,dR'\right]. \tag{14}$$

It might be well to mention that we have no real assurance that the exact steady-state solution of equations (1)–(4) converges uniformly as $\eta\to 0$ to the solution for $\eta=0$. This difficulty also occurs in boundary-layer theory in compressible hydrodynamics, which is likewise a high Reynolds number approximation (Pai 1956). As far as the author knows, no satisfactory theory exists for these problems, called "singular perturbation problems." One can only rely on plausibility arguments and comparison with observation, as in the following section.

IV. NUMERICAL RESULTS AND CONCLUSIONS

Since $R=0$ is surely a line of constant M, we may expect equation (13) to be very accurate near the polar axis. Thus let us expand the unknowns Ω and S in power series in R^2 and find the first non-trivial coefficients.

Substitution of $\Omega(R)=\Sigma_n a_n R^{2n}$ into equation (14) may be easily shown to lead to the result

$$a_1=a_0(s-1)\int_{R_i}^{R_0}dz\,\eta(z)z^{-2}/\,s\int_{R_i}^{R_0}dz\,\eta(z),$$

where R_i is the inside radius of the convection zone. But from Table 1 we see that $\eta(r)$ peaks near $R_i=R-\Delta R$, and we have approximately $a_1\simeq a_0(s-1)s^{-1}R_i^{-2}$, and therefore

$$\Omega(R)\simeq\Omega(0)\left[1+\frac{(s-1)}{s}\left(\frac{R}{R_i}\right)^2\right]. \tag{15}$$

We see that for the differential rotation to be of the observed sign and magnitude, we must have $s-1\sim\frac{1}{5}$. Kippenhahn (1963) also concluded that $s-1>0$ and mentioned that studies of turbulent convection in thin layers suggest that this is not unreasonable. Sakurai (1966) also finds $s-1\simeq\frac{1}{5}$.

One may also expand $S=f_0(z)R^2+f_1(z)R^4+\ldots$ and use equation (12) to conclude that, to first order in R^2,

$$-\partial S/\partial z=R\rho v_R\simeq\{[4sa_1a_0^{-1}-3(s-1)z^{-2}]\eta(z)+(1-s)z^{-1}d\eta(z)/dr\}R^2\,.$$

Therefore at the top and bottom of the convection zone, where $\eta=0$,

$$v_R\simeq\frac{(1-s)}{\rho z}R\frac{d\eta}{dr}. \tag{16}$$

Note that this expression is independent of the angular velocity.

We have seen that $s>1$, and thus at the top of the convection zone where $d\eta/dr<0$, equation (16) implies that $v_R>0$; whereas at the bottom of the convection zone,

$v_R < 0$. Thus the surface flow near the poles is from pole to equator, which conclusion agrees with the observations of Plaskett (1966).

To compute the order of magnitude v_R, we need to know $d\eta/dr$. Since variations over distances shorter than a mixing length are meaningless for our theory, we might choose from Table 1 the surface value $\rho^{-1}|d\eta/dr| \sim 5 \times 10^4$ cm/sec. At a latitude of 60° ($\theta = 30°$), where $R/z \simeq 0.8$, with $s - 1 = \frac{1}{5}$, equation (16) predicts $v_R \sim 8 \times 10^3$ cm/sec, in rough agreement with Plaskett's values $\sim 5 \times 10^3$ cm/sec for these latitudes. Since these observations are actually of photospheric velocities, where energy transfer has become mostly radiative, we would expect these velocities to have decayed to a value somewhat lower than those in the convection zone.

Measurement of sunspot drifts, which show meridional velocities of about 200 cm/sec (de Jager 1959), do not extend higher than a latitude of 40° and presumably represent an average over velocities at different depths. It is difficult to say just how to pick a representative depth for sunspots since there is no generally accepted model for them. Danielson (1965) concludes from Deinzer's (1965) sunspot models that equipartition between convective and radiative transport inside a sunspot with a surface field of 3000 gauss is reached at $D > 10^9$ cm. At a depth of $D \sim 3 \times 10^9$ cm in the convection zone we have $(s - 1)\rho^{-1}|d\eta/dr| \sim 800$ cm/sec, again in very rough agreement, since we might expect v_z at 30° to be of the order of v_R at 60°.

These same calculations have also been made from a convective zone model constructed under the assumption $\ell =$ pressure scale height (Baker and Temesvary, unpublished). Although the depth of the convection zone in this second model is considerably less than the one used here, the corresponding computed circulational velocities for the second model are only very slightly smaller than the ones computed here. Thus our results are insensitive to the particular convection zone model used.

We see that the theory stands up fairly well under comparison with observation, although it must be admitted that both are in great need of refinement. It would be desirable to include non-linear effects in integrating equations (1) and (2), although in our treatment equations (3) and (4) have not been linearized. However, as we have mentioned before, the fact that the mixing length is always of the order of the distance from the top of the convection zone may limit the accuracy of any theory involving a convective viscosity.

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