Tracer Age Symmetry in Advective-Diffusive Flows

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Abstract The "age" of a trace constituent is a common diagnostic of its transport in a geophysical flow. *Deleersnijder et al.* [2001a] and *Beckers et al.* [2001] analyzed tracers released from point sources in unbounded advective-diffusive flows with uniform coefficients and noted a surprising feature: the "mean tracer age" (the averaged elapsed time since tracer was injected) is symmetric about the source, despite the directionality of the flow. Although the majority of tracer is swept downstream, the small fraction that diffuses upstream does so at the same average rate. We explore this symmetry physically by examining the random walk trajectories that underlie the advective-diffusive description of transport. Using physical arguments we show that symmetry in the tracer age field is a natural consequence of symmetry in the velocity and diffusivity fields.

Keywords: tracers, age, ocean transport, advection-diffusion

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1 Introduction

"Age" is a diagnostic timescale of transport used in geophysical systems as diverse as the ocean, stratosphere, and ground water [e.g., the review of *Waugh and Hall*, 2002]. Common to these systems is the advective-diffusive nature of the transport. Superposed on bulk motions are mixing processes that necessitate a statistical treatment of transport. Not surprisingly, given the widely varying contexts, precise definitions vary. In one usage, age is a property of the tracer itself, and is defined as the elapsed time since tracer was injected from a source [e.g., *Deleersnijder et al.,* 2001b]. We refer to this age as "tracer age" to distinguish it from the "transit time" of an irreducible fluid element traveling to the interior from a specified boundary region, a property of the underlying fluid that *Deleersnijder et al.* [2001b] has called "water age" and is often simply called "age." As a result of mixing the tracer content of a macroscopic fluid parcel is comprised of a range of tracer ages, just as the parcel's irreducible fluid elements exhibit a range of transit times.

This note is largely motivated by recent work of *Deleersnijder et al.* [2001a] and *Beckers et al.* [2001], who analyzed tracer age in idealized unbounded advective-diffusive flows with uniform velocity and diffusivity. These authors noted the surprising result that the mean tracer age is symmetric about a point source, despite the strong asymmetry in the tracer concentration due to the directionality of the flow. *Beckers et al.* [2001] also noted the symmetry in numerical models of the North Sea. The symmetry is counterintuitive because one expects that the rate of tracer motion should reflect the relative difficulty of moving against the flow. Here, we present a physical explanation for this symmetry by analyzing the random walks that underlie advective-diffusive motion. We also compare and contrast different definitions of "age" in regards to this symmetry.

2 Tracer Age

2.1 Definitions

The concept of "age" as a diagnostic of transport is widely used in geophysics [*Waugh and Hall*, 2002]. However, definitions vary. The most direct definition in terms of tracer is what we call here "tracer age." Tracer age is defined to be a property of the tracer itself, rather than a property of the underlying fluid. (By contrast, *Hall and Plumb* [1994] and *Haine and Hall* [2002] define related diagnostics as properties of the fluid, independent of particular tracers.) Each tracer particle (e.g., molecule) is imagined to have a "clock" that is turned on at the time the tracer is injected into the fluid. A macroscopic fluid parcel contains many particles with a distribution of clock times, or "tracer ages." The tracer age distribution can be characterized by its temporal moments: the zeroeth moment (proportional to the tracer mole fraction), the first moment (the "mean tracer age"), and higher moments (e.g., the variance of tracer age). A tracer age distribution can be defined at each point in the domain. The distribution depends both on the underlying fluid flow and on the sources and sinks of the tracer.

As a concrete example, consider an inert passive tracer injected into an advective-diffusive flow by a point source at $\mathbf{r}'$ with time-dependent source strength $S(\mathbf{r}', t)$ having units of tracer mass per time. At position $\mathbf{r}$ and time t the tracer mole fraction $q(\mathbf{r}, t)$ is comprised of tracer injected at a range of past times. The contribution from the past time interval $t' + \delta t'$ is $S(\mathbf{r}', t')G(\mathbf{r}, t|\mathbf{r}', t')\delta t'$, where $G(\mathbf{r}, t|\mathbf{r}', t')$ is the response at $(\mathbf{r}, t)$ to an injection at $\mathbf{r}'$ at a single past time t'; that is, $S(\mathbf{r}', t) = \rho^{-1}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$, where ρ is the fluid density. G is the Green's function that carries tracer from $(\mathbf{r}', t)$ to $(\mathbf{r}, t)$. The concentration q is the sum of these contributions:

