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Resonance Cases and Small Divisors in a Third Integral of Motion. I

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In this paper a general discussion of the resonance cases in an axially-symmetric potential field is presented, when the unperturbed frequencies in the radial and z direction have a rational ratio. The general form of the third integral is not valid in these cases because of the appearance of divisors of the form (m^2P-n^2Q), which become zero in the resonance cases. However, a new isolating integral of the unperturbed case is available, and this can be used to construct a third integral in the form of a power series and eliminate all secular terms. Three cases are distinguished, (α) $m+n>4$, (β) $m+n=4$, and (γ) $m+n<4$. In the first case the orbits are rather similar to those of the general irrational case. In the third case the orbits show a quite peculiar character, which, however, can be explained rather accurately by a first-order theory of the third integral. Numerical integrations were made for the cases $P=16Q$, $4P=9Q$, and $P=4Q$. The third integral, given in first- or second-order approximation, is rather well conserved. Case β and the cases of small divisors, when m^2P-n^2Q is near zero but not equal to zero, are discussed in Paper II.

1. GENERAL THEORY

In a previous paper (Contopoulos, 1960, referred to hereafter as paper A) a formal third integral of motion is given in the axially symmetric field

$$U=-\frac{1}{2}\left(-\frac{C^2}{r^2}+P\xi^2+Qz^2-2b\xi z^2\right), \tag{1}$$

where r, θ, z are cylindrical coordinates, $\xi=r-r_0$ (r_0, $z_0=0$) is the initial point on the plane of symmetry, C is the angular momentum along the z axis, and P, Q, and b are constants.

Then if R, Z are the projections of the velocity v on the axes r and z, the energy integral is

$$F=\tfrac{1}{2}(P\xi^2+R^2+Qz^2+Z^2-2b\xi z^2)=\tfrac{1}{2}v_0^2. \tag{2}$$

The third integral is found in the form of a formal series

$$\begin{aligned}\Phi=&\tfrac{1}{2}(P\xi^2+R^2)+\frac{b}{4Q-P}[(P-2Q)\xi z^2-2\xi Z^2+2RzZ]\\&+\frac{b^2}{(4Q-P)(Q-P)}\left[-\frac{(Q-P)z^4}{2}+\frac{(2P+Q)R^2Z^2}{2PQ}\right.\\&+\frac{(2P-5Q)\xi^2Z^2}{2Q}-\frac{(4P-Q)R^2z^2}{2P}\\&\left.+\tfrac{3}{2}Q\xi^2z^2+6\xi RzZ\right]+\cdots\\=&\frac{R_0^2}{2}+\frac{b^2(2P+Q)R_0^2Z_0^2}{(4Q-P)(Q-P)2PQ}+\cdots.\end{aligned} \tag{3}$$

It can be seen that each term has a number of divisors of the form (m^2P-n^2Q). There is an infinite number of such divisors, so that if $P^{\frac{1}{2}}/Q^{\frac{1}{2}}=n/m=$rational, one such divisor becomes zero, and the above series is no longer valid. We call these cases "resonance cases" because then the frequencies of the oscillations along the axes r and z in the unperturbed case ($b=0$) are commensurable and the orbits are closed. We shall see that in these cases also there is a formal third integral of motion.

If Φ is an integral of motion then the Poisson bracket (Φ,F) is zero. If we write $F=F_0+bF_1$ and $\Phi=\Phi_0+b\Phi_1+b^2\Phi_2+\cdots$ we have

$$(\Phi_0,F_0)=0,$$

$$(\Phi_0,F_1)+(\Phi_1,F_0)=0,$$

and generally

$$(\Phi_\nu,F_1)+(\Phi_{\nu+1},F_0)=0,$$

or

$$R\frac{\partial\Phi_{\nu+1}}{\partial\xi}+Z\frac{\partial\Phi_{\nu+1}}{\partial z}-P\xi\frac{\partial\Phi_{\nu+1}}{\partial R}-Qz\frac{\partial\Phi_{\nu+1}}{\partial Z}=-(\Phi_\nu,F_1). \tag{4}$$

Thus $\Phi_{\nu+1}$ is found by solving (4) if Φ_0, $\Phi_1\cdots\Phi_\nu$ are known already. The corresponding system to Eq. (4) is

$$\frac{d\xi}{R}=\frac{dz}{Z}=\frac{dR}{-P\xi}=\frac{dZ}{-Qz}=\frac{d\Phi_{\nu+1}}{-(\Phi_\nu,F_1)}=dT, \tag{5}$$

where T is an auxiliary variable.

The solution of this system is

$$\begin{aligned}\xi&=\frac{(2\Phi_0)^{\frac{1}{2}}}{P^{\frac{1}{2}}}\sin P^{\frac{1}{2}}(T-T_0), & z&=\frac{(2V_0)^{\frac{1}{2}}}{Q^{\frac{1}{2}}}\sin Q^{\frac{1}{2}}T,\\ R&=(2\Phi_0)^{\frac{1}{2}}\cos P^{\frac{1}{2}}(T-T_0), & Z&=(2V_0)^{\frac{1}{2}}\cos Q^{\frac{1}{2}}T,\end{aligned} \tag{6}$$

where Φ_0, V_0 and T_0 are 3 integrals of the system (5), and $\Phi_0+V_0=F_0$. Then

$$\Phi_{\nu+1}=-\int(\Phi_\nu,F_1)dT+f(\Phi_0,V_0,T_0), \tag{7}$$

where f is an arbitrary function of Φ_0, V_0 and T_0.

In order to do the integration (7) ξ, R, z, Z are replaced by their values (6) and after the integration the sines and cosines are given again as functions of ξ, R,

z, Z. This is possible whenever (Φ_ν, F_1) does not contain a constant term, but only trigonometric terms of the form

$$\begin{matrix}\sin\\ \cos\end{matrix}[mP^{\frac{1}{2}}(T-T_0)-nQ^{\frac{1}{2}}T]. \tag{8}$$

If, however, a constant term appears, then the integration gives a secular term in T. It has been proved in Paper A that secular terms do not appear if $P^{\frac{1}{2}}/Q^{\frac{1}{2}}$ = irrational; but if $P^{\frac{1}{2}}/Q^{\frac{1}{2}}=n/m$, then there appear terms of the form

$$\begin{matrix}\sin\\ \cos\end{matrix}[mP^{\frac{1}{2}}(T-T_0)-nQ^{\frac{1}{2}}T]=\begin{matrix}\sin\\ \cos\end{matrix}(-mP^{\frac{1}{2}}T_0)=\text{const.}$$

in some (Φ_ν, F_1), that give rise to secular terms in $\Phi_{\nu+1}$.

In order to avoid such terms, Whittaker (1916, 1937) has used two methods, that in our notations can be described as follows:

(a) We may choose the function $f(\Phi_0,V_0,T_0)$ at each step in such a way that no secular terms appear in higher-order terms.

(b) The above method does not work if the critical terms appear in Φ_1, i.e., if $4Q-P=0$. In this case we assume first that $P\neq 4Q$ and multiply both members of Eq. (3) by $(4Q-P)$; in this way we have the integral

$$b\Phi'=(4Q-P)\Phi=(4Q-P)\Phi_0 +b[(P-2Q)\xi z^2-2\xi Z^2+2RzZ]+\cdots.$$

If now $4Q-P$ becomes zero, we get

$$\Phi'=(P-2Q)\xi z^2-2\xi Z^2+2zRZ+\cdots. \tag{9}$$

Whittaker gave two examples to illustrate his methods, but he did not prove that these methods always work, i.e., that we can always avoid the secular terms.

An indirect proof that in the resonance cases also there are formal integrals of motion was given by Cherry (1914). Cherry also gave some examples where new integrals are found.

We shall apply presently a general method, that may be considered as a combination of the two methods of Whittaker, to prove that in the case of the potential field (1) we always have a formal third integral of motion, whenever $P^{\frac{1}{2}}/Q^{\frac{1}{2}}=n/m$.

If $P^{\frac{1}{2}}/Q^{\frac{1}{2}}=n/m$ we have three isolating integrals of the system

$$\frac{d\xi}{R}=\frac{dz}{Z}=\frac{dR}{-P\xi}=\frac{dZ}{-Qz}. \tag{10}$$

namely Φ_0, V_0 and

$$S_0=(2\Phi_0)^{\frac{1}{2}m}(2V_0)^{\frac{1}{2}n}\sin(mP^{\frac{1}{2}}T_0) =(2\Phi_0)^{\frac{1}{2}m}(2V_0)^{\frac{1}{2}n}\sin[nQ^{\frac{1}{2}}T-mP^{\frac{1}{2}}(T-T_0)].$$

In fact using the solutions (6) we get

$$\sin[mP^{\frac{1}{2}}(T-T_0)] =\frac{1}{(2\Phi_0)^{\frac{1}{2}m}}[C^m{}_1P^{\frac{1}{2}}\xi R^{m-1}-C^m{}_3P^{\frac{3}{2}}\xi^3R^{m-3}+\cdots],$$

$$\cos[mP^{\frac{1}{2}}(T-T_0)]=\frac{1}{(2\Phi_0)^{\frac{1}{2}m}}[R^m-C^m{}_2P\xi^2R^{m-2}+\cdots],$$

$$\sin[nQ^{\frac{1}{2}}T]=\frac{1}{(2V_0)^{\frac{1}{2}n}}[C^n{}_1Q^{\frac{1}{2}}zZ^{n-1}-C^n{}_3Q^{\frac{3}{2}}z^3Z^{n-3}+\cdots],$$

$$\cos[nQ^{\frac{1}{2}}T]=\frac{1}{(2V_0)^{\frac{1}{2}n}}[Z^n-C^n{}_2Qz^2Z^{n-2}+\cdots],$$

hence

$$\begin{aligned}S_0=&\sum_{q_1=0}^{[\frac{1}{2}m]}\sum_{q_2=0}^{[\frac{1}{2}(n-1)]}(-1)^{q_1+q_2}\\ &\times C^m{}_{2q_1}C^n{}_{2q_2+1}(P^{\frac{1}{2}}\xi)^{2q_1}R^{m-2q_1}(Q^{\frac{1}{2}}z)^{2q_2+1}Z^{n-2q_2-1}\\ &-\sum_{q_1=0}^{[\frac{1}{2}(m-1)]}\sum_{q_2=0}^{[\frac{1}{2}n]}(-1)^{q_1+q_2}\\ &\times C^m{}_{2q_1+1}C^n{}_{2q_2}(P^{\frac{1}{2}}\xi)^{2q_1+1}R^{m-2q_1-1}(Q^{\frac{1}{2}}z)^{2q_2}Z^{n-2q_2}\end{aligned} \tag{11}$$

where the symbol $[A]$ means the biggest integer $\leq A$. Similarly

$$\begin{aligned}C_0=&(2\Phi_0)^{\frac{1}{2}m}(2V_0)^{\frac{1}{2}n}\cos[mP^{\frac{1}{2}}T_0]=(2\Phi_0)^{\frac{1}{2}m}(2V_0)^{\frac{1}{2}n}\\ &\times\cos[nQ^{\frac{1}{2}}T-mP^{\frac{1}{2}}(T-T_0)]=\sum_{q_1=0}^{[\frac{1}{2}m]}\sum_{q_2=0}^{[\frac{1}{2}n]}(-1)^{q_1+q_2}\\ &\times C^m{}_{2q_1}C^n{}_{2q_2}(P^{\frac{1}{2}}\xi)^{2q_1}R^{m-2q_1}(Q^{\frac{1}{2}}z)^{2q_2}Z^{n-2q_2}\\ &+\sum_{q_1=0}^{[\frac{1}{2}(m-1)]}\sum_{q_2=0}^{[\frac{1}{2}(n-1)]}(-1)^{q_1+q_2}C^m{}_{2q_1+1}C^n{}_{2q_2+1}(P^{\frac{1}{2}}\xi)^{2q_1+1}\\ &\times R^{m-2q_1-1}(Q^{\frac{1}{2}}z)^{2q_2+1}Z^{n-2q_2-1}.\end{aligned} \tag{12}$$

Hence S_0 and C_0 are two polynomial integrals of the system (10), but they are not independent. In fact

$$S_0^2+C_0^2=(2\Phi_0)^m(2V_0)^n. \tag{13}$$

These integrals have terms of degrees a, b, c, d, in ξ, R, z, Z, respectively, where $a+b=m$, $c+d=n$, and $a+c$ = odd in S_0, while $a+c$ = even in C_0. It is seen that n is always even, because $\gamma+\delta$ and Γ_j are always even (paper A, pp. 279–280).

