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Resonance Cases and Small Divisors in a Third Integral of Motion. III

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This paper discusses two cases where small divisors play an important role in the third integral. The Hamiltonian used is $H=\frac{1}{2}(X^2+Y^2+Ax^2+By^2)-\epsilon xy^2=h$. In the first case the two unperturbed frequencies are nearly equal. If ϵ is very small and we set $B=A+K\epsilon^2$ we find resonance phenomena when K is in the range $(-5h/3A^2, 10h/3A^2)$. For larger or smaller values of K all the orbits are boxes. This range is divided into four parts by the values $K=-2h/3A^2$, $K=5h/6A^2$, and $K=7h/3A^2$. The forms of the invariant curves are different in the four intervals. The corresponding orbits are either box type, or similar to the orbits of the resonance case $A=B$, except for the D-type orbits, which appear only in the third and fourth intervals.

In the second case one unperturbed frequency is almost the double of the other. If we set $4B=A+\epsilon k$ we find resonance phenomena when $-4(2h/A)^{\frac{1}{2}}<k<4(2h/A)^{\frac{1}{2}}$. The boundaries of the orbits are approximately arcs of three parabolas, as in the resonance case.

I. CASE $A\simeq B$. INVARIANT CURVES

This paper is a continuation of paper II (Contoupoulos 1965). We consider the Hamiltonian

$$H=\tfrac{1}{2}(X^2+Y^2+Ax^2+By^2)-\epsilon xy^2, \tag{1}$$

which corresponds to the potential

$$V=\tfrac{1}{2}(Ax^2+By^2)-\epsilon xy^2,$$

on the plane of symmetry of a nonaxisymmetric galaxy. Here x, y are coordinates with origin the "center" of the galaxy, X, Y the corresponding velocities, and A, B, ϵ are constants, where ϵ is small.

In Secs. I and II we consider the case when B is near, but not equal to A. Then the third integral of the general case ($A^{\frac{1}{2}}/B^{\frac{1}{2}}$=irrational) can be written (Contopoulos 1960)

$$\Phi_1=\Phi_{10}+\epsilon\Phi_{11}+\epsilon^2\Phi_{12}+\cdots, \tag{1$'$}$$

where

$$2\Phi_{10}=X^2+Ax^2,$$

$$\Phi_{11}=[1/(4B-A)][(A-2B)xy^2-2xY^2+2XyY],$$

$$\Phi_{12}=-\frac{1}{4B-A}\left[\frac{y^4}{2}+\frac{(2A+B)X^2Y^2}{2AB(A-B)}+\frac{(2A-5B)x^2Y^2}{2B(A-B)}-\frac{(4A-B)X^2y^2}{2A(A-B)}+\frac{3Bx^2y^2}{2(A-B)}+\frac{6xXyY}{A-B}\right],$$

etc.

When B is near A a small divisor appears in Φ_{12}. In order to avoid this small divisor we multiply Φ_1 by $(A-B)$.

We have seen in paper II that in the resonance case $A=B$ we have another zero-order integral

$$C_0=X^2Y^2-Ax^2Y^2-AX^2y^2+A^2x^2y^2+4AxXyY,$$

which is equal to the limit

$$C_0=\lim_{A\to B}(-\tfrac{2}{3}B)(4B-A)(A-B)\Phi_{12}.$$

Therefore if we set

$$B=A+\epsilon k \tag{2}$$

we can write

$$-\frac{(A-B)\Phi_1}{\epsilon}=k\Phi_1=\frac{k}{2}(X^2+Ax^2)-\frac{\epsilon k}{(3A+4\epsilon k)}[(A+2\epsilon k)xy^2+2xY^2-2XyY]+\frac{\epsilon}{2(3A+4\epsilon k)}\left[-\epsilon ky^4+\frac{(3A+\epsilon k)X^2Y^2}{A(A+\epsilon k)}-\frac{(3A+5\epsilon k)x^2Y^2}{(A+\epsilon k)}-\frac{(3A-\epsilon k)X^2y^2}{A}+3(A+\epsilon k)x^2y^2+12xXyY\right]+\cdots, \tag{3}$$

and if $k\to 0$ this integral becomes $\epsilon C_0/2A^2+\cdots$. In order to find the integral $(\epsilon/2A^2)(\varphi_0+\epsilon\varphi_1+\cdots)$ of the resonance case [Eq. (22) of paper II] for $k\to 0$ we add the terms $(\epsilon/2A^2)[-\frac{1}{3}(2\Phi_{10})^2+\frac{1}{12}(2\Phi_{20})^2]$ in Eq. (3), where $2\Phi_{20}=Y^2+By^2$. If we omit all terms of order higher than the second in ϵ we find

$$\begin{aligned}\bar\varphi=\bar\varphi_0+\epsilon\bar\varphi_1+\epsilon^2\bar\varphi_2={}&\tfrac{1}{2}k(X^2+Ax^2)+(\epsilon/2A^2)(X^2Y^2-Ax^2Y^2-AX^2y^2+A^2x^2y^2+4AxXyY-\tfrac{1}{3}(2\Phi_{10})^2+\tfrac{1}{12}(2\Phi_{20})^2)\\&-(\epsilon k/3A)(Axy^2+2xY^2-2XyY)-(\epsilon^2/9A^3)(4A^2xy^4+4A^2x^3y^2+2Ax^3Y^2+22Ax^2XyY-20AxX^2y^2+14xX^2Y^2\\&-14X^3yY+13AXy^3Y-7xY^4+7XyY^3-9Axy^2Y^2)+(\epsilon^2k/6A^3)(-6X^2Y^2+2Ax^2Y^2+5AX^2y^2-A^2x^2y^2\\&-A^2y^4-16AxXyY)+(2\epsilon^2k^2/9A^2)(-Axy^2+4xY^2-4XyY)=\bar\varphi_{;0}=\bar\varphi_{0;0}+\epsilon\bar\varphi_{1;0}+\epsilon^2\bar\varphi_{2;0},\end{aligned} \tag{4}$$

where the subscript 0 after the semicolon indicates the initial point.

The invariant curves are found if we set $y=y_0=0$ and $Y^2=2h-X^2-Ax^2$ in Eq. (4):

$$f_0(x,X^2)+\epsilon f_1(x,X^2)+\cdots = f_0(x_0,X_0^2)+\epsilon f_1(x_0,X_0^2)+\cdots, \quad (5)$$

where

$$f_0(x,X^2)=\tfrac{1}{2}k(X^2+Ax^2) \quad (6)$$

and

$$f_1(x,X^2)=-(1/24A^2)(15X^4-2X^2(10h-3Ax^2) -9A^2x^4+28hAx^2)-(2kx/3A)(2h-x^2-Ax^2). \quad (7)$$

If k is of order 1 and ϵ is a small quantity the invariant curves are distorted ellipses around a point near the center (Contopoulos 1965).

If k is of order ϵ, namely

$$k=\epsilon K, \quad \text{hence} \quad B=A+\epsilon^2K, \quad (8)$$

then the invariant curves are given by the equation

$$\theta(Ax^2,X^2)=15X^4-2X^2(10h-3Ax^2+6KA^2) -9A^2x^4+2Ax^2(14h-6KA^2) -15X_0^4+2X_0^2(10h-3Ax_0^2+6KA^2) +9A^2x_0^4-2Ax_0^2(14h-6KA^2). \quad (9)$$

The discussion of this equation is made in the same way as that of Eq. (31) of paper II. The inequality (32) is replaced by

$$\mathcal{J}(Ax^2)=144A^2x^4-24Ax^2(20h-6KA^2)+100h^2+225X_0^4 -30X_0^2(10h-3Ax_0^2+6KA^2)-135A^2x_0^4+420hAx_0^2 -180KA^3x_0^2+120KA^2h+36K^2A^4\geqslant 0, \quad (10)$$

and Eqs. (33) and (34) are replaced by

$$\theta(Ax^2,0)=-9A^2x^4+2Ax^2(14h-6KA^2) -15X_0^4+2X_0^2(10h-3Ax_0^2+6KA^2) +9A^2x_0^4-2Ax_0^2(14h-6KA^2), \quad (11)$$

and

$$\theta(Ax^2,\, 2h-Ax^2)=(10h+15X_0^2-9Ax_0^2-12KA^2)Y_0^2 =\theta(0,2h). \quad (12)$$

We consider first the case

$$\text{(i)} \quad 10h+15X_0^2-9Ax_0^2-12KA^2>0. \quad (13)$$

Then $\theta(0,2h)>0$ and we have two real roots x^2 of Eq. (9) smaller than $2h-Ax^2$ if $\mathcal{J}(Ax^2)>0$ and

$$(10h-3Ax^2+6KA^2)/15<2h. \quad (14)$$

The last inequality is verified for every positive Ax^2 if $KA^2\leqslant 10h/3$. If, however, $KA^2>10h/3$ the inequality (13) is not verified inside the limiting curve

$$X_0^2+Ax_0^2=2h. \quad (15)$$

In fact the hyperbola

$$15X_0^2-9Ax_0^2=12KA^2-10h \quad (16)$$

intersects the X_0 axis at two points $\pm[(12KA^2-10h)/15]^{\frac{1}{2}}$ inside the limiting curve if $10h/3>KA^2>5h/6$. If $KA^2>10h/3$ it is outside the limiting curve. If $KA^2=5h/6$ the curve (16) is reduced to two straight lines through the origin. If $5h/6>KA^2>-2h/3$ the hyperbola (16) intersects the $A^{\frac{1}{2}}x_0$ axis at the points $\pm[(10h-12KA^2)/9]$. If $KA^2<-2h/3$ then the hyperbola is outside the limiting curve, and all initial positions inside the limiting curve satisfy inequality (13).