$$q(\mathbf{r}, t) = \int_{-\infty}^{t} dt'\, S(\mathbf{r}', t') G(\mathbf{r}, t|\mathbf{r}', t') \tag{1}$$

$$= \int_{0}^{\infty} d\xi S(\mathbf{r}', t - \xi) G(\mathbf{r}, t|\mathbf{r}', t - \xi) \ , \tag{2}$$

where $\xi \equiv t - t'$ is the elapsed time since a contribution was injected into the flow, the "tracer age" of the contribution. The fraction of $q(\mathbf{r}, t)$ with tracer age in the interval ξ to $\xi + \delta\xi$ is

$$Z(\mathbf{r}, t|\mathbf{r}', t - \xi)\delta\xi = \frac{S(\mathbf{r}', t - \xi) G(\mathbf{r}, t|\mathbf{r}', t - \xi)}{q(\mathbf{r}, t)}\,\delta\xi \ , \tag{3}$$

thereby defining $Z(\mathbf{r}, t|\mathbf{r}', t - \xi)$, the "tracer age distribution." This formulation is discussed in detail by *Holzer and Hall* [2000] and for linear tracers is related to the "concentration distribution function," $c(\mathbf{r}, t|\mathbf{r}', t - \xi)$, of *Deleersnijder et al.* [2001b] simply by $Z = c/q$. By construction, $\int_0^\infty Z d\xi = 1$. The "mean tracer age" is the first moment of this distribution: $A(\mathbf{r}, t, \mathbf{r}') = \int_0^\infty \xi Z d\xi$.

We emphasize that Z and A are in general distinct from the "age spectrum," $\mathcal{G}$, and "mean age," Γ, of *Hall and Plumb* [1994] (also called the "transit time distribution" and "mean transit time," respectively). $\mathcal{G}$ and its first moment Γ are descriptors of the underlying fluid transport and are independent of the properties of any particular tracer, while Z depends explicitly on the tracer source, as seen in (3). $\mathcal{G}$ is the distribution of transit times since a fluid parcel made last contact with some specified region, Ω, and Γ is the mean of the distribution. The relationship of Z and A to underlying timescales of the flow and the conditions under which $Z \approx \mathcal{G}$ are laid out by *Holzer and Hall* [2000]. For example, *Holzer and Hall* [2000] show that for a constant uniform source on a region Ω in a bounded domain $\Gamma(\mathbf{r}) \approx 2(A(\mathbf{r}) - A(\Omega))$. In more general cases there is no such simple relationship. Further contrast and comparison of tracer and fluid age is made in the summary and discussion section.

2.2 Examples

In order to gain insight to tracer age and illustrate tracer age symmetry in a simple context we consider the following model: a passive inert tracer injected into an unbounded advective-diffusive flow with uniform and constant velocity $\mathbf{v}$ and diffusivity κ. Tracer age in these simple models is also analyzed by *Deleersnijder et al.* [2001b] and *Beckers et al.* [2001]. The point source is taken to be the origin, and $t' = 0$. The Green's function is

$$G(\mathbf{r}, t) = \left(\frac{1}{\sqrt{4\pi\kappa t}}\right)^n \exp\left(-\frac{|\mathbf{r} - \mathbf{v}t|^2}{4\kappa t}\right) \ , \tag{4}$$

where n is the dimensionality of the flow, and it is assumed that $\rho = 1$, giving G units of L^{-n}. We consider the cases $n = 1$ and $n = 3$ because they permit easy analytic solution.

If the source is $S(t) = s_0$ for $t \geq 0$ and $S(t) = 0$ for $t < 0$, then the steady-state response of the tracer is $q(\mathbf{r}) = \int_0^\infty G(\mathbf{r}, t) dt$. One finds in 3-D

$$q(\mathbf{r}) = \frac{s_0}{4\pi\kappa r}\exp\left(-\frac{(rv - \mathbf{r}\cdot\mathbf{v})}{2\kappa}\right) \tag{5}$$

$$Z(\mathbf{r},\xi) = \frac{r}{\sqrt{4\pi\kappa\xi^3}}\exp\left(-\frac{(r - v\xi)^2}{4\kappa\xi}\right) \tag{6}$$