It can be easily proven now that every polynomial integral of the system (10) is a polynomial of Φ_0, V_0, S_0 and C_0. In fact, such an integral can be written as a polynomial of the quantities (6). If we express the powers of sines and cosines by means of sines and cosines of multiples of $x=P^{\frac{1}{2}}(T-T_0)$ and $y=Q^{\frac{1}{2}}T$ we get terms of the form

$$q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}\,{}^{\cos}_{\sin}(\alpha-2p_1)x\cos(\beta-2p_2)x$$
$$\times{}^{\cos}_{\sin}(\gamma-2p_3)y\cos(\delta-2p_4)y, \quad (14)$$

where we have cosine or sine whenever α or γ is even or odd respectively; α, β, γ, δ are positive integers and p_1, p_2, p_3, p_4 are integers absolutely equal or smaller than α, β, γ, δ. The terms (14) are sums of terms of the form

$$q'(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}\,{}^{\cos}_{\sin}(A_ix\pm\Gamma_jy), \quad (15)$$

where

$$A_i=(\alpha-2p_1)\pm(\beta-2p_2),$$
$$\Gamma_j=(\gamma-2p_3)\pm(\delta-2p_4),$$

hence

$$A_ix\pm\Gamma_jy=(A_iP^{\frac{1}{2}}\pm\Gamma_jQ^{\frac{1}{2}})T-A_iP^{\frac{1}{2}}T_0.$$

As the integral is independent of T, all terms of the form (15) will be zero except those that have

$A_iP^{\frac{1}{2}}\pm\Gamma_jQ^{\frac{1}{2}}=0$, i.e., $A_i=\pm km$, $\Gamma_j=\pm kn$. Then $\alpha+\beta=km+2\nu_1^*$, $\gamma+\delta=kn+2\nu_2^*(k>0, \nu_1^*\geq 0, \nu_2^*\geq 0)$.

Therefore the integral is a sum of terms of the form

$$\pm q'(2\Phi_0)^{\frac{1}{2}(2\nu_1^*)}(2V_0)^{\frac{1}{2}(2\nu_2^*)}$$
$$\times\left[(2\Phi_0)^{\frac{1}{2}km}(2V_0)^{\frac{1}{2}kn}\,{}^{\cos}_{\sin}(kmP^{\frac{1}{2}}T_0)\right] \quad (16)$$

or $\pm q'(2\Phi_0)^{\nu_1^*}(2V_0)^{\nu_2^*}\times$(polynomial in S_0, C_0 of degree k), i.e., it is a polynomial in Φ_0, V_0, S_0, C_0.

If now in formula (7), (Φ_ν,F_1) includes a term $\xi^\alpha R^\beta z^\gamma Z^\delta$ with $\alpha+\beta=m+2\nu_1$, $\gamma+\delta=n+2\nu_2(\nu_1\geq 0, \nu_2\geq 0)$, then $\Phi_{\nu+1}$ contains secular terms of the form

$$\pm q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}\,{}^{\cos}_{\sin}(mP^{\frac{1}{2}}T_0)T;$$

in fact if $\alpha=$even, $\gamma=$even we have terms of the form (paper A, p. 279)

$$2q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}\int\cos A_ix\cos\Gamma_jydT$$
$$=q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}$$
$$\times\int[\cos(A_ix+\Gamma_jy)+\cos(A_ix-\Gamma_jy)]dT.$$

A_i, Γ_j take all the values from $(\alpha+\beta)$ to $-(\alpha+\beta)$ and from $(\gamma+\delta)$ to $-(\gamma+\delta)$, respectively. Therefore, there are values of A_i, Γ_j for which $A_i=\pm m$, $\Gamma_j=\pm n$, i.e., we have secular terms of the form

$$q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}\cos(mP^{\frac{1}{2}}T_0)T.$$

If $\alpha=$odd, $\gamma=$odd, we have terms of the form

$$2q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}\int\sin A_ix\sin\Gamma_jydT$$
$$=q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}$$
$$\times\int[\cos(A_ix-\Gamma_jy)-\cos(A_ix+\Gamma_jy)]dT,$$

hence we have secular terms of the form

$$\pm q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}\cos(mP^{\frac{1}{2}}T_0)T. \quad (17)$$

If $\alpha=$even, $\gamma=$odd we have

$$2q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}\int\cos A_ix\sin\Gamma_jydT$$
$$=q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}$$
$$\times\int[\sin(\Gamma_jy+A_ix)+\sin(\Gamma_jy-A_ix)]dT,$$

i.e., the secular terms are of the form

$$\pm q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}\sin(mP^{\frac{1}{2}}T_0)T, \quad (18)$$

and if $\alpha=$odd, $\gamma=$even we have

$$2q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}\int\sin A_ix\cos\Gamma_jydt$$
$$=q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}$$
$$\times\int[\sin(A_ix+\Gamma_jy)+\sin(A_ix-\Gamma_jy)]dT,$$

i.e., we have secular terms of the form (18).

Therefore if $\alpha+\gamma=$even we have secular terms of the form (17) and if $\alpha+\gamma=$odd we have secular terms of the form (18)

If $\alpha+\beta=2m+2\nu_1'$, $\gamma+\delta=2n+2\nu_2'(\nu_1'\geq 0, \nu_2'\geq 0)$ we also have secular terms of the form

$$\pm q(2\Phi)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)}\,{}^{\cos}_{\sin}(2mP^{\frac{1}{2}}T_0)T, \quad (18a)$$

etc.

If we calculate successively the terms Φ_0, Φ_1, $\Phi_2\cdots$ of the third integral, where Φ_ν is of degree $\nu+2$ in ξ, R, z, Z, then the first secular term is in Φ_ν with $\nu=m+n-2$.

If we replace Φ_0 by S_0 or C_0, where $a+b=m$, we get the integral

$$S=S_0+bS_1+b^2S_2+\cdots,\quad \text{or}\quad C=C_0+bC_1+b^2C_2+\cdots.$$

Then there will appear two secular terms in S_2 or C_2:

$$q_1(2\Phi_0)^{\frac{1}{2}(m+2)}(2V_0)^{\frac{1}{2}n}\genfrac{}{}{0pt}{}{\cos}{\sin}(mP^{\frac{1}{2}}T_0)T$$

$$+q_2(2\Phi_0)^{\frac{1}{2}m}(2V_0)^{\frac{1}{2}(n+2)}\genfrac{}{}{0pt}{}{\cos}{\sin}(mP^{\frac{1}{2}}T_0)T, \quad (19)$$

unless it happens that some coefficients q are zero.

In the terms (19) we have cosines for S_2 and sines for C_2 because in S_0 it is $a+c=$odd, hence in S_2, $S_4\cdots$: $a+c=$odd, but in S_1, $S_3\cdots$: $a+c=$even, therefore in (S_1,F_1): $\alpha+\gamma=$even (Paper A, p. 278). Similarly in C_2, $C_4\cdots$ we have $a+c=$even, and in C_1, $C_3\cdots$: $a+c=$odd, hence in (C_1,F_1): $\alpha+\gamma=$odd.

In order to eliminate the secular terms (19) we may take an integral of the form

$$\varphi=\varphi_0+b\varphi_1+b^2\varphi_2+\cdots, \quad (20)$$

where

$$\varphi_0=x_1(2\Phi_0)^2+x_2(2\Phi_0)(2V_0)+x_3(2V_0)^2, \quad (21)$$

and in the term φ_{m+n-4} either S_0, or C_0 is included; in fact φ_k is of degree $k+4$ in ξ, R, z, Z. Further, as φ_0, φ_2, $\cdots$ have $a+c=$even and φ_1, φ_3, $\cdots$ have $a+c=$odd, S_0 is used in φ_{m+n-4} if $m+n=$odd, and C_0 if $m+n=$even.

Then we define x_1, x_2, x_3 so that the secular terms in φ_{m+n-2} disappear. In fact the terms (21) give also two secular terms of the form (19) in φ_{m+n-2}, which should cancel the corresponding secular terms due to S_0 or C_0. Thus we have two linear equations for the three unknowns x_1, x_2, x_3.

If x_{10}, x_{20}, x_{30} is one solution of these equations then the other solutions are $x_1=x_{10}+x_1'$, $x_2=x_{20}+x_2'$, $x_3=x_{30}+x_3'$, where x_1', x_2', x_3' satisfy two linear homogeneous equations. Then $x_1'=\frac{1}{2}x_2'=x_3'$; in fact we see that if the quantity

$$x_1'(2F_0)^2=x_1'[(2\Phi_0)^2+2(2\Phi_0)(2V_0)+(2V_0)^2]$$

is included in φ_0, this gives terms up to Φ_2, but not higher order terms; its contribution to the secular terms is zero. This means that if we set $x_2'=2x_1'$, $x_3'=x_1'$, we have a solution of the above system of two homogeneous equations in x_1', x_2', x_3'. But this system has only one solution with respect to x_2'/x_1', x_3'/x_1'; i.e., this is the solution given above.

The secular terms of S_4 or C_4 are eliminated in a similar way by adding

$$[y_1(2\Phi_0)^3+y_2(2\Phi_0)^2(2V_0)+y_3(2\Phi_0)(2V_0)^2+y_4(2V_0)^3]$$

in φ_2. Then both this quantity and S_0 or C_0 give three secular terms in φ_{m+n} of the form

$$\pm q(2\Phi_0)^{\frac{1}{2}(m+4)}(2V_0)^{\frac{1}{2}m}\genfrac{}{}{0pt}{}{\cos}{\sin}(mP^{\frac{1}{2}}T_0)T,$$

$$\pm q(2\Phi_0)^{\frac{1}{2}(m+2)}(2V_0)^{\frac{1}{2}(n+2)}\genfrac{}{}{0pt}{}{\cos}{\sin}(mP^{\frac{1}{2}}T_0)T, \quad (22)$$

$$\pm q(2\Phi_0)^{\frac{1}{2}m}(2V_0)^{\frac{1}{2}(n+4)}\genfrac{}{}{0pt}{}{\cos}{\sin}(mP^{\frac{1}{2}}T_0)T.$$

If we require that these two sets of secular terms cancel each other, we have three equations for the four quantities y_1, y_2, y_3, y_4. It can be proved as above that if y_{10}, y_{20}, y_{30}, y_{40} is one solution of these equations, then the general solution is $y_1=y_{10}+y_1'$, $y_2=y_{20}+y_2'$, $y_3=y_{30}+y_3'$, $y_4=y_{40}+y_4'$, where $y_1'=\frac{1}{3}y_2'=\frac{1}{3}y_3'=y_4'$.