Equation (10) has two real roots Ax_a^2 and $Ax_b^2(\geqslant Ax_a^2)$ if inequality (13) is satisfied. It is proved, as in the resonance case, that we must have $Ax^2\leqslant Ax_a^2$ in order to have $\mathcal{J}(Ax^2)>0$ and that Ax_a^2 is between the roots Ax_1^2, $Ax_2^2(x_1^2\leqslant x_2^2)$ of $\theta(Ax^2,0)=0$, which are

$$Ax_i^2=\frac{14h-6KA^2}{9}\pm\tfrac{1}{9}\{[14h-9(X_0^2+Ax_0^2)-6KA^2]^2 +216X_0^2(2h-X_0^2-Ax_0^2)\}^{\frac{1}{2}}. \quad (17)$$

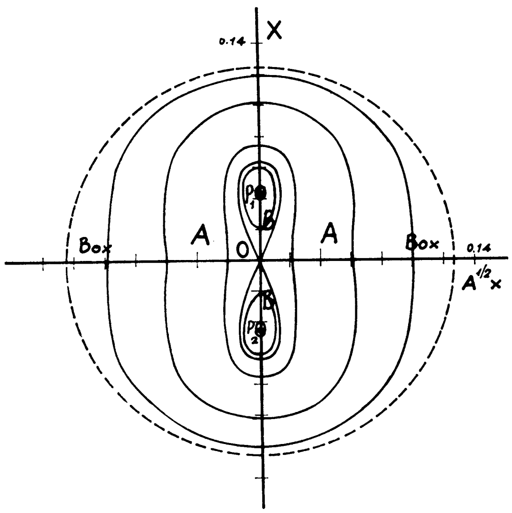
If $Ax_1^2=Ax_a^2$ we have a double root $X^2=0$ for $Ax^2=Ax_a^2$.

Equation (9) has two positive roots if $\theta(Ax^2,0)>0$ and $10h-3Ax^2+6KA^2>0$, no positive root if $\theta(Ax^2,0)>0$ and $10h-3Ax^2+6KA^2<0$ and one positive root is $\theta(Ax^2,0)<0$. Therefore if $Ax^2<Ax_1^2$ we have one acceptable root, and if $Ax_1^2<Ax^2<Ax_a^2$ we have two acceptable roots if we have also $Ax^2<\frac{1}{3}(10h+6KA^2)$ and no acceptable root if $Ax^2>\frac{1}{3}(10h+6KA^2)$.

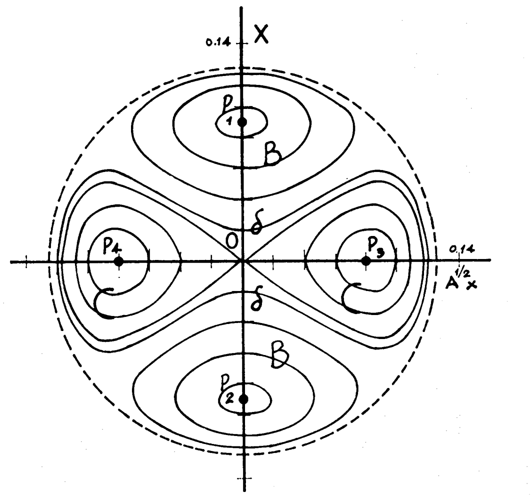
Let us assume first that $0\leqslant Ax_a^2<\frac{1}{3}(10h+6KA^2)$. Then if $Ax_1^2>0$ we have invariant curves of type A [Fig. 1(a); one root for X^2 if $0<Ax^2<Ax_1^2$ and two roots if $Ax_1^2<Ax^2<Ax_a^2$]. If $Ax_1^2<0$ we have invariant curves of type B (two real roots for X^2), which surround one of the invariant points $P_1,P_2\{x=0, X=\pm[(10h+6KA^2)/15]^{\frac{1}{2}}\}$, [if $2h>(10h+6KA^2)/15>0$]. In fact $\mathcal{J}(0)=25(2h-3X_0^2-Ax_0^2+6KA^2/5)^2+20Ax_0^2(26h-3X_0^2-8Ax_0^2-6KA^2)$. This cannot become negative if inequality (13) is satisfied inside the limiting curve; therefore it cannot be $Ax_a^2<0$, if $KA^2<10h/3$.

If $Ax_a^2=0$ we must have $Ax_0^2=0$ and $X_0^2=(10h+6KA^2)/15$.

If $KA^2\rightarrow 10h/3$ the points P_1, P_2 tend to the limiting curve. If $KA^2\rightarrow -5h/3$ the points P_1, P_3 tend to coincide at the origin.

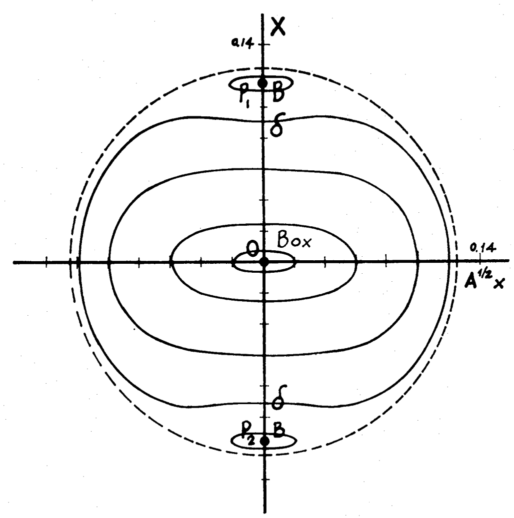


(a)



(b)

FIG. 1. Invariant curves in the cases: (a) $A=0.1$, $\epsilon\rightarrow 0$, $h=0.00765$, $k=-h/A^2=-0.765$. (b) $A=0.1$, $\epsilon\rightarrow 0$, $h=0.00765$, $k=h/A^2=0.765$. (c) $A=0.1$, $\epsilon\rightarrow 0$, $h=0.00765$, $k=8h/3A^2=2.04$.



(c)

The separation between type A and type B invariant curves is the curve with $Ax_1^2=0$, i.e.,

$$\theta(0,0)=-15X_0^4+2X_0^2(10h-3Ax_0^2+6KA^2)\\ +9A^2x_0^4-2Ax_0^2(14h-6KA^2). \quad (18)$$

The branch of this curve going through the origin intersects the X axis at the point $X_0=\pm[(20h+12KA^2)/15]^{\frac{1}{2}}$ if $h>(10h+6KA^2)/15>0$.

If $5h/6>KA^2>-5h/3$ this curve is a figure eight; the B-type curves are inside it $[\theta(0,0)>0]$ and the A-type curves outside $[\theta(0,0)<0]$.

If $KA^2\leqslant -5h/3$ the whole curve shrinks to the origin.

If $KA^2=5h/6$ we have

$$\theta(0,0)=(2h-X_0^2-Ax_0^2)(15X_0^2-9Ax_0^2)=0,$$

i.e., this curve is composed of the limiting curve and the two straight lines, to which is reduced then the curve (16). For $KA^2>5h/6$ this curve takes the form of the symbol ∞, but it is outside the region defined by inequality (13) [Fig. 1(b)]. Therefore if $KA^2>5h/6$ there are no invariant curves of type A.