and

$$A(\mathbf{r}) = \frac{r}{v} \tag{7}$$

where $r = |\mathbf{r}|$ and $v = |\mathbf{v}|$. In 1-D the solutions are

$$q(x) = \frac{s_0}{v}\exp\left(-\frac{v(|x| - x)}{2\kappa}\right) \tag{8}$$

$$Z(r,\xi) = \frac{v}{\sqrt{4\pi\kappa\xi}}\exp\left(-\frac{(|x| - v\xi)^2}{4\kappa\xi}\right) \tag{9}$$

and

$$A(x) = \frac{|x|}{v} + \frac{2\kappa}{v^2}\ . \tag{10}$$

(See also *Deleersnijder et al.* [2001a,b] and *Beckers et al.* [2001] for these and other related solutions.)

Several features are worth noting. The first point, constituting the main focus of this work, concerns the differences between q on the one hand and Z and A on the other. The concentration, q, is highly asymmetric. For example, in 1-D, tracer completely fills the downstream domain (i.e., $q(x) = 1$ for $x > 0$), while upstream tracer falls as $e^{-|x|v/2\kappa}$. By contrast, both Z and A are symmetric in x. This symmetry, noted by *Beckers et al.* [2001] in a study of North Sea models and by *Deleersnijder et al.* [2001a] and *Beckers et al.* [2001] in the 1-D solutions above, is counterintuitive. One expects that it is harder to move against the flow than with the flow, and therefore it should take longer. One finds, instead, that while only a small fraction of the tracer moves against the flow, this fraction requires no more time to travel an equal distance than the larger fraction moving with the flow. This symmetry is discussed in detail in the next section.

Before addressing the symmetry, however, we also note the qualitative difference between 1-D and 3-D. Tracer completely fills the domain downstream in 1-D; that is, if one waits long enough, $q = 1$ anywhere downstream. This is not the case in 3-D, where even directly downstream $q \propto 1/r$. In 3-D there is too much space to be filled by a point source. In 1-D $A(x)$ is non-zero everywhere, including at the point source. Because of diffusive motion tracer can make an excursion downstream or upstream from the point source and then return to the source, causing $A(0) > 0$. In 3-D, however, there is too much available space, and recirculation back to the origin has infinitesimal influence. (See Appendix C of *Holzer and Hall* [2000] for a related discussion.)

3 Tracer Age Symmetry

In order to understand physically the counterintuitive tracer age symmetry we consider a Lagrangian description of transport. Advective-diffusive transport arises from the continuum limit of such a description. Diffusion represents the aggregate effect of random motions of particles. Advection is the net drift of particles in a direction of preferred probability for individual particle steps.

For simplicity, we consider particles that move in discrete steps of unit magnitude every time step δt, selecting randomly among the six possible directions $\pm\hat{x}$, $\pm\hat{y}$, and $\pm\hat{z}$ in a 3-D rectilinear lattice; that is, the single step probability density function (pdf) consists of six spikes, one for each direction. Take the direction of the macroscopic velocity to be $+\hat{x}$. Particles are more likely to take $+\hat{x}$ steps than $-\hat{x}$ steps (i.e., the $+\hat{x}$ spike of the pdf has greater magnitude). Steps in $\pm\hat{y}$ and $\pm\hat{z}$ all have equal probability. Step probabilities are assumed to be spatially uniform, resulting in uniform macroscopic velocity and diffusivity. Because volume elements have unit magnitude, the particle concentration at $\mathbf{r} = (x, y, z)$ is equal to the particle number at $\mathbf{r}$. The mean tracer age, $A(\mathbf{r})$, is the average over the particles at $\mathbf{r}$ of the elapsed times since they were injected at a source, which we take to be a point source at $\mathbf{r}' = (x', y', z')$ of magnitude $S(\mathbf{r}', t)$ (particle number per time).

Clearly q, Z, and A are symmetric in y and z, since there is no preferred direction in this plane. However, it may seem surprising that Z and A are symmetric in x, despite the directionality of the velocity (i.e., the preferred single step probability). Our physical argument for the symmetry requires two ingredients: (1) For each sequence of particle steps (a "trajectory") connecting $\mathbf{r}'$ to $\mathbf{r}$ there is a "reflection" trajectory connecting $\mathbf{r}'$ to $-\mathbf{r}$. The reflection is obtained by reversing the sign of all the steps. The existence of a reflection requires that the probability for a step in an opposite direction be nonzero, although it can be arbitrarily close to zero. (For example, the limit of small probability of a $-\hat{x}$ step is the limit of small x diffusion. The tracer age is still symmetric in x, but it is realized by a vanishingly small amount of tracer at points $x < x'$.) (2) The steps comprising a trajectory are statistically independent. In the argument that follows this independence allows us to reorder a step sequence with no impact on its overall probability. Note that if steps were not statistically independent, but instead had a finite decorrelation time, one could accumulate a sequence of steps over the decorrelation time and consider the net displacement of the accumulation as the fundamental step.