In general $\varphi_{m+n+2k-4}$ includes S_{2k} or C_{2k}, that has $k+1$ secular terms

$$\pm q(2\Phi_0)^{\frac{1}{2}(m+2k)}(2V_0)^{\frac{1}{2}n}\genfrac{}{}{0pt}{}{\cos}{\sin}(mP^{\frac{1}{2}}T_0)T,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \quad (23)$$

$$\pm(2\Phi_0)^{\frac{1}{2}m}(2V_0)^{\frac{1}{2}(n+2k)}\genfrac{}{}{0pt}{}{\cos}{\sin}(mP^{\frac{1}{2}}T_0)T$$

and these are eliminated by adding the quantity

$$\omega_1(2\Phi_0)^{k+1}+\omega_1(2\Phi_0)^k(2V_0)+\cdots+\omega_{k+2}(2V_0)^{k+1}$$

in φ_{2k-2}, which gives again $k+1$ secular terms in $\varphi_{m+n+2k-4}$; by setting the coefficients of the secular terms of $\varphi_{m+n+2k-4}$ equal to zero, we have $k+1$ equations for the $k+2$ unknowns ω_1, ω_2, $\cdots\omega_{k+2}$. If ω_{10}, ω_{20}, $\cdots\omega_{k+2,0}$ is one solution, then all other solutions are

$$\omega_1=\omega_{10}+\omega_1',\quad \omega_2=\omega_{20}+\omega_2',\cdots\omega_{k+2}=\omega_{k+2,0}+\omega_{k+2}',$$

where

$$\omega_1'=\frac{\omega_2'}{C^{k+1}{}_1}=\frac{\omega_3'}{C^{k+1}{}_2}=\cdots=\omega_{k+2}'.$$

In order to eliminate the secular terms of the form (18a) in $\Phi_{2m+2n+2k-4}$ we add a quantity

$$[\bar{\omega}_1(2\Phi_0)^{k+1}+\bar{\omega}_2(2\Phi_0)^k(2V_0)+\cdots+\bar{\omega}_{k+2}(2V_0)^{k+1}]\genfrac{}{}{0pt}{}{S_0}{C_0}$$

in $\Phi_{m+n+2k-2}$; again we take S_0 if $m+n=$odd, and C_0 is $m+n=$even. Then both this quantity and S_0 (or C_0)

give $k+1$ secular terms of the form (18a) in $\Phi_{2m+2n+2k-4}$:

$$\begin{aligned}
&\pm q(2\Phi_0)^{\frac{1}{2}(2m+2k)}(2V_0)^{\frac{1}{2}(2n)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(2mP^{\frac{1}{2}}T_0)T,\\
&\pm q(2\Phi_0)^{\frac{1}{2}(2m+2k-2)}(2V_0)^{\frac{1}{2}(2n+2)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(2mP^{\frac{1}{2}}T_0)T, \qquad (24)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\pm q(2\Phi_0)^{\frac{1}{2}(2m)}(2V_0)^{\frac{1}{2}(2n+2k)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(2mP^{\frac{1}{2}}T_0)T;
\end{aligned}$$

by requiring that these terms cancel each other we get $k+1$ equations for the $k+2$ unknowns $\bar{\omega}_1,\cdots, \bar{\omega}_{k+2}$. The general solution is again of the form

$$\bar{\omega}_1=\bar{\omega}_{10}+\bar{\omega}_1',\cdots \bar{\omega}_{k+2}=\bar{\omega}_{k+2,0}+\bar{\omega}_{k+2}'$$

where $\bar{\omega}_{10}, \cdots, \bar{\omega}_{k+2,0}$ is a special solution, and

$$\bar{\omega}_1'=\bar{\omega}_2'/C^{k+1}{}_1=\cdots=\bar{\omega}_{k+2}'.$$

In fact the integral

$$I_0'=\begin{smallmatrix}S_0\\ C_0\end{smallmatrix}(2F_0)^{k+1}$$

does not give any higher-order secular terms beyond those given by S_0 or C_0; in particular it does not give any secular terms of the form (24) in $\Phi_{2m+2n+2k-4}$. If I_1' is the next higher order term after I_0', then

$$(I_1',F_0)=-(I_0',F_1)=-\left(\begin{smallmatrix}S_0\\ C_0\end{smallmatrix},F_1\right)(2F_0)^{k+1} -2(k+1)(F_0,F_1)\begin{smallmatrix}S_0\\ C_0\end{smallmatrix}(2F_0)^k,$$

i.e.,

$$I_1'=\begin{smallmatrix}S_1\\ C_1\end{smallmatrix}(2F_0)^{k+1}+2(k+1)F_1\begin{smallmatrix}S_0\\ C_0\end{smallmatrix}(2F_0)^k,$$

because

$$-\int(F_0,F_1)dT=F_1.$$

Similarly

$$I_2'=\begin{smallmatrix}S_2\\ C_2\end{smallmatrix}(2F_0)^{k+1}+2(k+1)F_1\begin{smallmatrix}S_1\\ C_1\end{smallmatrix}(2F_0)^k,$$

because $F_2=0$, and finally

$$I_{m+n-2}'=\begin{smallmatrix}S_{m+n-2}\\ C_{m+n-2}\end{smallmatrix}(2F_0)^{k+1}+2(k+1)F_1\begin{smallmatrix}S_{m+n-3}\\ C_{m+n-3}\end{smallmatrix}(2F_0)^k$$

and this term is of degree $2m+2n+2k$ but has no secular terms of the form (24).

In the same way the secular terms of the most general form

$$\pm q(2\Phi_0)^{\frac{1}{2}(\alpha+\beta)}(2V_0)^{\frac{1}{2}(\gamma+\delta)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(jmP^{\frac{1}{2}}T_0)T$$

are eliminated by adding terms of the form

$$[\Omega_1(2\Phi_0)^{k+1}+\Omega_2(2\Phi_0)^k(2V_0)+\cdots+\Omega_{k+2}(2V_0)^{k+1}]\begin{smallmatrix}S_0{}^{j-1}\\ C_0{}^{j-1}\end{smallmatrix}$$

in $\varphi_{(j-1)(m+n)+2k-2}$.

Therefore we can always eliminate the secular terms in the third integral, written in the form (20).

Now we may distinguish a number of cases.

(α) If $m+n>4$, then Φ_0 depends only on Φ_0 and V_0; S_0 (or C_0) has as a coefficient some positive power of b, therefore if b is small the influence of this term is small. This means that if $m+n>4$ the form of the third integral depends mainly on Φ_0 and $V_0=F_0-\Phi_0$.

(β) If $m+n=4$, then Φ_0 includes both $x_1(2\Phi_0)^2+x_2(2\Phi_0)(2V_0)+x_3(2V_0)^2$ and C_0. In fact, then the secular term of Φ appears first in Φ_2. Then $m=n=2$, i.e., $P^{\frac{1}{2}}=Q^{\frac{1}{2}}$.

(γ) If $m+n<4$, we write φ in the form

$$\varphi=S_0+b[x_1(2\Phi_0)^2+x_2(2\Phi_0)(2V_0)+x_3(2V_0)^2] +b^2(\cdots)+\cdots. \qquad (25)$$

Then the secular term of Φ appears in Φ_1, i.e., $4Q-P=0$, or $P^{\frac{1}{2}}/Q^{\frac{1}{2}}=2/1$, $m=1$, $n=2$.

It is seen that $m+n$ cannot become smaller than 3, as Φ_1 already is of the third degree. In fact, n is always even, but neither m nor n becomes zero.

This distinction between the cases α, β, and γ shows that higher order commensurabilities (i.e., if $P^{\frac{1}{2}}/Q^{\frac{1}{2}}$ is not equal to 1/1 or 2/1) do not give any appreciable change of the over-all form of the orbits from the general irrational case, if b is small, while if $P^{\frac{1}{2}}/Q^{\frac{1}{2}}=1/1$ or 2/1 we should expect to find important differences in the general forms of the orbits. This is, in fact, what has been observed in the calculated orbits.

2. GENERAL RATIONAL CASE

Some orbits have been calculated in the case (I) $P^{\frac{1}{2}}/Q^{\frac{1}{2}}=4/1$ ($m+n=5$) and (II) $P^{\frac{1}{2}}/Q^{\frac{1}{2}}=2/3$ ($m+n=5$). In these cases Φ_0 gives secular terms in Φ_3.

Case (I) $P=16Q$

In this case $m=1$, $n=4$, $m+n=5$, therefore we must use the integral S_0, or

$$\bar{S}_0=-S_0/4Q^{\frac{1}{2}}=Q^2\xi z^4-6Q\xi z^2Z^2 +\xi Z^4-RzZ^3+QRz^3Z \qquad (26)$$

of the unperturbed system.

In the general form of the expansion of the third integral in powers of b, some third-order terms (Barbanis

1962) have zero divisors. If we multiply Φ_3 (the coefficient of b^3) by $(P-16Q)/180$ and then set $P=16Q$, we find again $\bar{S}_0$. This should be expected because the only integral of fifth degree of the system (10) in this case is S_0 or a multiple of it.

If then we calculate $\bar{S}_1$ by the formula

$$(\bar{S}_1,F_0)+(\bar{S}_0,F_1)=0, \tag{27}$$

we get

$$\begin{aligned}\bar{S}_1=&\frac{(R^2+P\xi^2)}{256Q^2}(-11Z^4-54Qz^2Z^2+29Q^2z^4)\\&+\frac{(R^2-P\xi^2)}{768Q^2}[8Z^4-144Qz^2Z^2+16Q^2z^4]\\&+\frac{2\xi RzZ}{3Q}(Z^2-2Qz^2)-Q\xi^2z^4-\frac{Qz^6}{6}\\&-\frac{(Z^2-Qz^2)}{24Q^2}[Z^4+4Qz^2Z^2+Q^2z^4],\end{aligned} \tag{28}$$

and in the calculation of S_2 there appear the secular terms

$$\frac{A}{192Q^3}-\frac{11B}{192Q^3}, \tag{29}$$

where

$$A=T(2\Phi_0)^{3/2}(2V_0)^2\cos P^{1/2}T_0$$

and

$$B=T(2\Phi_0)^{1/2}(2V_0)^3\cos P^{1/2}T_0.$$

On the other hand, the terms

$$x_1(2\Phi_0)^2+x_1(2\Phi_0)(2V_0)+x_3(2V_0)^2 \tag{30}$$

in φ_0 give

$$8x_1\Phi_0\Phi_1+4x_2(\Phi_0V_1+V_0\Phi_1)+8x_3V_0V_1 \tag{31}$$

in φ_1, where $V_1=F_1-\Phi_1$, and

$$\begin{aligned}4x_1(\Phi_1^2+2\Phi_0\Phi_2)+4x_2(\Phi_0V_2+\Phi_1V_1+\Phi_2V_0)\\+4x_3(V_1^2+2V_0V_2)\end{aligned} \tag{32}$$

in φ_2, where $V_2=-\Phi_2$ (because $V=F-\Phi$ and $F_2=0$).

In φ_3 the terms (30) give the secular terms

$$\frac{x_1A}{96Q^4}+\frac{x_2}{Q^4}\left(-\frac{A}{192}+\frac{B}{192}\right)-\frac{x_3B}{96Q^4}. \tag{33}$$

Adding the secular terms (29) and (33) we must have zero, i.e.,

$$\frac{A}{192Q^4}(Q+2x_1-x_2)+\frac{B}{192Q^4}(-11Q+x_2-2x_3)=0. \tag{34}$$

Hence one solution for x_1, x_2, x_3 is

$$x_{10}=-\tfrac{1}{2}Q,\quad x_{20}=0,\quad x_{30}=-(11/2)Q \tag{35}$$

and the other solutions are $x_1=x_{10}+x_1'$, $x_2=x_{20}+x_2'$, $x_3=x_{30}+x_3'$, where $x_1'=\frac{1}{2}x_2'=x_3'$, in agreement with the general theory.