Let us consider now the case $\frac{1}{3}(10h+6KA^2)<Ax_a^2$. We have

$$\mathcal{G}[\tfrac{1}{3}(10h+6KA^2)]=5\sigma(Ax_0^2,X_0^2)\\ =-15\theta[\tfrac{1}{3}(10h+6KA^2),0], \quad (19)$$

where

$$\sigma(Ax_0^2,X_0^2)\\ =-3(2h-X_0^2-Ax_0^2)(10h+15X_0^2-9Ax_0^2-12KA^2)\\ +20(2h+3KA^2)^2. \quad (20)$$

Therefore if $\frac{1}{3}(10h+6KA^2)=Ax_a^2$, then $Ax_1^2=Ax_a^2$. If $\frac{1}{3}(10h+6KA^2)<Ax_a^2$, then $\sigma(Ax_0^2,X_0^2)>0$ and from Eq. (19) we find that $\frac{1}{3}(10h+6KA^2)$ is outside the interval (Ax_1^2, Ax_2^2). Further $\frac{1}{3}(10h+6KA^2) < \frac{1}{6}(10h-3KA^2)$, hence $KA^2 < -\frac{2}{3}h$, and $\frac{1}{3}(10h+6KA^2) < \frac{1}{9}(14h-6KA^2)$, i.e., $\frac{1}{3}(10h+6KA^2) < Ax_1^2$.

In this case we have one acceptable (positive) root for X^2 if $0 < Ax^2 < Ax_1^2$ and no positive root if $Ax_1^2 < Ax^2$. The invariant curves are then simple closed curves (distorted ellipses). They correspond to box-type orbits of the general nonresonance case.

Such orbits appear if $KA^2 < -\frac{2}{3}h$. The box and A-type invariant curves are separated by the curve

$$\sigma(Ax_0^2, X_0^2) = 0. \tag{21}$$

If $KA^2 < -2h/3$, the inequality (13) is satisfied for all points inside the limiting curve; hence the curve (21) is inside the limiting curve. It intersects the $A^{\frac{1}{2}}x$ axis at the points $A^{\frac{1}{2}}x_0 = \pm[\frac{1}{3}(10h+6KA^2)]^{\frac{1}{2}}$ and the X axis at the points

$$X_0 = \pm(\{10h+6KA^2+12[-KA^2(10h+6KA^2)]^{\frac{1}{2}}\}/15)^{\frac{1}{2}},$$

which are at a distance smaller than $(2h)^{\frac{1}{2}}$ from the origin if $-2h/3 > KA^2 > -5h/3$.

When $KA^2 \to -\frac{2}{3}h$ these points tend to the limiting curve. For $KA^2 = -\frac{2}{3}h$ the curve (21) coincides with the limiting curve.

When $KA^2 \to -5h/3$ it can be proved that the whole curve (21) shrinks to the origin.

When $KA^2 < -5h/3$ this curve becomes imaginary. Then all orbits are boxes as in the general nonresonance case.

We consider now the case

$$\text{(ii)}\quad 10h+15X_0^2-9Ax_0^2-12KA^2 < 0. \tag{22}$$

Then we have one acceptable solution X^2 whenever $\theta(Ax^2,0) \geqslant 0$.

But

$$\theta(Ax^2,0) = 9(Ax_2^2 - Ax^2)(Ax^2 - Ax_1^2). \tag{23}$$

The roots (17) coincide only if $X_0^2 = 0$ and $Ax_0^2 = \frac{1}{9}(14h-6KA^2)$. Therefore we have two invariant points P_3, P_4 $\{A^{\frac{1}{2}}x = \pm[\frac{1}{9}(14h-6KA^2)]^{\frac{1}{2}}, X=0\}$ if $0 < \frac{1}{9}(14h-6KA^2) < 2h$, or $7h/3 > KA^2 > -2h/3$. These invariant points are in the region (22) if $KA^2 > -2h/3$.

When $KA^2 \to -2h/3$ the points P_3, P_4 tend to the limiting curve and when $KA^2 \to 7h/3$ they tend to coincide at the origin.

If the hyperbola (16) intersects the $A^{\frac{1}{2}}x$ axis, i.e., if $\frac{5}{6}h > KA^2 > -\frac{2}{3}h$, then the invariant curves in the region (22) are closed curves surrounding the points P_3 or P_4 (Fig. 1 of paper II). In fact, then the roots Ax_1^2, Ax_2^2 are between $\frac{1}{9}(10h-12KA^2)$ and $2h$, because

$$\theta[\tfrac{1}{9}(10h-12KA^2),0] = \theta(2h,0) < 0;$$

it is also $\theta(Ax^2,0) \geqslant 0$ for

$$Ax_1^2 \leqslant Ax^2 \leqslant Ax_2^2, \tag{24}$$

and $Ax_1^2 \leqslant \frac{1}{9}(14h-6KA^2) \leqslant Ax_2^2$. These invariant curves are of type C.

If the hyperbola (16) intersects the X axis, i.e., if $10h/3 > KA^2 > 5h/6$, then we may have invariant curves surrounding both points P_3, P_4, which are like the A-type invariant curves rotated by 90° [type δ; Fig. 1(b)]. The maximum values of X^2 are X_a^2 and they are larger than the value $X^2 = X_1^2$ for $x=0$. The C- and δ-type invariant curves are separated by the curve (18). The C-type invariant curves are inside it [$\theta(0,0)<0$] and the δ-type invariant curves outside it [$\theta(0,0)>0$]. This curve intersects the $A^{\frac{1}{2}}x$ axis at the points $A^{\frac{1}{2}}x_0 = \pm[\frac{1}{9}(28h-12KA^2)]^{\frac{1}{2}}$ if $h > \frac{1}{9}(14h-6KA^2) > 0$, i.e., $7h/3 > KA^2 > 5h/6$. These points tend to the limiting curve if $KA^2 \to 5h/6$ and to the origin if $KA^2 \to 7h/3$. If $KA^2 \geqslant 7h/3$ the curve (18) is reduced to the point 0.

A discussion of Eq. (9) considered as a function of Ax^2 gives a distinction between the invariant curves of types C and δ in the same way as we distinguished above between the invariant curves of types B and A. If $10/3 > KA^2 > 7h/3$ the points P_3, P_4 do not exist any more. Then we have near the origin invariant curves for which the maximum X^2 occurs at $x=0$. They are simple invariant curves (distorted ellipses) similar to the invariant curves of the box-type orbits [Fig. 1(c)]. These invariant curves are separated from the δ-type invariant curves by the curve

$$\begin{aligned}\bar{\sigma}(Ax_0^2,X_0^2) &= (2h-X_0^2-Ax_0^2)(10h+15X_0^2-9Ax_0^2-12KA^2)\\ &\quad +(20h-6KA^2)^2=0,\end{aligned} \tag{25}$$

which is analogous to the curve (21). In fact we prove that simple invariant curves occur if $\frac{1}{3}(-14h+6KA^2) > X_1^2$, where X_1^2 is the smaller root of the equation $\theta(0,X^2)=0$. Then it is proved that $\frac{1}{3}(-14h+6KA^2) > X_a^2$, where X_a^2 is the smaller root of $\bar{\mathfrak{J}}(X^2)$, i.e., the discriminant of Eq. (9). In fact we see that

$$\theta[0, -\tfrac{1}{3}(14h+6KA^2)] = \bar{\sigma}(Ax_0^2,X_0^2) = \tfrac{1}{9}\bar{\mathfrak{J}}[-\tfrac{1}{3}(14h+6KA^2)]$$

and we proceed in the same way as above.

When $KA^2 \to 7h/3$ the whole curve (25) shrinks to the origin and for $KA^2 < 7h/3$ it becomes imaginary. That is, there are no simple invariant curves then near the origin. If $10h/3 > KA^2 > 7h/3$ we have simple invariant curves near the origin [Fig. 1(c)]. When $KA^2 \to 10h/3$ the curve (25) tends to the limiting curve. For $KA^2 > 10h/3$ all invariant curves are simple.

The conclusion is that the resonance phenomena occur when KA^2 is in the interval $(-5h/3, 10h/3)$. This interval is divided into the intervals $(-5h/3, -2h/3)$, $(-2h/3, 5h/6)$, $(5h/6, 7h/3)$, and $(7h/3, 10h/3)$. The corresponding sets of invariant curves are given in Figs. 1(a), Fig. 1 of paper II, 1(b), 1(c).

These considerations are valid when ϵ is very small, tending to zero. If ϵ is finite the invariant curves and

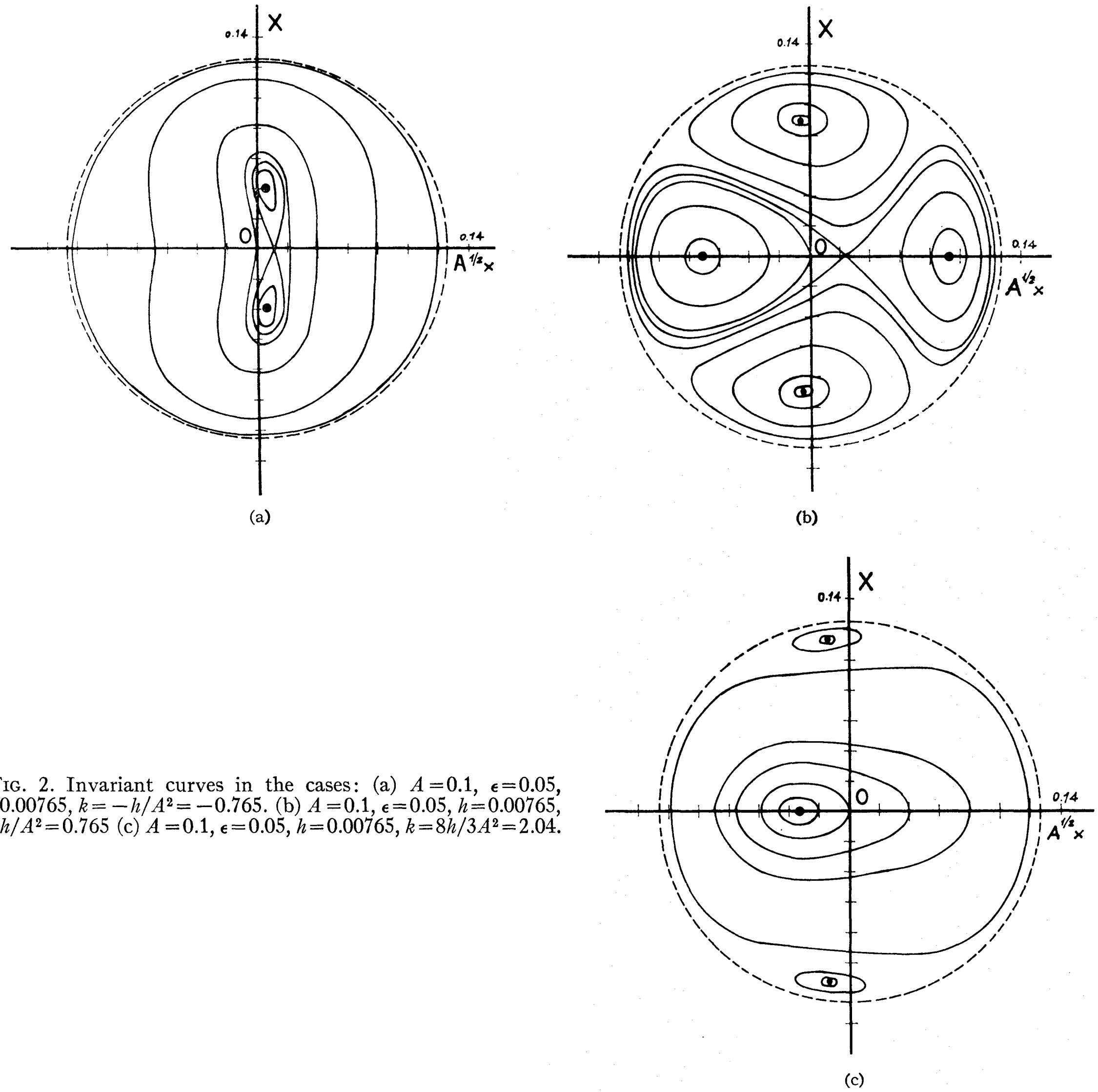


FIG. 2. Invariant curves in the cases: (a) $A=0.1$, $\epsilon=0.05$, $h=0.00765$, $k=-h/A^2=-0.765$. (b) $A=0.1$, $\epsilon=0.05$, $h=0.00765$, $k=h/A^2=0.765$ (c) $A=0.1$, $\epsilon=0.05$, $h=0.00765$, $k=8h/3A^2=2.04$.

the subdivisions of the resonance region are somewhat altered. However the main features of the classification remain the same. This is seen in Figs. 2(a)–(c), which correspond to the Figs. 1(a)–(c), but have $\epsilon=0.05$. The invariant curves are given in first-order approximation in ϵ. Higher-order approximations of the invariant curves corresponding to Fig. 1 of paper II (for $k=0$), have been considered in paper II. When $\epsilon=0.05$ the first-order theoretical invariant curves are very near the observed invariant curves, which are found by numerical integration of orbits.

II. BOUNDARIES OF ORBITS ($A\simeq B$)

In order to find the boundary of an orbit we eliminate X, Y between Eq. (4) and the equations

$$X^2+Y^2+Ax^2+By^2-2\epsilon xy^2=2h \tag{26}$$

and

$$J=(\partial\varphi/\partial X)Y-(\partial\varphi/\partial Y)X=0, \tag{27}$$

which is written

$$XY[-6kA^2+\epsilon(20h-12Ax^2-3Ay^2-8kAx)-15\epsilon Y^2]$$
$$=4\epsilon A(k+3x)y(2Y^2-2h+Ax^2+Ay^2), \tag{28}$$

if only terms up to the first order in ϵ are included.

Therefore XY is of order ϵ and the boundaries are near the lines $x=\pm(2\Phi_{1;0}/A)^{\frac{1}{2}}$ and $y=\pm(2\Phi_{2;0}/B)^{\frac{1}{2}}$.

If $x_0=y_0=0$ we find, in first-order approximation the boundaries

$$x=\pm\frac{X_0}{A^{\frac{1}{2}}}\left[1+\frac{\epsilon(Y_0^2-Ay^2)}{kA^2}\right]+\frac{\epsilon(2Y_0^2-Ay^2)}{3A^2} \tag{29}$$

and

$$y=\pm\frac{Y_0}{B^{\frac{1}{2}}}\left[1+\frac{2\epsilon x}{3A}+\frac{\epsilon}{kA^2}(Ax^2-X_0^2)\right]. \tag{30}$$

These formulas differ from those of the general irrational case (Contopoulos 1960) by the terms that have k in the denominator.

It is evident that these formulas are not valid if k is very small.

If $x_0=c_0\epsilon$ (small) and $X_0=0$, the boundary that intersects the x axis perpendicularly is found if we set $X=\chi\epsilon$, $x=c\epsilon$. Then Eqs. (28) and (4) become

$$XY=-2\epsilon y(2h-Ay^2)/3A \tag{31}$$

and

$$\frac{\epsilon^2k}{2}[\chi^2+A(c^2-c_0^2)]-\frac{\epsilon^2k}{3A}[Acy^2+2c(2h-Ay^2)-4hc_0+\frac{4y^2}{3A}(2h-Ay^2)]-\frac{\epsilon^2kA^2y^4}{6A^3}=0, \tag{32}$$

or

$$\chi^2=4y^2(2h-Ay^2)/9A^2 \tag{33}$$

and

$$c^2-\frac{2}{3A^2}(4h-Ay^2)c-c_0^2+\frac{8hc_0}{3A^2}-\frac{y^2}{9A^3}(8h-Ay^2)=0, \tag{34}$$

hence

$$c_1=c_0-\frac{y^2}{3A},\quad c_2=-c_0+\frac{8h-Ay^2}{3A^2}. \tag{35}$$

We have a periodic orbit if

$$x_0=4h\epsilon/3A^2. \tag{36}$$

The solutions (35), (36) are the same as in the general irrational case (Contopoulos 1965, Appendix A).

If $k=\epsilon K$ we have in zero order:

(i) The energy integral

$$X^2=C_1-Y^2, \tag{37}$$

where

$$C_1=2h-A(x^2+y^2). \tag{38}$$

(ii) The third integral [Eq. (4)] then becomes

$$48AxyXY=15Y^4-2C_2Y^2+C_3, \tag{39}$$

where

$$C_2=20h-3A(4x^2+y^2)-6KA^2, \tag{40}$$

$$C_3=16h^2+8hAy^2-3A^2y^2(8x^2+3y^2)+12\varphi_{0;0}+12KA^2(Ay^2-Y_0^2-Ay_0^2), \tag{41}$$

and

$$\varphi_{0;0}=X_0^2Y_0^2-Ax_0^2Y_0^2-AX_0^2y_0^2+A^2x_0^2y_0^2+4Ax_0X_0y_0Y_0-\tfrac{1}{3}(X_0^2+Ax_0^2)^2+\tfrac{1}{12}(Y_0^2+Ay_0^2)^2 \tag{42}$$

as in the resonance case.

(iii) Equation (28) is now written

$$XY(-15Y^2+C_2)=12Axy(2Y^2-C_1). \tag{43}$$

Equations (38), (39), and (43) have the same form as Eqs. (46), (47), and (50) of the resonance case (paper II), therefore they are solved in the same way. The boundary is given by Eq. (58) of paper II. The only difference is that C_2 and C_3 are not the same as in the resonance case.

We find a few characteristic points of the boundary. Because the method is in general similar to that used in paper II detailed calculations are not given, except when they differ essentially from those of paper II.