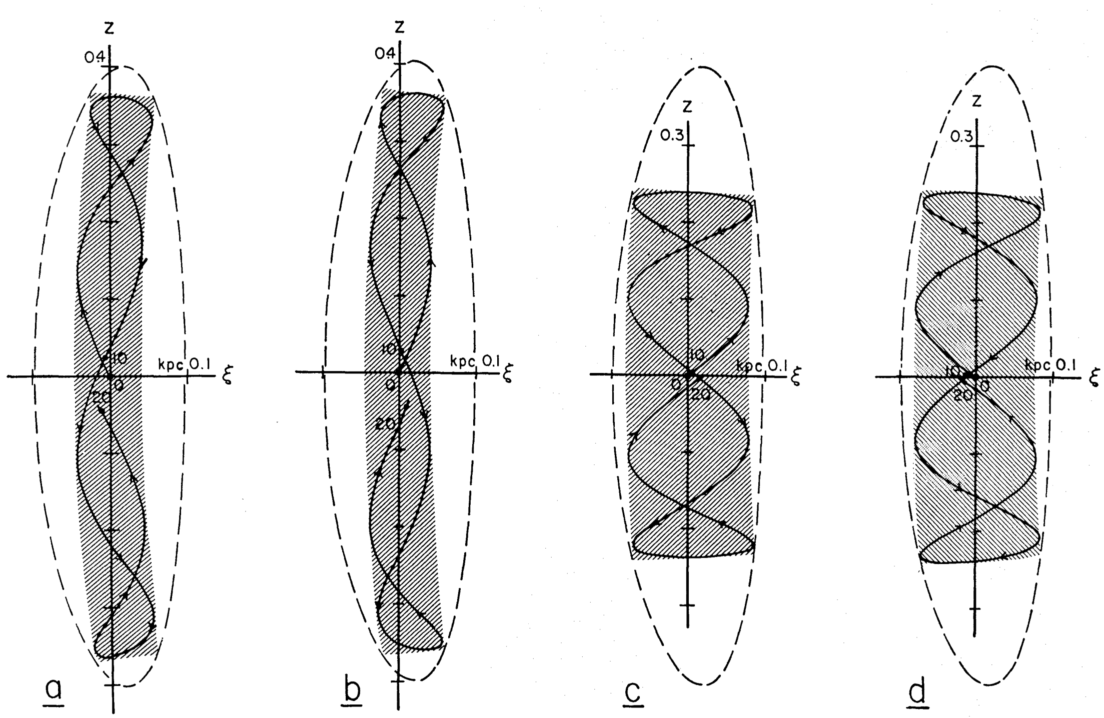

FIG. 1. Orbits in the case $P=16\,Q$.

TABLE I. Characteristics of the orbits represented in Fig. 1.

	ξ_0	z_0	R_0	Z_0	$2F$	$\xi_{\min}$ $(z=0)$	$\xi_{\max}$ $(z=0)$	ξ_1	z_1	ξ_2	z_2
a	0.	0.	−0.09850	0.07482	0.015300282	−0.079	0.074	−0.070	±0.242	0.085	±0.234
b	0.	0.	0.09850	0.07482	0.015300282	−0.079	0.075	−0.069	±0.242	0.085	±0.234
c	0.	0.	0.05120	0.11260	0.015300199	−0.044	0.036	−0.022	±0.365	0.057	±0.358
d	0.	0.	−0.05120	0.11260	0.015300199	−0.044	0.036	−0.022	±0.365	0.058	±0.358

Therefore the third integral is

$$\varphi=-\tfrac{1}{2}Q(2\Phi_0)^2-(11/2)Q(2V_0)^2 +b(-4Q\Phi_0\Phi_1-44QV_0V_1+\bar{S}_0) +b^2(-2Q(\Phi_1^2+2\Phi_0\Phi_2) -22Q(V_1^2+2V_0V_2)+\bar{S}_1)+\cdots \tag{36}$$

and no secular term appears now in φ_3.

Four orbits were calculated in the potential field (1) with $P=1.6$ $(10^7 \text{ yr})^{-2}$, $Q=0.1$ $(10^7 \text{ yr})^{-2}$, $b=0.2$ kpc^{-1} $(10^7 \text{ yr})^{-2}$; C is not needed in our calculation.

This field does not represent any real stellar system; in fact it should represent approximately a prolate stellar system. The results, however, are typical of any case of higher-order commensurability between $P^{\frac{1}{2}}$ and $Q^{\frac{1}{2}}$.

The four orbits are represented in Fig. 1 (a, b, c, d). Their characteristics are given in Table I: In this table ξ_0, z_0, R_0, Z_0 represent the initial coordinates and velocities; $\xi_{\min}(z=0)$, $\xi_{\max}(z=0)$ are the minimum and maximum values of ξ when the curve crosses the ξ axis; (ξ_1,z_1) and (ξ_2,z_2) are the angular points of the boundary; and F is the value of the energy. The units used are kpc for ξ and z, kpc $(10^7 \text{ yr})^{-1}$ for R and Z, and kpc^2 $(10^7 \text{ yr})^{-2}$ for F. We have

$$2F=R^2+16Q\xi^2+Z^2+Qz^2-2b\xi z^2=v_0^2$$

and the measure of the initial velocity is approximately the same: $v_0=0.123694$ kpc $(10^7 \text{ yr})^{-1}$.

The calculations of the orbits were made by means of Runge–Kutta method, in double precision. Whenever a step 0.02 (10^7 yr) was used in integration, the accuracy of the energy integral was within 1 unit in the ninth decimal place for 3×10^9 yr (15000 steps). Whenever the step was 0.05 (10^7 yr) the energy integral was conserved within two units in the eighth decimal place for 3×10^9 yr. A comparison of the individual values of coordinates and velocities for the same set of data and different steps showed that at least five decimals were the same for an interval of 2×10^9 yr.

When single precision with step 0.05 (10^7 yr) was used, the error in the energy constant was in one case one unit at the sixth decimal place after 1.1×10^9 yr. In another case, when a step 0.1 (10^7 yr) was used, one unit at the sixth decimal place was lost after 1.7×10^9 yr.

A comparison of the individual values of coordinates and velocities between single and double precision showed an agreement within two units in the fifth decimal place for 2×10^9 yr, when a step 0.1 (10^7 yr) was used, and within one unit in the fifth decimal place when a step 0.05 (10^7 yr) was used, although in this case the energy integral was conserved with less accuracy.

From Fig. 1 and Table I it is seen that the boundaries of the orbits are curvilinear "parallelograms" whose right and left boundaries are slightly concave to the right. The apices of these curvilinear parallelograms were found from an inspection of the tables of the values of the coordinates for every 0.05×10^7 yr, which are not given here. These apices lie, within an accuracy of ±0.001, on the curve of zero velocity, i.e., the curve found by setting $R=Z=0$ in Eq. (2). In our case the curve of zero velocity

$$P\xi^2+Qz^2-2b\xi z^2=v_0^2 \tag{37}$$

is represented by a dashed line in Fig. 1. Its deviation from the ellipse

$$P\xi^2+Qz^2=v_0^2$$

is measured by the maximum value of $2b\xi z^2/v_0^2$. If we write

$$\omega=2b\xi z^2,$$

then

$$\omega=\frac{2b\xi(v_0^2-P\xi^2)}{Q-2b\xi},$$

and this is maximum if

$$4bP\xi^3-3PQ\xi^2+v_0^2Q=0.$$

For $P=0.4$, $Q=0.1$, $b=0.2$, and $v_0^2=0.0153$, the maximum value of ω is $\omega_{\max}=0.0030$. Therefore the maximum value of ω/v_0^2 is 20%.

The curve of zero velocity intersects the ξ axis at the points $\xi=\pm v_0/P^{\frac{1}{2}}=\pm0.098$, and the z axis at the points $z=\pm v_0/Q^{\frac{1}{2}}=\pm0.391$. The maximum value of z is for ξ given by the equation

$$bP\xi^2-PQ\xi+bv_0^2=0.$$

The solution of this equation that is smaller than $v_0/p^{\frac{1}{2}}$ is

$$\xi=\frac{2bv_0^2/PQ}{\{1+[1-(4b^2v_0^2/PQ^2)]^{\frac{1}{2}}\}}=\frac{bv_0^2}{PQ}\left(1+\frac{b^2v_0^2}{PQ^2}+\cdots\right).$$

For $P=1.6$, $Q=0.1$, $b=0.2$, $v_0^2=0.0153$ we have $z_{\max}=0.399$ for $\xi=0.020$.

TABLE II. Approximate values of the third integral.

	φ_0 Initial	φ_0 Max	φ_0 Min	$\varphi_0+b\varphi_1$ Initial	$\varphi_0+b\varphi_1$ Max	$\varphi_0+b\varphi_1$ Min	$\varphi_0+b\varphi_1+b^2\varphi_2$ Initial	$\varphi_0+b\varphi_1+b^2\varphi_2$ Max	$\varphi_0+b\varphi_1+b^2\varphi_2$ Min	T
a	−0.22	−0.19	−0.24	−0.219	−0.217	−0.219	−0.2188	−0.2185	−0.2191	4×10^9 yr
b	−0.22	−0.20	−0.25	−0.219	−0.217	−0.219	−0.2188	−0.2186	−0.2191	5×10^9 yr
c	−0.89	−0.87	−0.99	−0.888	−0.884	−0.956	−0.8894	−0.8858	−0.8938	3×10^9 yr
d	−0.89	−0.80	−0.99	−0.888	−0.884	−0.957	−0.8894	−0.8846	−0.8931	5×10^9 yr

Equation (37) gives

$$z^2=(v_0^2-P\xi^2)/(Q-2b\xi).$$

In order that the curve of zero velocity should be closed, so that the orbits should not extend to infinity, we must have $Q-2b\xi>0$ whenever $v_0^2-P\xi^2\geq 0$. Hence

$$Q-2bv_0/P^{\frac{1}{2}}>0,$$

or

$$b<QP^{\frac{1}{2}}/2v_0.$$

In the above case $b<0.51$.

If $b=0$, the boundaries are exactly parallelograms with sides

$$\xi=\pm R_0/4Q^{\frac{1}{2}},\quad z=\pm Z_0/Q^{\frac{1}{2}},$$

i.e.

orbits a, b: $\xi=\pm0.078,\quad z=\pm0.237;$

orbits c, d: $\xi=\pm0.040,\quad z=\pm0.356.$

Therefore the deviations of the actual boundaries from the corresponding parallelograms of the unperturbed cases are rather small.

The third integral can be written

$$\varphi=\varphi_0+b\varphi_1+b^2\varphi_2+\cdots,\tag{38}$$

where

$$\varphi_0=-\tfrac{1}{2}Q(R^2+16Q\xi^2)^2-(11/2)Q(Z^2+Qz^2)^2,\tag{39}$$

$$\begin{aligned}\varphi_1=Q^2\xi z^4-6Q\xi z^2Z^2+\xi Z^4-RzZ^3+QRz^3Z\\ +\tfrac{1}{3}(R^2+16Q\xi^2)(7Q\xi z^2-\xi Z^2+RzZ)\\ -(11/3)(Z^2+Qz^2)(Q\xi z^2-\xi Z^2+RzZ)\end{aligned}\tag{40}$$

and

$$\begin{aligned}\varphi_2=&\frac{(R^2+16Q\xi^2)}{256Q^2}(-11Z^4-54Qz^2Z^2+29Q^2z^4)+\frac{(R^2-16Q\xi^2)}{96Q^2}(Z^4-18Qz^2Z^2+2Q^2z^4)+\frac{2\xi RzZ}{3Q}(Z^2-2Qz^2)\\ &-Q\xi^2z^4-\frac{Qz^6}{6}-\frac{(Z^2-Qz^2)}{24Q^2}(Z^4+4Qz^2Z^2+Q^2z^4)-\frac{1}{18Q}[(7Q\xi z^2-\xi Z^2+RzZ)^2+11(Q\xi z^2-\xi Z^2+RzZ)^2]\\ &-\tfrac{1}{6}[R^2+16Q\xi^2-11(Z^2+Qz^2)]\cdot\left(\frac{z^4}{2}+\frac{11R^2Z^2}{160Q^2}+\frac{9\xi^2Z^2}{10Q}-\frac{21R^2z^2}{160Q}+\frac{\xi^2z^2}{10}+\frac{2\xi RzZ}{5Q}\right).\end{aligned}\tag{41}$$

The values of φ_0, $b\varphi_1$, $\varphi_0+b\varphi_1$, $b^2\varphi_2$, $\varphi_0+b\varphi_1+b^2\varphi_2$ were printed every 10^8 yr. Table II gives the initial values of φ_0, $\varphi_0+b\varphi_1$, $\varphi_0+b\varphi_1+b^2\varphi_2$ and their maximum and minimum values for the time intervals given in the last column, in units of 10^{-4} kpc^2 (10^7 yr)$^{-4}$. It is seen that if we include terms up to φ_2 in the third integral its accuracy is fairly good of intervals of at least some billion years.

Further a great similarity between orbits a and b and between orbits c and d is to be remarked (Tables I and II and Fig. 1). This indicates that two orbits with velocities symmetric with respect to the axes ξ and z have the same general behavior (the same boundaries, etc.).

Case (II) $9P=4Q$

Five orbits were calculated in this case with $P=0.4$ (10^7 yr)$^{-2}$, $Q=0.9$ (10^7 yr)$^{-2}$ and

orbit a: $b=0.2$ kpc^{-1} (10^7 yr)$^{-2}$, $\xi_0=z_0=0$,

$R_0=0.11260$ kpc (10^7 yr)$^{-1}$, $Z_0=0.05120$ kpc (10^7 yr)$^{-1}$;

orbit b: $b=0.4$ kpc^{-1} (10^7 yr)$^{-2}$, $\xi_0=z_0=0$,

$R_0=0.11260$ kpc (10^7 yr)$^{-1}$, $Z_0=0.05120$ kpc (10^7 yr)$^{-1}$;

orbit c: $b=0.4$ kpc^{-1} (10^7 yr)$^{-2}$, $\xi_0=z_0=0$,

$R_0=0.05120$ kpc (10^7 yr)$^{-1}$, $Z_0=0.11260$ kpc (10^7 yr)$^{-1}$;

orbit d: $b=1$ kpc^{-1} (10^7 yr)$^{-2}$, $\xi_0=z_0=0$,

$R_0=0.11260$ kpc (10^7 yr)$^{-1}$, $Z_0=0.05120$ kpc (10^7 yr)$^{-1}$;

orbit e: $b=2$ kpc^{-1} (10^7 yr)$^{-2}$, $\xi_0=z_0=0$,

$R_0=0.11260$ kpc (10^7 yr)$^{-1}$, $Z_0=0.05120$ kpc (10^7 yr)$^{-1}$.

In all these cases the velocity is the same: $v_0=0.123694$ kpc (10^7 yr)$^{-1}$.

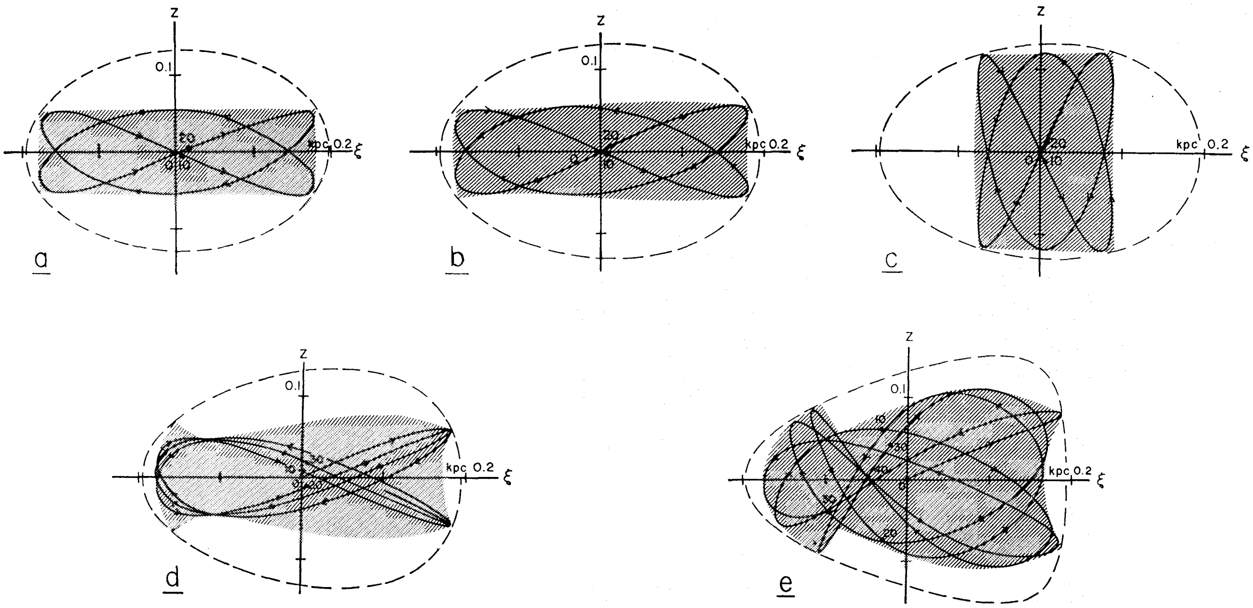

FIG. 2. Orbits in the case $9P=4Q$.

The calculations were made again by means of Runge–Kutta method in double precision. The energy integral $2F(=0.015300199)$ was conserved with the following accuracy for the corresponding time intervals:

orbit a:	3×10^{-9}	for	3×10^{9} yr;
orbit b:	16×10^{-9}	for	12×10^{9} yr;
orbit c:	36×10^{-9}	for	9×10^{9} yr;
orbit d:	36×10^{-9}	for	9×10^{9} yr;
orbit e:	9×10^{-9}	for	5×10^{9} yr.

The forms of the orbits are represented in Fig. 2(a–e).

Orbit a ($b=0.2$) remains quite near the periodic orbit of the unperturbed case ($b=0$) for a few billion years; but as the perturbation increases the orbits deviate more and more from the unperturbed case; orbit e does not resemble the periodic case at all.

In all cases, the orbits do not fill all the space inside the curve of zero velocity

$$0.4\xi^2+0.9z^2-2b\xi z^2=0.0153,$$

which is represented by a dashed line in Fig. 2. In order that this curve should be closed we must have $b<2.30$; case e is near the limiting case.

In cases a, b, c, the boundary of the orbit is approximately a parallelogram with sides

$$\xi=\pm R_0/P^{\frac{1}{2}},\quad z=\pm Z_0/Q^{\frac{1}{2}},$$

i.e.,

orbits a, b: $\xi=\pm0.178,\quad z=\pm0.054;$

orbit c: $\xi=\pm0.081,\quad z=\pm0.119.$

The actual boundaries are curved in the same way as in cases c and d but in a smaller degree. The characteristics of the orbits are given in Table III. The boundary is symmetric with respect to the ξ axis. Both left and right side of the boundary are concave to the right. The points where these sides intersect the ξ axis are given by $\xi_{\min}(z=0)$ and $\xi_{\max}(z=0)$. The upper and lower sides join the left and right sides at the usual angular points of the boundary (ξ_1,z_1) and (ξ_2,z_2). These points lie on the curve of zero velocity with an accuracy of about ±0.001 in both coordinates.

The points (ξ_3,z_3) and (ξ_4,z_4) are the minimum and maximum of the upper boundary, and correspondingly the maximum and minimum of the lower boundary. The accuracy of the values ξ_3 and ξ_4 is only about ±0.01.

The point (ξ_4,z_4) is always a regular point of the boundary. The same seems to be the case for the point

TABLE III. Characteristics of the orbits represented in Fig. 2.

	$\xi_{\min}$ ($z=0$)	$\xi_{\max}$ ($z=0$)	ξ_1	z_1	ξ_2	z_2	ξ_3	z_3	ξ_4	z_4
b	-0.179	$+0.177$	-0.172	±0.056	$+0.181$	±0.054	-0.10	±0.050	$+0.12$	±0.060
c	-0.079	$+0.089$	-0.073	±0.116	$+0.092$	±0.119	...	...	...	...
d	-0.178	$+0.172$	-0.158	±0.067	$+0.184$	±0.058	-0.11	±0.045	$+0.08$	±0.074
e	-0.174	$+0.166$	-0.104	±0.091	$+0.190$	±0.081	-0.08	±0.055	$+0.09$	±0.110

orbit a: $\xi_{\min}=-0.178$, $\xi_{\max}=+0.179$; $z_{\max}$ and $z_{\min}$ vary from ±0.052 to ±0.055.

(ξ_3,z_3) in cases b, c, d; in case e, however, this minimum seems to be an angular point of the boundary.

The curves of zero velocity intersect the ξ axis at the points $\xi=\pm v_0/P^{\frac{1}{2}}=\pm 0.196$, and the z axis at the points $z=\pm v_0/Q^{\frac{1}{2}}=\pm 0.130$. The maximum and minimum values of z are:

case a $(b=0.2)$:	$z=\pm 0.131$	for $\xi=0.009$;
cases b, c $(b=0.4)$:	$z=\pm 0.131$	for $\xi=0.017$;
case d $(b=1)$:	$z=\pm 0.134$	for $\xi=0.045$;
case e $(b=2)$:	$z=\pm 0.151$	for $\xi=0.114$.

The deviation of the curve of zero velocity from the ellipse

$$0.4\xi^2+0.9z^2=v_0^2$$

is again given by the maximum value of $2b\xi z^2/v_0^2$, which is: case a: 3.5%, cases b, c: 7.5%, case d: 23%, case e: 82%. It is seen that in the last case the deviation is considerable; however, even in this case the existence of an accurate boundary of the orbit indicates that a third integral exists, if not for all times, at least for times of the order of many billion years.

The exact form of the third integral was not calculated in this case.

3. THE CASE $P/Q=4/1$

In this case the third integral φ can be given by Eq. (25), where S_0 is replaced by

$$\bar{S}_0=S_0/2Q^{\frac{1}{2}}=\xi(Qz^2-Z^2)+RzZ. \quad (42)$$

This is the limit of $\Phi_1(4Q-P)/2$ when $P\rightarrow 4Q$.

If we calculate $\bar{S}_1$ by the formula (27) we find

$$\bar{S}_1=-\frac{1}{2}\left(\frac{z^4}{2}-\frac{\xi^2Z^2}{Q}-\frac{z^2R^2}{Q}-\xi^2z^2+\frac{2\xi RzZ}{Q}\right). \quad (43)$$

This is the limit of

$$\Phi_2\frac{(4Q-P)}{2}+\frac{(2P+Q)}{4PQ(P-Q)}(2\Phi_0)(2V_0)$$

when $P\rightarrow 4Q$.

Now $\bar{S}_2$ has the secular terms

$$-5A/64Q^3+5B/16Q^3, \quad (44)$$

where

$$A=T(2\Phi_0)^{3/2}(2V_0)\cos P^{1/2}T_0$$

and

$$B=T(2\Phi_0)^{1/2}(2V_0)^2\cos P^{1/2}T_0.$$

On the other hand, the term $x_1(2\Phi_0)^2+x_2(2\Phi_0)(2V_0)+x_3(2V_0)^2$ of φ_1 gives the secular terms

$$\frac{x_1A}{Q}+x_2\left(\frac{-A}{2Q}+\frac{B}{2Q}\right)-\frac{x_3B}{Q} \quad (45)$$

in φ_2.