A. Section with the Curve of Zero Velocity

Equation (59) of paper II is replaced here by

$$C(Ax^2)=15A^2x^4-(20h+12KA^2)Ax^2-4h^2+12\varphi_{0;0}+12KA^2(2h-Y_0^2-Ay_0^2)=0. \tag{44}$$

The discriminant is

$$4\{160h^2-180\varphi_{0;0}+60KA^2[-4h+3(Y_0^2+Ay_0^2)]+36K^2A^4\}=4\{[5(2X_0^2-Y_0^2+2Ax_0^2-Ay_0^2)-6KA^2]^2+360A(x_0Y_0-X_0y_0)^2\}\geqslant 0. \tag{45}$$

It is zero only for the periodic orbits, where

$$\frac{y_0}{x_0}=\frac{Y_0}{X_0}=\pm\left(\frac{20h-6KA^2}{10h+5KA^2}\right)^{\frac{1}{2}}. \tag{46}$$

The periodic orbits are two straight lines through the origin in zero-order approximation. They exist for $10h/3>KA^2>-5h/3$, i.e., in the whole resonance region. If $KA^2\to 10h/3$ these orbits tend to the periodic orbit $y=0$. If $KA^2\to -5h/3$ they tend to the periodic orbit $x=0$ (in zero order).

The periodic orbits (46) are stable, while the periodic orbits $y=0$ and $x=0$ are unstable inside the resonance region and stable outside it. We may say that the periodic orbits (46), take away the stability of the orbits $y=0$ and $x=0$.

In higher approximation the resonance periodic orbits do not pass through the origin, as it is indicated by the corresponding invariant curves [Figs. 2(a)–(c)].

We assume $y_0=0$; then $C(0)=\theta(0,0)$ and

$$C(2h)=\theta(0,2h)=\theta(0,0)+20h^2>C(0).$$

If $\theta(0,0)>0$ we have two real roots of Eq. (43) if $KA^2>-5h/3$ (B-type orbits).

If $\theta(0,2h)<0$, we have no real root (C-type orbits), and if $\theta(0,0)<0$, $\theta(0,2h)>0$ we have one real root for Ax^2 (A-, D- or box-type orbits).

B. Section with the x Axis (if $XY\neq 0$)

Equations (79) and (81) of paper II are replaced here by

$$Y^2=C_2/15=(20h-12Ax^2-6KA^2)/15 \quad (47)$$

and

$$D(Ax^2)=144A^2x^4-24Ax^2(20h-6KA^2)+100h^2+225X_0^4 -30X_0^2(10h-3Ax_0^2+6KA^2)-135A^2x_0^4+420hAx_0^2 -180KA^3x_0^2+120KA^2h+36K^2A^4=\mathcal{G}(Ax^2)=0. \quad (48)$$

As we have seen above, Eq. (48) has real roots only if inequality (13) is satisfied inside the limiting curve. The smaller root Ax_a^2 is acceptable if

$$0\leqslant Ax_a^2\leqslant 2h \quad \text{and} \quad 0\leqslant X^2\leqslant 2h, \quad 0\leqslant Y^2\leqslant 2h,$$

hence

$$0\leqslant Ax_a^2\leqslant \tfrac{1}{3}(10h+6KA^2)$$

and

$$Ax_a^2\leqslant \tfrac{1}{12}(20h-6KA^2).$$

The last relation is satisfied whenever we have real roots of Eq. (47) because their mean is $\frac{1}{12}(20h-6KA^2)$. We have also seen that when inequality (13) is satisfied we cannot have $\mathcal{G}(0)<0$. Therefore the only condition is

$$\mathcal{G}[\tfrac{1}{3}(10h+6KA^2)]\geqslant 0.$$

We have already seen that if this inequality is satisfied we have A-type orbits, otherwise we have box-type orbits (no angular point of the boundary on the x axis).

C. Branch Perpendicular to the x Axis ($X=0$)

Equation (84) of paper II is replaced here by

$$E(Ax^2)=9A^2x^4-2Ax^2(14h-6KA^2)+4h^2-12\varphi_{0;0} -12KA^2(2h-Y_0^2-Ay_0^2)=0. \quad (49)$$

The discriminant is

$$4\{[2(X_0^2+Ax_0^2)-7(Y_0^2+Ay_0^2)+6KA^2]^2 +216(X_0Y_0+Ax_0y_0)^2\}\geqslant 0. \quad (50)$$

It is zero only for the periodic orbit where

$$\frac{A^{\frac{1}{2}}y_0}{X_0}=-\frac{Y_0}{A^{\frac{1}{2}}x_0}=\pm\left(\frac{2h+3KA^2}{7h-3KA^2}\right)^{\frac{1}{2}}. \quad (51)$$

The periodic orbit is the ellipse

$$\frac{Ax_0^2}{7h-3KA^2}+\frac{Ay_0^2}{2h+3KA^2}=\frac{2}{9}. \quad (52)$$

It exists whenever $7h/3>KA^2>-2h/3$, i.e., when there are C-type orbits.

We find, as in the resonance case, that the boundaries of the A-, B- and C-type orbits have 1, 0, and 2 positive roots for $X=0$.

In case A there is a double root for $x=x_0=0$. In second approximation the double root (corresponding to an unstable periodic orbit) is given by Eq. (36).

D. Section with the y Axis (if $XY\neq 0$)

Equation (106) of paper II is replaced here by

$$F(Ay^2)=144A^2y^4-24Ay^2(10h+6KA^2)+160h^2 -180\varphi_{0;0}+60KA^2(-4h+3(Y_0^2+Ay_0^2)) +36K^2A^4=0. \quad (53)$$

We have real roots only if

$$\theta(0,0)\equiv -4h^2+12\varphi_{0;0}+12KA^2(2h-Y_0^2-Ay_0^2)\geqslant 0.$$

This happens in cases B, δ, and for some "box" orbits. Further, we must have $0\leqslant Ay^2\leqslant 2h$ and

$$0\leqslant Y^2=(20h-3Ay^2-6KA^2)/15\leqslant 2h-Ay^2.$$

These relations are reduced to

(i) $0\leqslant Ay^2\leqslant \frac{1}{12}(10h+6KA^2)$ if $KA^2<7h/3$,

and

(ii) $0\leqslant Ay^2\leqslant \frac{1}{3}(20h-6KA^2)$ if $KA^2>7h/3$.

We have always

$$F(0)=160h^2-180\varphi_{0;0} +60KA^2[-4h+3(Y_0^2+Ay_0^2)]+36K^2A^4\geqslant 0, \quad (54)$$

as given in Eq. (45).

If $KA^2<7h/3$ and Eq. (53) has real roots, then one of them is smaller than the mean of the roots $\frac{1}{12}(10h+6KA^2)$; i.e., then we have always one acceptable root (cases B and δ only).

If $KA^2>7h/3$ we must also have

$$F[\tfrac{1}{3}(20h-6KA^2)]=-15\bar{\sigma}(Ax_0^2,X_0^2) -40(20h-6KA^2)(-14h+6KA^2)<0, \quad (55)$$

because the mean of the roots is then larger than $\frac{1}{3}(20h-6KA^2)$.

The curve

$$15\bar{\sigma}(Ax_0^2,X_0^2)+40(20h-6KA^2)(-14h+6KA^2)=0 \quad (56)$$

is inside the curve (25) if $10h/3>KA>7h/3$.

The orbits with initial conditions inside the curve (56) are "box" orbits; their boundary does not have an angular point on the y axis. The orbits with an angular point on the y axis are called D-type orbits. We see that if $KA^2<7h/3$ then the D-type orbits correspond to δ-type invariant curves. If, however, $KA^2>7h/3$, only part of the D-type orbits correspond to invariant curves of δ type; if the initial conditions are between the curves (56) and (25) the orbits are D type while the invariant curves are of the simple (ellipse like) type.

The curve (56) intersects the X axis at the points

$$X_0=\pm((1/15)\{10h+6KA^2-4[3(6KA^2-10h)(10h-3KA^2)]^{\frac{1}{2}}\})^{\frac{1}{2}}$$

and the $A^{\frac{1}{2}}x$ axis at the points

$$A^{\frac{1}{2}}x_0=\pm[\tfrac{1}{3}(-14h+6KA^2)]^{\frac{1}{2}}.$$

These points are inside the corresponding intersections of the curve (25); they tend to the origin if $KA^2\rightarrow 7h/3$ and to the limiting curve if $KA^2\rightarrow 10h/3$. The curves (56) and (25) both shrink to one point if $KA^2\rightarrow 7h/3$ and tend to coincide with the limiting curve if $KA^2\rightarrow 10h/3$.