Adding the terms (44) and (45) we must have zero. Hence

$$\frac{A}{64Q^3}(-5+64Q^2x_1-32Q^2x_2)+\frac{B}{16Q^3}(5+8Q^2x_2-16Q^2x_3)=0. \quad (46)$$

One solution of this equation is

$$x_{10}=-\frac{1}{64Q^2},\quad x_{20}=-\frac{3}{16Q^2},\quad x_{30}=\frac{7}{32Q^2},$$

and all other solutions are $x_1=x_{10}+x_1'$, $x_2=x_{20}+x_2'$, $x_3=x_{30}+x_3'$, where $x_1'=\frac{1}{2}x_2'=x_3'$.

Therefore the third integral becomes

$$\varphi=\xi(Qz^2-Z^2)+RzZ-\frac{b}{2}\left(\frac{z^4}{Q}-\frac{\xi^2Z^2}{Q}-\frac{z^2R^2}{Q}-\xi^2z^2+\frac{2\xi RzZ}{Q}+\frac{1}{32Q^2}[(R^2+P\xi^2)^2+12(R^2+P\xi^2)(Z^2+Qz^2)-14(Z^2+Qz^2)^2]\right)+\cdots. \quad (47)$$

This can be written

$$\varphi=Q\xi z^2-\xi Z^2+RzZ+b\left\{-\frac{\xi^4}{4}-\frac{\xi^2z^2}{4}-\frac{z^4}{32}-\frac{\xi^2R^2}{8Q}-\frac{\xi^2Z^2}{4Q}+\frac{5R^2z^2}{16Q}+\frac{7z^2Z^2}{16Q}-\frac{\xi RzZ}{Q}-\frac{R^4}{64Q^2}-\frac{3R^2Z^2}{16Q^2}+\frac{7Z^4}{32Q^2}\right\}+\cdots$$

$$=-\frac{b}{64Q^2}(R_0^4+12R_0^2Z_0^2-14Z_0^4)+\cdots, \quad (48)$$

where R_0, Z_0 are the components of the initial velocity v_0, when $\xi_0=z_0=0$.

The same result can be found also in the following way. The coefficient Φ_3 of b^3 (Barbanis 1962) can be written after some operations:

$$\Phi_3=\frac{Q}{(4Q-P)^2(P-Q)P}\bar{S}_0\left[\frac{96(2\Phi_0)}{(9P-4Q)}-\frac{504(2V_0)}{(P-16Q)}\right]+\text{terms with coefficient}\frac{1}{(4Q-P)}.$$

If we multiply the integral Φ by $\frac{1}{2}(4Q-P)$ the only critical terms in $\frac{1}{2}[\phi_3(4Q-P)]$ are

$$\frac{Q\bar{S}_0}{2(4Q-P)(P-Q)P}\left[\frac{96(2\Phi_0)}{9P-4Q}-\frac{504(2V_0)}{P-16Q}\right]. \quad (49)$$

But if we add

$$\frac{Q}{(P-Q)P}\left[\frac{y_1(2\Phi_0)^2}{9P-4Q}+\frac{y_2(2V_0)^2}{P-16Q}\right]$$

in $\frac{1}{2}[\Phi_2(4Q-P)]$, we get the further terms

$$\frac{4Q}{(P-Q)P}\left[\frac{y_1\Phi_1(2\Phi_0)}{9P-4Q}+\frac{y_2V_1(2V_0)}{P-16Q}\right]$$

in $\frac{1}{2}[\Phi_3(4Q-P)]$, and these give the further critical terms

$$\frac{4Q\Phi_1}{(P-Q)P}\left[\frac{y_1(2\Phi_0)}{9P-4Q}-\frac{y_2(2V_0)}{P-16Q}\right], \tag{50}$$

because $V_1=-\xi z^2-\Phi_1$, and Φ_1 is critical.

If we add (49) and (50) and set $\bar{S}_0/(4Q-P)\to\frac{1}{2}\phi_1$ when $P\to 4Q$, we must have zero. This is effected if $y_1=-6$ and $y_2=-63/2$.

Thus $\varphi_0=\bar{S}_0$ and φ_1 is the limit of

$$\Phi_2\frac{(4Q-P)}{2}-\frac{Q}{(P-Q)P}\left[\frac{6(2\Phi_0)^2}{9P-4Q}+\frac{63(2V_0)^2}{2(P-16Q)}\right].$$

By adding and subtracting $[(2P+Q)/4PQ(P-Q)]\times(2\Phi_0)(2V_0)$ we find for $P=4Q$

$$\varphi_1=\bar{S}_1-\frac{3}{16Q^2}(2\Phi_0)(2V_0)-\frac{(2\Phi_0)^2}{64Q^2}+\frac{7(2V_0)^2}{32Q^2},$$

i.e., the same result as above.

In the same way as above we can eliminate the secular terms from φ_3, etc. The general form of φ_2 is

$$\varphi_2=\lim_{P\to 4Q}\left[\frac{\Phi_3(4Q-P)}{2}+\frac{4Q}{(P-Q)P}\left\{-\frac{6\Phi_1(2\Phi_0)}{9P-4Q}-\frac{63V_1(2V_0)}{2(P-16Q)}\right\}\right]+f(\Phi_0,V_0,\bar{S}_0)=\frac{1}{24Q^2}\left\{\frac{13}{4}Q\xi^3z^2+\frac{75}{4}\xi^3Z^2-\frac{55}{4}\xi^2RzZ\right.$$
$$\left.-\frac{61}{16}\xi R^2z^2+\frac{85}{16Q}\xi R^2Z^2-\frac{85}{16Q}R^3zZ-\frac{35}{2}Q\xi z^4-21\xi z^2Z^2+\frac{25}{2Q}\xi Z^4-\frac{5}{2}Rz^3Z-\frac{25}{2Q}RzZ^3\right\}+f(\Phi_0,V_0,\bar{S}_0), \tag{51}$$

where f_0 is an arbitrary function of Φ_0, V_0 and $\bar{S}_0$.

The secular terms in φ_3 due to the explicitly written terms in (51) are

$$475B/6144Q^{9/2}$$

where $B=T(2\Phi_0)(2V_0)^2\sin 2P^{1/2}T_0$. If we add

$$f(\Phi_0,V_0,\bar{S}_0)=\bar{S}_0[\bar{\boldsymbol{x}}_1(2\Phi_0)+\bar{\boldsymbol{x}}_2(2V_0)] \tag{52}$$

in φ_2, we find also the secular terms $(\bar{\boldsymbol{x}}_1-\bar{\boldsymbol{x}}_2)B/8Q^{3/2}$. Hence if

$$\bar{x}_1-\bar{x}_2=-\frac{475}{768Q^3},$$

the secular terms are eliminated.

We take $\bar{x}_{10}=0$, $\bar{x}_{20}=475/768Q^3$ and any other solution is $\bar{x}_1=\bar{x}_{10}+\bar{x}_1'$, $\bar{x}_2=\bar{x}_{20}+\bar{x}_2'$, where $\bar{x}_1'=\bar{x}_2'$. If we set

$$f(\Phi_0,V_0,\bar{S}_0)=(475/768Q^3)\bar{S}_0(2V_0) \tag{53}$$

in formula (51), we find

$$\varphi_2=\frac{1}{24Q^2}\left\{\frac{13}{4}Q\xi^3z^2+\frac{75}{4}\xi^3Z^2-\frac{55}{4}\xi^2RzZ-\frac{61}{16}\xi R^2z^2\right.$$
$$+\frac{85}{16Q}\xi R^2Z^2-\frac{85}{16Q}R^3zZ-\frac{85}{32}Q\xi z^4-21\xi z^2Z^2$$
$$\left.-\frac{75}{32Q}\xi Z^4+\frac{395}{32}QRz^3Z+\frac{75}{32Q}RzZ^3\right\}. \tag{54}$$

The third integral permits the calculation of the boundary of the orbit on the ξ,z plane. The method used is the same as in paper A.

The energy integral (2) together with the third integral (48) define a surface (φ,F) in the four-dimensional phase space; the projection of this surface on the plane ξ,z defines the space occupied by the orbit on this plane. The equation of the boundary of the orbit is then found by eliminating R and Z between Eqs. (2), (48) and

$$J=\frac{\partial\varphi}{\partial R}Z-\frac{\partial\varphi}{\partial Z}R=0. \tag{55}$$

We will derive presently the equation of the boundary in first approximation with respect to b; in all the calculations below the terms of degree higher than the first in b are always omitted.

Equation (2) gives

$$R^2=-Z^2-4Q\xi^2-Qz^2+2b\xi z^2+v_0^2, \tag{56}$$

and if we set this value in Eq. (48) and write

$$\Omega=(1/32Q^2)(R_0^4+12R_0^2Z_0^2-14Z_0^4), \tag{57}$$

we find

$$RZ\left(z-\frac{b\xi z}{Q}\right)=\xi(Z^2-Qz^2)+b\left\{-\frac{25Z^4}{64Q^2}-\frac{\xi^2Z^2}{2Q}-\frac{9z^2Z^2}{32Q}\right.$$
$$\left.+\frac{5v_0^2Z^2}{32Q^2}+\frac{3\xi^2z^2}{2}+\frac{23z^4}{64}-\frac{11v_0^2z^2}{32Q}+\frac{v_0^4}{64Q^2}-\frac{\Omega}{2}\right\}. \tag{58}$$

On the other hand,

$$J=z(Z^2-R^2)+2\xi RZ+b\left\{RZ\left(\frac{\xi^2}{4Q}-\frac{z^2}{4Q}+\frac{5R^2}{16Q^2}-\frac{5Z^2}{4Q^2}\right)+\frac{\xi z}{Q}(R^2-Z^2)\right\}=0, \quad (59)$$

and if we set the value (56) in this equation we find in first approximation

$$RZ\left(2\xi+b\left\{-\frac{25Z^2}{16Q^2}-\frac{\xi^2}{Q}-\frac{9z^2}{16Q}+\frac{5v_0^2}{16Q^2}\right\}\right)=z(-2Z^2-4Q\xi^2-Qz^2+v_0^2)+b\xi z\left(\frac{2Z^2}{Q}+4\xi^2+3z^2-\frac{v_0^2}{Q}\right). \quad (60)$$

If we eliminate RZ between (58) and (60) we find

$$Z^4\left(-\frac{75b\xi}{32Q^2}\right)+Z^2\left(2(\xi^2+z^2)+\frac{b\xi}{Q}\left\{-2\xi^2-\frac{57z^2}{16}+\frac{5v_0^2}{8Q}\right\}\right)+2Q\xi^2z^2+Qz^4-v_0^2z^2+b\xi\left\{-4\xi^2z^2-\frac{87z^4}{32}+\frac{v_0^2z^2}{Q}+\frac{v_0^4}{32Q^2}-\Omega\right\}=0. \quad (61)$$

If we set $b=0$, we find the zero-order solution for Z^2:

$$Z_{(0)}^2=\frac{z^2(v_0^2-2Q\xi^2-Qz^2)}{2(\xi^2+z^2)} \quad (62)$$

and if we set

$$Z^2=Z_{(0)}^2+bX_b \quad (63)$$

in Eq. (61), we find

$$X_b=\frac{\xi}{2(\xi^2+z^2)}\left\{4\xi^2z^2+\frac{87z^4}{32}-\frac{v_0^2z^2}{Q}-\frac{v_0^4}{32Q^2}+\Omega+\frac{Z_{(0)}^2}{Q}\left(2\xi^2+\frac{57z^2}{16}-\frac{5v_0^2}{8Q}\right)+\frac{75Z_{(0)}^4}{32Q^2}\right\}. \quad (64)$$

Then the relation (56) gives

$$R^2=-Z_{(0)}^2-4Q\xi^2-Qz^2+v_0^2+b(2\xi z^2-X_b). \quad (65)$$

From Eq. (59) we find

$$R^2Z^2\left(4\xi^2+4b\xi\left\{\frac{\xi^2}{4Q}-\frac{z^2}{4Q}+\frac{5R^2}{16Q^2}-\frac{5Z^2}{4Q^2}\right\}\right)=z^2(R^2-Z^2)^2\left(1-\frac{2b\xi}{Q}\right), \quad (66)$$

and if we insert in this equation the values (63) and (65) we find the equation of the boundary of the orbit on the ξ, z plane:

$$-\frac{\xi^2z^2}{(\xi^2+z^2)}(Q^2z^4-4Qv_0^2\xi^2-2Qv_0^2z^2+v_0^4)+4b\xi^2X_b(4Q\xi^2+3Qz^2-v_0^2)+2b\xi\left\{\left[\frac{\xi^4(4Q\xi^2+3Qz^2-v_0^2)}{4(\xi^2+z^2)^2}-\frac{(4Q\xi^2+Qz^2-v_0^2)^2}{4}\right]\times\left[\frac{-25z^2(v_0^2-2Q\xi^2-Qz^2)}{16Q^2(\xi^2+z^2)}-\frac{2\xi^2}{Q}-\frac{9z^2}{8Q}+\frac{5v_0^2}{8Q^2}\right]-\frac{\xi^4z^2(4Q\xi^2+3Qz^2-v_0^2)^2}{Q(\xi^2+z^2)^2}-4Q\xi^2z^4\right\}. \quad (67)$$

In zero-order approximation this equation gives the following solutions:

$$\text{(a)}\quad z_{(0)}^2=0, \quad (68)$$

$$\text{(b)}\quad \xi_{(0)}^2=(v_0^2-Qz^2)^2/4Qv_0^2, \quad (69)$$

$$\text{(c)}\quad \xi_{(0)}=0. \quad (70)$$

(a) If $z_{(0)}^2=0$, we set

$$z^2=bY_b \quad (71)$$

in (67) and find

$$(v_0^2-4Q\xi^2)(v_0^2Y_b+4\xi^2X_b)=0, \quad (72)$$

where now

$$X_b=\frac{1}{2\xi}\left[-\frac{v_0^4}{32Q^2}+\Omega\right]. \quad (73)$$

Hence if $v_0^2-4Q\xi^2\neq0$,

$$Y_b=-\frac{2\xi}{v_0^2}\left[-\frac{v_0^4}{32Q^2}+\Omega\right]=\frac{5Z_0^2(3Z_0^2-2R_0^2)}{16Q^2v_0^2}\xi \quad (74)$$

and

$$z^2=\frac{5bZ_0^2(3Z_0^2-2R_0^2)}{16Q^2v_0^2}\xi. \quad (75)$$

This equation represents a parabola symmetric with respect to the axis ξ and going through the origin. If $b>0$ and $3Z_0^2-2R_0^2>0$ the parabola is to the right of the origin; if $3Z_0^2-2R_0^2<0$ the parabola is to the left of the origin; and if $3Z_0^2-2R_0^2=0$ the parabola vanishes. Thus one can explain the fact that the orbits a, A, b, B in Fig. 3 leave an open space to the right, while the orbits c, C, d, D leave an open space to the left. In case E, where $3Z_0^2=2R_0^2(=\frac{6}{5}v_0^2)$ it is seen that no open space is left either to the right or to the left. In this case

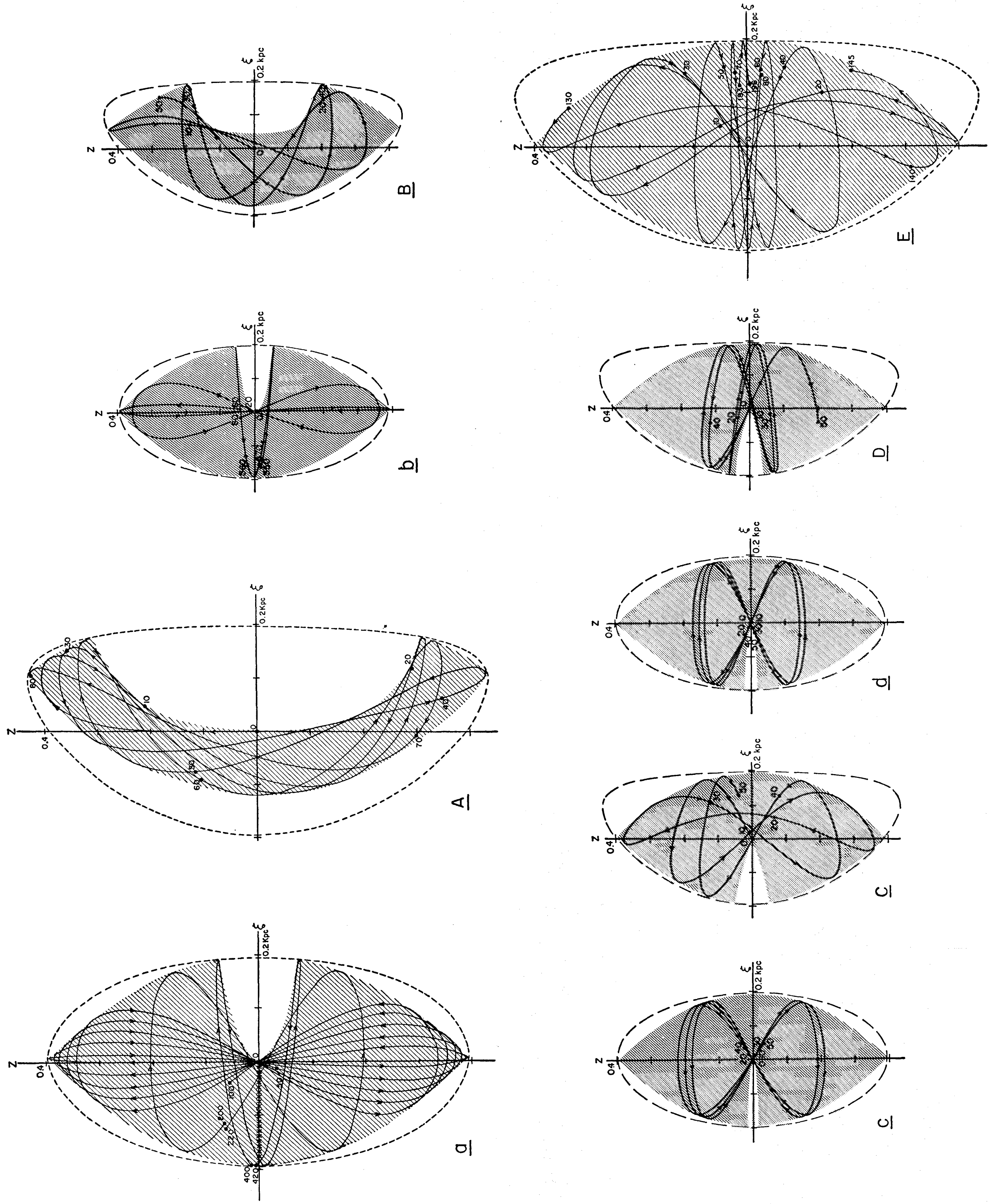

FIG. 3. Orbits in the case P=4Q.

the angle of the velocity vector with the ξ axis is $\tan^{-1}(2/3)=33°41'24''$.

If $b=0$ all orbits are periodic, of the form of a figure eight. If b is small ($b=0.02$, cases a, b, c, d) the moving point forms approximately figure eights, which, however, change gradually. If $R_0/Z_0<0$ the loops become more elongated, until they become very thin; then they become flatter again, and so on. If $R_0/Z_0>0$ the loops become first more and more flat; then they begin to expand, and so on. In case a the orbit needs 4.4×10^9 yr (22 loops) to go from the most elongated figure eight to the most flat one or vice versa. We may call this time "the long-range half-period," to distinguish it from the mean period of one loop which is given in Table V. The long-range half-period in case b is approximately 6×10^9 yr (30 loops) and in cases c and d 6.3×10^9 yr (32 loops).

If b is larger ($b=0.2$, cases A, B, C, D, E) the orbits are again composed of loops, but these are not similar to closed figure eights. In this case we can only loosely define a long-range half-period of the order of 10–70 $\times10^7$ yr, comprising from 0.5 to 3.5 loops. Case E has the greatest number of loops in one long-range half-period, between 3 to 3.5.

The characteristics of all the curves are given in Table IV, which is of the same form as Table I.

The boundaries of the calculated orbits are composed of four arcs, except in case E. Every two consecutive arcs intersect at a right angle on the curve of zero velocity.

The inner arc is approximately a parabola. A comparison of Tables IV and V shows that for $b=0.02$ the empirically found inner boundary coincides rather well with the parabola (75). But if $b=0.2$ the coincidence is not good enough. Evidently higher-order terms in b are in this case important. In cases A and B the opening to the right is larger than indicated by the parabola (75), while in cases C and D the opening to the left is smaller than the corresponding parabola. In case E the parabola does not become a double line $z^2=0$, but vanishes completely.

Orbits with the same absolute values of the components of the initial velocity have the same boundaries. In fact, the equation of the boundary (67) depends only on the squares of R_0 and Z_0 and not on their sign. Therefore, although initially two orbits with velocities symmetric with respect to the z axis behave differently, finally the two orbits fill the same space and look very similar to each other. This is quite clear in cases c and d, and somewhat less clear in cases C, D, because the angular points of the boundary cannot be defined quite accurately in these cases.

The maximum value of $Z_0^2(3Z_0^2-2R_0^2)$ for a given value of the total velocity $v_0=(R_0^2+Z_0^2)^{\frac{1}{2}}$, is realized for $Z_0^2=v_0^2$, $R_0=0$; then

$$z^2=(15bv_0^2/16Q^2)\xi$$

and the opening to the right is maximum (cases a, A).

The minimum value of $Z_0^2(3Z_0^2-2R_0^2)$ is realized for $Z_0^2=v_0^2/5$, $R_0^2=4v_0^2/5$. Then $|Z_0/R_0|=\frac{1}{2}$, i.e., the angle between the velocity vector and the ξ axis is $\pm26°33'54''$. Then the parabola is

$$z^2=-(bv_0^2/16Q^2)\xi$$

and the opening to the left is maximum (cases c, C, d, D).

The opening of the parabola is roughly proportional to b, for the same values of $|R_0|$ and $|Z_0|$. This explains the difference between pairs of orbits (a,A), (b,B), etc. Table V gives the coefficients of ξ in formula (75) as well as the value of z given by this formula for $\xi=\pm v_0/2Q^{\frac{1}{2}}=\pm0.196$.

(b) The zero-order solution (69) gives two branches

$$\xi_1=\frac{(v_0^2-Qz^2)}{2v_0Q^{\frac{1}{2}}},\quad \xi_2=-\frac{(v_0^2-Qz^2)}{2v_0Q^{\frac{1}{2}}},\tag{76}$$

symmetric with respect to the ξ axis. These branches meet at the points $z=\pm v_0/Q^{\frac{1}{2}}$; they intersect at right angles, because

$$\frac{d\xi_1}{dz}=\frac{-Q^{\frac{1}{2}}z}{v_0},\quad \frac{d\xi_2}{dz}=\frac{Q^{\frac{1}{2}}z}{v_0}\tag{77}$$

Table IV. Characteristics of the orbits represented in Fig. 3.