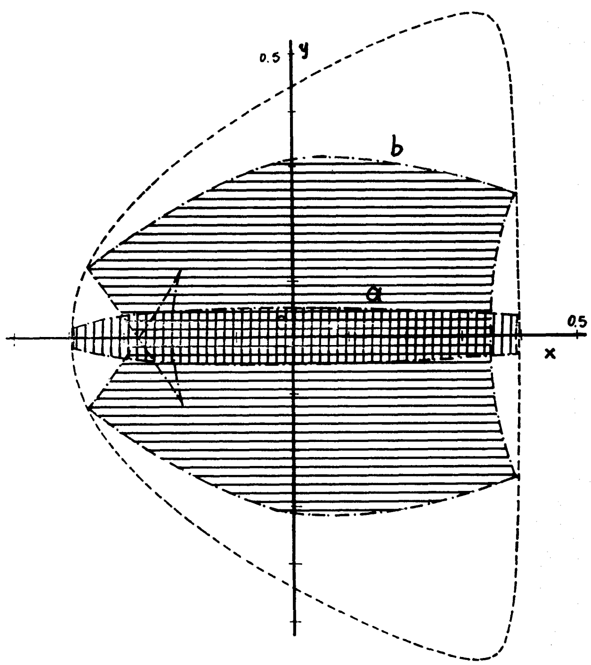


FIG. 3. Two orbits in the case $A=0.1$, $B=0.08$, $\epsilon=0.1$ ($h\simeq 0.00765$). Initial conditions = (a) $x_0=0.39$, $y_0=X_0=0$, $Y_0=0.009487$ (box type). (b) $x_0=0.35$, $y_0=X_0=0$, $Y_0=0.055227$ (A type on the left, box type on the right).

E. Branch Perpendicular to the y Axis ($Y=0$)

Equation (109) of paper II is replaced here by

$$G(Ay^2)=9A^2y^4-2Ay^2(4h+6KA^2)-16h^2-12\varphi_{0;0}+12KA^2(Y_0^2+Ay_0^2)=0. \quad (57)$$

The discriminant is the same as that of Eq. (49); i.e., it is positive except for the type-C periodic orbit, when it is zero.

As in paper II it is seen that no solution of Eq. (57) is acceptable in case B, two solutions in case C, and one solution in case A and D and for box-type orbits.

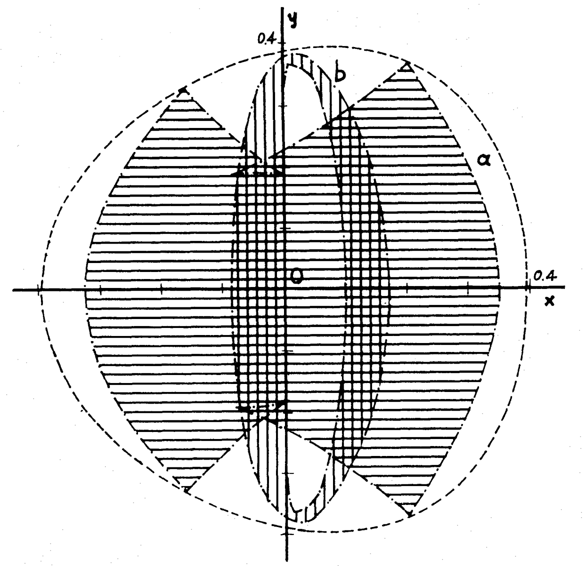


FIG. 4. Two orbits in the case $A=0.1$, $B=0.104$, $\epsilon=0.05$, ($h\simeq 0.00765$). Initial conditions: (a) $x_0=0.35$, $y_0=X_0=0$, $Y_0=0.055227$ (D type). (b) $x_0=0.1$, $y_0=X_0=0$, $Y_0=0.1195826$ (C type).

Figures 3–5, give some examples of boundaries of orbits in various cases. The boundaries are marked as dash-dotted lines and the areas covered by the orbits are shaded. The curves of zero velocity are marked as dashed lines, on which lie the apexes of the boundaries of the A-, B- and D-type orbits.

In all cases we take $A=0.1$, $h\simeq 0.00765$; hence the intervals $(-5h/3, -2h/3)$, $(-2h/3, 5h/6)$, $(5h/6, 7h/3)$ and $(7h/3, 10h/3)$ for KA^2 are $(-0.01275, -0.0051)$, $(-0.0051, 0.006375)$, $(0.006375, 0.01785)$, $(0.01785,$

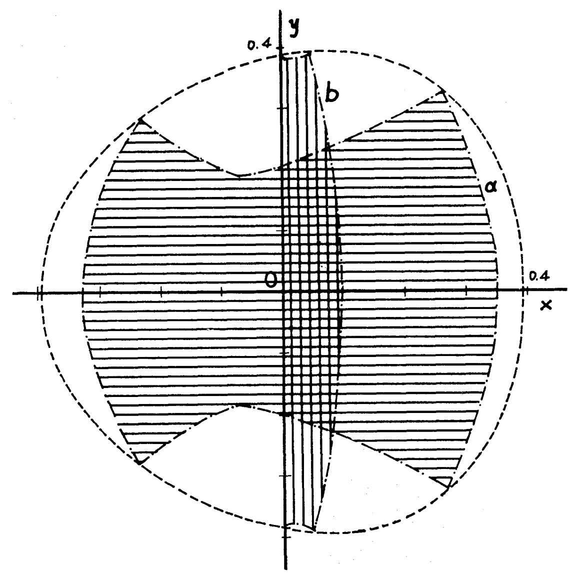


FIG. 5. Two orbits in the case $A=0.1$, $B=0.106$, $\epsilon=0.05$, ($h\simeq 0.00765$). Initial conditions: (a) $x_0=0.35$, $y_0=X_0=0$, $Y_0=0.55227$ (D type). (b) $x_0=0.1$, $y_0=X_0=0$, $Y_0=0.1195826$ (box type).

0.0255). The corresponding intervals for B for $\epsilon=0.1$ are: (i) (0.08725, 0.0949), (ii) (0.0949, 0.106375), (iii) (0.106375, 0.11785), (iv) (0.11785, 0.1255); and for $\epsilon=0.05$ they are: (i) (0.0968125, 0.098725), (ii) (0.98725, 0.10159375), (iii) (0.10159375, 0.1044625), (iv) (0.1044625, 0.106375). Of course, these intervals are given in first approximation and in practice they are somewhat different, especially in the case $\epsilon=0.1$.

Figure 3 gives two orbits when B is in the interval (i). Both orbits have their initial points on the x axis and their initial velocities perpendicular to this axis. Orbit a is of box type. Orbit b is A type at left, i.e. the boundary has an angular point on the x axis and forms a triangle inside the space covered by the orbit. On the right, the boundary does not have an angular point. This is due to the fact that when ϵ is large the invariant curves are distorted so that the A-type character (two maxima for $|x|$, which do not happen for $X=0$) appears only on the left, while on the right the maximum x occurs for $X=0$ [Fig. 2(a)].

If B is in the interval (ii) the orbits are similar to those of paper II, because the value $K=0$ belongs to this interval.

Figure 4 gives two orbits when B is in the interval (iii). Orbit a is of D type, while orbit b is of C type, as it should be expected from the form of the invariant curves [Fig. 2(b)].

Figure 5 gives two orbits when B is in the interval (iv). Orbit a is of D type while orbit b is of box type, as should be expected from the theory above.

Similar results are found when $\epsilon=0.1$. Then, however, the D-type orbits have the angular points of the upper and lower boundaries more to the left of the y axis than the corresponding orbits in the case $\epsilon=0.05$.

When B is outside the resonance range all the orbits are boxes.

III. CASE $A\simeq 4B$. INVARIANT CURVES

This case is similar to the case $A=4B$ of paper I (Contopoulos 1963). (The notations of papers II and III are different from those used in paper I. Namely, we here use the symbols $x, y, X, Y, A, B, \epsilon, \Phi_{10}, \Phi_{20}, \cdots$ instead of $\xi, z, R, Z, P, Q, b, \Phi_0, V_0, \cdots$.)

If $A\simeq 4B$ a small divisor appears in the term Φ_{11} of the third integral (1′).

We can construct an integral valid near the resonance case $A=4B$ if we set

$$4B=A+\epsilon k \tag{58}$$

and multiply Φ_1 by $2(4B-A)/\epsilon$. Then

$$\begin{aligned}\varphi=\varphi_0+\epsilon\varphi_1+\cdots=k(X^2+Ax^2)+Axy^2-4xY^2+4XyY\\ +(\epsilon/A^2)(-kA^2xy^2-A^2x^4-A^2x^2y^2-\tfrac{1}{8}A^2y^4-2Ax^2X^2\\ -4Ax^2Y^2+5AX^2y^2+7Ay^2Y^2-16AxXyY-X^4\\ -12X^2Y^2+14Y^4)+\cdots=\varphi_{0;0}+\epsilon\varphi_{1;0}+\cdots,\end{aligned} \tag{59}$$

where we have included in φ_1 the terms

$$-(1/A^2)[(2\Phi_{10})^2+12(2\Phi_{10})(2\Phi_{20})-14(2\Phi_{20})^2]$$

of the resonance case [the present terms are four times larger than in Eqs. (47), (48) of paper I].

The invariant curves on the $A^{\frac{1}{2}}x_0$, X_0 plane for $y=y_0=0$ are found if we set

$$Y^2=2h-X^2-Ax^2 \tag{60}$$

in Eq. (59):

$$\begin{aligned}\psi(Ax^2,X^2)=k(X^2+Ax^2)-4x(2h-X^2-Ax^2)\\ +(\epsilon/A^2)(25X^4-2X^2(40h-21Ax^2)\\ +17A^2x^4-64hAx^2)=\psi(Ax_0^2,X_0^2)=\text{const.}\end{aligned} \tag{61}$$

If $\epsilon\rightarrow 0$ the invariant curves become

$$X^2=[\psi+4x(2h-Ax^2)-kAx^2]/(k+4x). \tag{62}$$

These curves for various values of k and $A^{\frac{1}{2}}x_0, X_0$ are shown in Fig. 6.

All the invariant curves are inside the limiting curve

$$X_0^2+Ax_0^2=2h. \tag{63}$$

The invariant curves cannot cross the line $A^{\frac{1}{2}}x=-\frac{1}{4}kA^{\frac{1}{2}}$.

If $-\frac{1}{4}kA^{\frac{1}{2}}<-(2h)^{\frac{1}{2}}$ or $-\frac{1}{4}kA^{\frac{1}{2}}>(2h)^{\frac{1}{2}}$ this line is outside the limiting curve and the invariant curves are simple distorted ellipses as in the general nonresonance case. We have resonance phenomena if

$$-4(2h/A)^{\frac{1}{2}}<k<4(2h/A)^{\frac{1}{2}}. \tag{64}$$

We must always have $0\leqslant X^2\leqslant 2h-Ax^2$; if $k+4x>0$ these relations are written: $4Ax^3+kAx^2-8hx-\psi<0$ and $\psi<2hk$, or

$$\begin{aligned}\omega(x)\equiv 4A(x^3-x_0^3)+kA(x^2-x_0^2)-8h(x-x_0)\\ -X_0^2(k+4x_0)<0,\end{aligned} \tag{65}$$

and $k+4x_0>0$. The function $\omega(x)$ is positive for $x=-\frac{1}{4}k$, $x=(2h/A)^{\frac{1}{2}}$ and $x\rightarrow\infty$, and negative for $x=x_0$; therefore it has three real roots x_1, x_2, x_3, where $-\frac{1}{4}k<x_1<x_0<x_2<(2h/A)^{\frac{1}{2}}<x_3$, and we must have $x_1\leqslant x\leqslant x_2$. For $x=x_1$ or $x=x_2$ we have $X^2=0$.

The two roots x_1, x_2 coincide if they are equal to one root of the equation $\omega'(x)\equiv 12Ax^2+2kAx-8h=0$, or

$$x=-\tfrac{1}{12}k\{1\pm[1+(96h/k^2A)]^{\frac{1}{2}}\}. \tag{66}$$

In order that these roots should be between $\pm(2h/A)^{\frac{1}{2}}$ we must have the relations (64). Then we have always two acceptable solutions (two invariant points that correspond to two periodic orbits that cross the x axis perpendicularly). We can check that then we have $X^2_{\max}=0$.

Therefore we have two sets of closed invariant curves around the points (66). If $k=0$ the two sets are symmetric with respect to the X axis.

If k is outside the limits (64), e.g., if $|k|A^{\frac{1}{2}}>4(2h)^{\frac{1}{2}}$

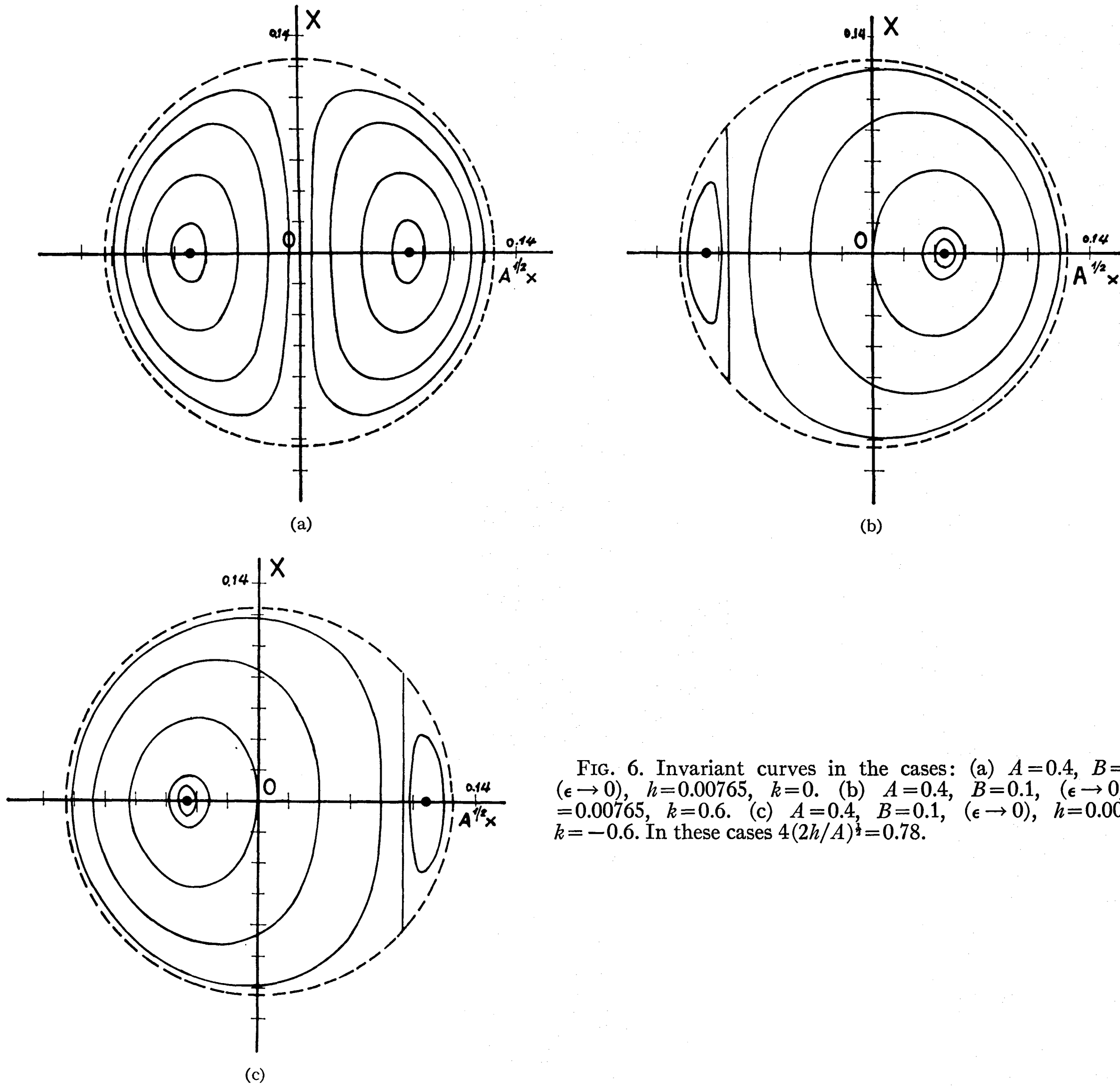


FIG. 6. Invariant curves in the cases: (a) $A=0.4$, $B=0.1$, $(\epsilon\to 0)$, $h=0.00765$, $k=0$. (b) $A=0.4$, $B=0.1$, $(\epsilon\to 0)$, $h=0.00765$, $k=0.6$. (c) $A=0.4$, $B=0.1$, $(\epsilon\to 0)$, $h=0.00765$, $k=-0.6$. In these cases $4(2h/A)^{\frac{1}{2}}=0.78$.

we have only one set of closed invariant curves around the point $x=-\frac{1}{12}k[1-(1+(96h/k^2A)^{\frac{1}{2}}]$.

This explains why for values of A near $4B$ we have two periodic orbits perpendicular to the x axis, while in general we have only one such orbit (Contoupoulos 1965).

A numerical application of the approximate formula (66) for $h=0.00765$, $B=0.1$, $\epsilon=0.1$, and $A=0.39$ $(k=0.1)$, of $A=0.41$ $(k=-0.1)$ gives the periodic orbits at $x=0.106$, $x=-0.123$ for $A=0.39$ and at $x=0.120$, $x=-0.104$ for $A=0.41$, while the periodic orbits found empirically are at $x=0.121$, $x=-0.106$ for $A=0.39$ and at $x=0.133$, $x=-0.085$ for $A=0.41$.