	ξ_0	z_0	R_0	Z_0	$2F$	ξ_{min} ($z=0$)	ξ_{max} ($z=0$)	ξ_1	z_1	ξ_2	z_2
a	0.	0.	0.	0.12370	0.015301690	−0.192	0.	+0.008	±0.391	+0.192	±0.077
A	0.	0.	0.	0.12370	0.015301690	−0.118	0.	+0.110	±0.432	+0.177	±0.309
b	0.	0.	0.05532	0.11064	0.015301511	−0.194	0.	+0.005	±0.391	+0.194	±0.055
B	0.	0.	0.05532	0.11064	0.015301511	−0.168	+0.043	+0.056	±0.425	+0.189	±0.202
c	0.	0.	0.11064	0.05532	0.015301511	0.	+0.195	−0.195	±0.020	0.	±0.391
C	0.	0.	0.11064	0.05532	0.015301511	∼0.	+0.193	−0.191	±0.057	∼0.	±0.391
d	0.	0.	−0.11064	0.05532	0.015301511	0.	+0.195	−0.195	±0.020	0.	±0.391
D	0.	0.	−0.11064	0.05532	0.015301511	∼0.	+0.193	−0.193	±0.043	+0.007	±0.396
E	0.	0.	0.09581	0.07823	0.015299489	−0.195	+0.195		$\xi=+0.010$	$z=\pm0.398$	

and for $z=\pm v_0/Q^{\frac{1}{2}}$,

$$\left(\frac{d\xi_1}{dz}\right)\left(\frac{d\xi_2}{dz}\right)=-1. \tag{78}$$

In first-order approximation we set

$$\xi^{(i)}=\xi_i+b\Xi_i \tag{79}$$

in Eq. (67) and find after a number of operations:

$$\Xi_1=\Xi_2\equiv\frac{1}{128Q^2v_0^4}\times\left\{-84Q^2v_0^2z^4+Q[36R_0^4+62R_0^2Z_0^2+51Z_0^4]z^2 +5v_0^2Z_0^2(3Z_0^2-2R_0^2)+\frac{100Q^3v_0^2z^6}{v_0^2+Qz^2}\right\}. \tag{80}$$

In the special case when $3Z_0^2-2R_0^2=0$, we have

$$\Xi_i=\frac{z^2}{32Qv_0^2}\left[-21Qz^2+9v_0^2+\frac{25Qz^4}{v_0^2+Qz^2}\right]. \tag{81}$$

For $z=0$, it is then $\Xi_i=0$, i.e.,

$$\xi_i=\pm v_0/2Q^{\frac{1}{2}}. \tag{82}$$

The curve of zero velocity

$$4Q\xi^2+Qz^2-2b\xi z^2=v_0^2 \tag{83}$$

intersects the ξ axis at the points $\xi=\pm 0.196$ and the z axis at the points $z=\pm 0.391$. The maximum value of z is $z_{\max}=0.391$ for $\xi=0.008$ if $b=0.02$, and $z_{\max}=0.434$ for $\xi=0.094$ if $b=0.2$. This curve is represented by a dashed line in Fig. 3.

In case E, $(3Z_0^2-2R_0^2=0)$, the curve of zero velocity and the boundary of the orbit are tangent at the points $\xi_i=\pm v_0/2Q^{\frac{1}{2}}$, $z=0$.

In the general case

$$\xi^{(i)}=\pm\frac{(v_0^2-Qz^2)}{2v_0Q^{\frac{1}{2}}}+b\Xi_i. \tag{84}$$

For $z=0$, we have

$$\xi^{(i)}=\pm\frac{v_0}{2Q^{\frac{1}{2}}}+\frac{5bZ_0^2(3Z_0^2-2R_0^2)}{128Q^2v_0^2}. \tag{85}$$

Table V gives the values of $b\Xi_i$ and $\xi_1+b\Xi_1$, $\xi_2+b\Xi_2$ for $z=0$, on the side where the boundary actually intersects the ξ axis. These values coincide very well with the values $\xi_{\min}(z=0)$ or $\xi_{\max}(z=0)$ in Table IV for $b=0.02$, but less satisfactorily for $b=0.2$.

For $z=\pm v_0/Q^{\frac{1}{2}}$, we have $\xi_i=0$ and

$$\xi=b(R_0^2-4Z_0^2)^2/64Q^2v_0^2. \tag{85a}$$

The last value is always positive (or zero) if $b>0$; this indicates that the point where the two branches of the envelope meet is to the right of the axis z.

This point lies on the curve of zero velocity (83) to an accuracy of first order in b. The accurate boundaries of the calculated orbits also show that the points of intersection of the different branches of the boundaries lie on the curve of zero velocity within an accuracy of ± 0.001. The value (85a) however, is not accurate, because if $\xi_i=0$ the approximation formula (79) is no more convenient.

The points of intersection of the curves (79) and (75) are found by setting the value (75) in ξ_i and $z^2=0$ in Ξ_i, and solving for ξ. Then

$$\xi=(\pm)\frac{v_0}{2Q^{\frac{1}{2}}}-\frac{5bZ_0^2(3Z_0^2-2R_0^2)}{128Q^2v_0^2}, \tag{86}$$

where we have the plus sign if $3Z_0^2-2R_0^2>0$ and the minus sign if $3Z_0^2-2R_0^2<0$.

If we set the value (86) in Eq. (83) we find

$$z^2=(\pm)\frac{5bZ_0^2(3Z_0^2-2R_0^2)}{32Q^{5/2}v_0}. \tag{87}$$

This value is the same as (75) for $\xi=(\pm)(v_0/2Q^{\frac{1}{2}})$. Therefore the intersection lies on the curve of zero velocity to this approximation.

TABLE V. Data for the orbits represented in Fig. 3.

	$3Z_0^2-2R_0^2$	Coeff. of ξ in (75)	z for $\xi=\pm 0.196$	$b\Xi_1=b\Xi_2$ $(z=0)$	$\xi_1+b\Xi_1$ $(z=0)$	$\xi_2+b\Xi_2$ $(z=0)$	Mean period of a loop
a	+0.0459	+0.0287	±0.075	+0.0036	−0.192	...	20.0×10^7 yr
A	+0.0459	+0.287	±0.237	+0.036	−0.160	...	24.4×10^7 yr
b	+0.0306	+0.0153	±0.055	+0.0019	−0.195	...	20.0×10^7 yr
B	+0.0306	+0.153	±0.173	+0.019	−0.176	...	23.0×10^7 yr
c	−0.0153	−0.0019	±0.019	−0.0002	...	+0.195	19.8×10^7 yr
C	−0.0153	−0.019	±0.061	−0.002	...	+0.193	19.1×10^7 yr
d	−0.0153	−0.0019	±0.019	−0.0002	...	+0.195	19.8×10^7 yr
D	−0.0153	−0.019	±0.061	−0.002	...	+0.193	19.1×10^7 yr
E	0.	0.	0.	0.	0.	0.	20.2×10^7 yr

TABLE VI. Approximate values of the third integral.

	$\Delta(2F)$	φ_0 Initial	Max	Min	$\varphi_0+b\varphi_1$ Initial	Max	Min	$\varphi_0+b\varphi_1+b^2\varphi_2$ Initial	Max	Min	T
a	2×10^{-9}	0.	+1.2	0.	+1.024	+1.028	+1.024	+1.024	+1.027	+1.024	5×10^9 yr
A	1×10^{-9}	0.	+19.6	0.	+10.2	+16.9	+10.2	+10.2	+15.8	+10.2	6×10^9 yr
b	1×10^{-9}	0.	+0.6	−0.5	+0.5122	+0.5145	+0.5110	0.5122	+0.5140	+0.5107	6×10^9 yr
B	1×10^{-9}	0.	+9.3	−5.4	+5.12	+8.19	+3.29	+5.12	+6.84	+3.05	6×10^9 yr
c	4×10^{-9}	0.	0.	−1.1	−0.146	−0.145	−0.153	−0.146	−0.144	−0.149	10×10^9 yr
C	4×10^{-9}	0.	0.	−10.0	−1.46	+1.02	−3.13	−1.46	−0.07	−3.40	10×10^9 yr
d	2×10^{-9}	0.	0.	−1.1	−0.146	−0.145	−0.153	−0.146	−0.144	−0.149	11×10^9 yr
D	1×10^{-9}	0.	+0.7	−9.3	−1.46	+1.38	−2.55	−1.46	−0.01	−2.81	5×10^9 yr
E	2×10^{-9}	0.	+0.6	−8.8	−0.73	+1.87	−2.23	−0.73	+0.74	−1.92	6×10^7 yr

At the intersection the tangent of the curve (75) is given by

$$\frac{d\xi}{dz}=\frac{32Q^2v_0^2z}{5bZ_0^2(3Z_0^2-2R_0^2)} \tag{88}$$

and the tangent of the curve (79) by

$$\frac{d\xi'}{dz}=(\pm)\frac{Q^{\frac{1}{2}}z}{v_0}+b[\cdots]. \tag{89}$$

Thus

$$\left(\frac{d\xi}{dz}\right)\left(\frac{d\xi'}{dz}\right)=(\pm)\frac{32Q^{5/2}v_0z^2}{5bZ_0^2(3Z_0^2-2R_0^2)}+\cdots \tag{90}$$

and if we set the value (87) of z^2 we get

$$\left(\frac{d\xi}{dz}\right)\left(\frac{d\xi'}{dz}\right)=-1 \tag{91}$$

to a zero-order approximation; we cannot go to a higher approximation because we have not calculated the higher-order terms in the expansion of z^2.

(c) The zero-order solution (70) can be used in the same way as solution (68) to give a corresponding first-order solution, if we set

$$\xi=b\psi.$$

Then Eq. (67) gives

$$b^2\psi^2[\cdots]+2b^2\psi[\cdots]+\cdots=0, \tag{92}$$

But now, in order to find ψ we need the terms with b^2 in the equation of the boundary, and these have not been calculated in this paper; therefore we leave this point open. It is probable, however, that some characteristics of the orbits, namely the fact that the boundary does not go through 0 in case B and similar cases, could be explained by the existence of another branch of the boundary, that corresponds to the zero-order solution $\xi_{(0)}=0$.

We finish with a discussion of the accuracy of the third integral in the calculated cases. Table VI gives first the accuracy with which the energy integral was conserved for the time interval indicated in the last column. Part of the inaccuracy is due to the fact that the calculation was eventually stopped and resumed later while only eight significant figures of the coordinates and velocities were available. The actual accuracy of the double precision calculations is about 1×10^{-9} kpc^2 (10^7 yr)$^{-2}$ in 6×10^9 yr.

The third integral is given by Eq. (20), where $\varphi_0+b\varphi_1$ is given by Eq. (48) and φ_2 is given by Eq. (54).

The values of φ_0, $\varphi_0+b\varphi_1$, and $\varphi_0+b\varphi_1+b^2\varphi_2$ were printed every 10^8 yr. Table VI gives the initial values, as well as the maximum and minimum values of the above quantities in units of 0.0001 kpc^3 (10^7 yr)$^{-2}$.

It is seen that whenever $b=0.02$ (cases a, b, c, d) the third integral is rather well conserved in second-order approximation, and even in first-order approximation, for time intervals of the order of 5–10×10^9 yr at least.

The changes of the third integral to this approximation seem to be almost periodic, therefore it is not probable that much larger changes will appear later.

If $b=0.2$ the third integral is not conserved very well, although its values are of the same order of magnitude. This is probably because higher-order terms are important in this case.

A great similarity between the maximum and minimum values of the third integral in cases c and d and a smaller similarity between the corresponding values in cases C and D is to be remarked. This verifies the fact that orbits with initial velocities symmetric with respect to the axes ξ and z behave similarly in the long run.

Case $P=Q$ and the cases of small divisors, when m^2P-n^2Q is near zero, but not equal to zero, will be considered in Paper II.

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