If we take into account the terms of first order in ϵ the invariant curves change a little, but their topology is the same

In particular we note that the straight line $A^{\frac{1}{2}}x=-\frac{1}{4}kA^{\frac{1}{2}}$, which separates the sets of invariant curves around the two invariant points P_1, P_2, becomes curved (for $k>0$ it is concave to the left).

IV. BOUNDARIES OF ORBITS $(A\simeq 4B)$

We find the boundaries of the orbits in zero-order approximation. As usually we eliminate X and Y between the zero-order energy integral

$$X^2=2h-Y^2-Ax^2-\tfrac{1}{4}Ay^2, \tag{67}$$

the third integral

$$\varphi_0=k(X^2+Ax^2)+x(Ay^2-4Y^2)+4yXY=\varphi_{0;0}, \tag{68}$$

and

$$J_0\equiv(\partial\varphi_0/\partial X)Y-(\partial\varphi_0/\partial Y)X = 2XY(k+4x)+4y(Y^2-X^2)=0. \tag{69}$$

Equations (68) and (69) become

$$4yXY=Y^2(k+4x)-Axy^2-k(2h-\tfrac{1}{4}Ay^2)+\varphi_{0;0}, \tag{70}$$

and

$$(k+4x)XY=2y(-2Y^2+2h-Ax^2-\tfrac{1}{4}Ay^2). \tag{71}$$

Eliminating XY we find

$$Y^2=[(k+4x)^2+16y^2]^{-1}[2y^2(8h-4Ax^2-Ay^2) + (k+4x)(Axy^2-\varphi_{0;0})+\tfrac{1}{4}k(k+4x)(8h-Ay^2)]. \quad (72)$$

If we raise both members of Eq. (71) to the square and replace X^2 and Y^2 by their values (67) and (72) we find after some operations

$$A^3y^6-2A^2y^4(8h-kAx)-Ay^2[(2h-Ax^2)(k^2A-32h) + (2hk-\varphi_{0;0})(-kA+12Ax)] -4A(2hk-\varphi_{0;0})(4Ax^3+kAx^2-8hx-\varphi_{0;0})=0. \quad (73)$$

If $X_0=y_0=0$ we have for $y=0$:

$$(x-x_0)[4Ax^2+(4x_0+k)Ax +4Ax_0^2-8h+kAx_0]=0, \quad (74)$$

hence we have three roots, x_0, x_1, x_2. If $4x_0+k>0$ we have

$$x_2<-(2h/A)^{\frac{1}{2}}<-\tfrac{1}{8}(4x_0+k)<x_1<(2h/A)^{\frac{1}{2}},$$

and if $4x_0+k<0$ we have

$$-(2h/A)^{\frac{1}{2}}<x_2<-\tfrac{1}{8}(4x_0+k)<(2h/A)^{\frac{1}{2}}<x_1.$$

If $X_0=0$ Eq. (73) is written

$$A^2[y^2-(4x_0+k)(x-x_0)] \times[y^2-(4x_1+k)(x-x_1)][(y^2-(4x_2+k)(x-x_2)]=0, \quad (75)$$

i.e., the boundary is composed of three parabolas, as in the resonance case (Fig. 8, in Contopoulos 1965).

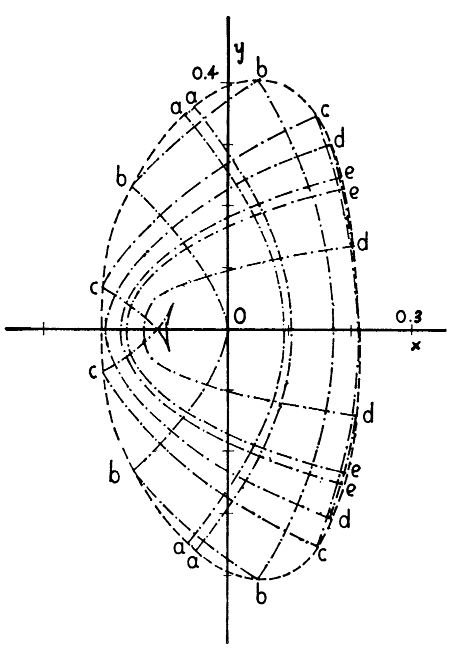


FIG. 7. Boundaries of orbits in the case: $A=0.34$, $B=0.1$, $\epsilon=0.1$ ($k=0.6$), $h=0.00765$, and initial conditions $y_0=X_0=0$ and (a) $A^{\frac{1}{2}}x_0=0.06$, (b) $A^{\frac{1}{2}}x_0=0$, (c) $A^{\frac{1}{2}}x_0=-0.06$, (d) $A^{\frac{1}{2}}x_0=-0.08$, $A^{\frac{1}{2}}x_0=-0.10$.

We have periodic orbits if $x_0=x_1$ or $x_0=x_2$, i.e.,

$$x_0=-\tfrac{1}{12}k\{1\pm[1+(96h/k^2A)]^{\frac{1}{2}}\},$$

as given by Eq. (66).

The periodic orbits in this approximation are parabolas

$$y^2-(4x_0+k)(x-x_0)=0. \quad (76)$$

If $k\to -4(2h/A)^{\frac{1}{2}}$ we have periodic orbits for $x_0\to (2h/A)^{\frac{1}{2}}$ and $x_0\to -\frac{1}{3}(2h/A)^{\frac{1}{2}}$; then the first parabola becomes $y^2=0$, i.e., it coincides with the x axis. If $k\to 4(2h/A)^{\frac{1}{2}}$ we find similar results.

Some orbits calculated in the case $k=0.6$ are shown in Fig. 7. The boundaries of the orbits are approximately three parabolas. All orbits surround one of the two main periodic orbits.

In case c we see that the left boundary has an angular point on the x axis and there is also an inner triangle formed by extending the two arcs of the boundary a little beyond the angular point, and by a small arc perpendicular to the x axis. In this respect the orbit has some similarity with the A-type orbits of the case $A\simeq B$. The corresponding invariant curve has two minima with respect to x which do not lie on the x axis. This happens because this invariant curve goes near the limiting line that separates the two sets of invariant curves around the two invariant points P_1, P_2, and this limiting line is concave to the left.

V. CONCLUSIONS

We see that the resonance phenomena do not appear only at the resonance itself, i.e., when the ratio of the unperturbed frequencies $A^{\frac{1}{2}}/B^{\frac{1}{2}}$ has a rational value, but also for a range of values of $A^{\frac{1}{2}}/B^{\frac{1}{2}}$.

The main resonance effect is the appearance of new periodic orbits besides the periodic orbit near the axis $x=0$ and the orbit $y=0$. When one of the new periodic orbits is stable there appear "tube" orbits near it. The invariant curves corresponding to the "tube" orbits are closed curves around the stable invariant points, which represent the periodic orbits. As the ratio $A^{\frac{1}{2}}/B^{\frac{1}{2}}$ increases or decreases the invariant points move until they disappear, either at the limiting curve (which corresponds to the periodic orbit $y=0$) or by reaching another invariant point, e.g., the invariant point near the center (which corresponds to the periodic orbit near $x=0$).

In the case $A^{\frac{1}{2}}/B^{\frac{1}{2}}\simeq 2$ there are two periodic orbits (approximately two parabolas) crossing perpendicularly the x axis one on each side of the origin. As $A-4B$ increases (decreases) the periodic orbit to the right (to the left) tends to coincide with the orbit $y=0$. The periodic orbit $y=0$ is unstable in the resonance region and stable outside it.

In the case $A^{\frac{1}{2}}/B^{\frac{1}{2}}\simeq 1$ the structure of the resonance region is more complicated. There are four different patterns of invariant curves which correspond to four consecutive intervals of values of $A-B$. One of these intervals includes the value $A-B=0$. The invariant points P_1, P_2 appear in all four patterns, while the invariant points P_3, P_4 appear only in the central two patterns.

The theory of the third integral, developed in zero order with respect to ϵ, gives the main characteristics of the invariant curves and of the various types of orbits. Higher approximations give results which are closer to the accurate results found by numerical integration of the orbits.

One new form of resonant orbits, namely the D-type orbits, appears for a range of values of $A-B(<0)$, while it is not present for $A=B$. This is one case of a resonance phenomenon which appears near but not at the resonance itself.

[We would like to indicate a few typographical errors in paper II: In Eq. (9) write $\varphi=$ instead of $\varphi_0=$; in Eq. (33) write $-3Ax_0^2)$ instead of $-3Ax_0)^2$; in Eq. (41) write $9A^2x_0^4$ instead of $9Ax_0^4$; in Eq. (59) write A instead of B; in Eq. (60) write $32h^2$ instead of 32^2; in Table I, col. 4 write 1.0 instead of 1.7; and in p. 831, column 2, line 12 interchange X_0 and Y_0.]

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