Consider a sequence of n steps $w_1, ..., w_n$ forming a trajectory W from $\mathbf{r}'$ to $\mathbf{r}$. If $p(w_j)$ is the probability of the j^{th} step, then $P(W) = p(w_1)\cdots p(w_n)$ is the probability that trajectory W is sampled by a particle. Now, in every trajectory there must be a subset of steps that, when taken in sequence, forms a "sub-trajectory" directly from $\mathbf{r}'$ to $\mathbf{r}$. The remaining set of steps form a sub-trajectory of zero net displacement. We consider the reordered sequence

$$W = \overbrace{w_1,\, ..,\, w_{n-m}}^{\mathbf{r}'\rightarrow\mathbf{r}'},\ \overbrace{w_{n-m+1},\, ...\, ,\, w_n}^{\mathbf{r}'\rightarrow\mathbf{r}}, \tag{11}$$

where $R \equiv w_1, ..., w_{n-m}$ is a "recirculation sub-trajectory" of zero net displacement and step number $n - m$, and $D \equiv w_{n-m+1}, ..., w_n$ goes directly to $\mathbf{r}$ in the minimum number of steps $m = |x|/\delta x$. Because of the statistical independence of steps the reordering does not affect the overall probability of W, and so $P(W) = P(R)P(D)$.

Each permutation of steps in (11) is also a trajectory from $\mathbf{r}'$ to $\mathbf{r}$ and has the same probability. To obtain the full probability, $\mathcal{P}$, of traveling from $\mathbf{r}'$ to $\mathbf{r}$ in n steps, $P(W)$ must be multiplied by a factor $B(n,m)$, the number of distinct permutations of n steps that result in a net m steps in one direction. That is,

$$\mathcal{P} = B(n,m)P(R)P(D) \ . \tag{12}$$

$P(D)$ depends on both the magnitude of $\mathbf{r}-\mathbf{r}'$ (a longer sequence of steps is required to reach a greater $|\mathbf{r}-\mathbf{r}'|$) and its direction (steps against the flow are less likely than steps with the flow). However, $P(D)$ does not depend on the total step number n or, equivalently, on the total duration of the trajectory $n\delta t$, as long as $m \geq n$. Every trajectory to $\mathbf{r}$ must have the sub-trajectory D, regardless of the total step number. The additional steps affect $P(R)$ but not $P(D)$.

The expected number of particles q at $\mathbf{r}$ is the sum of the probabilities of all trajectories to $\mathbf{r}$ of all step numbers n (equivalently, durations $n\delta t$) multiplied by the particle number, $S(\mathbf{r}', t-n\delta t)\delta t$, emitted at the time the trajectory started at $\mathbf{r}'$. That is,

$$q(\mathbf{r},t) = \sum_{n=m}^{\infty} S(\mathbf{r}', t-n\delta t)\delta t B(n,m)P(R)P(D) \tag{13}$$

(Note that, compared to Section 2, q and S here have units of particle number and particle number per time, respectively.) The quantity $S(\mathbf{r}', t-n\delta t)B(n,m)P(R)P(D)\delta t$ is the number of particles that took time $n\delta t$ to travel $\mathbf{r}'$ to $\mathbf{r}$. Therefore, the tracer age distribution, following (3), is

$$Z(\mathbf{r},t) = \frac{S(\mathbf{r}', t-n\delta t)B(n,m)P(R)P(D)}{\sum_{n=m}^{\infty} \delta t S(\mathbf{r}', t-n\delta t)B(n,m)P(R)P(D)} \ . \tag{14}$$

We now exploit the fact that $P(D)$ does not depend on the length of a trajectory by moving it outside the summation, leaving

$$Z(\mathbf{r},t) = \frac{S(\mathbf{r}', t-n\delta t)B(n,m)P(R)}{\sum_{n=m}^{\infty} \delta t S(\mathbf{r}', t-n\delta t)B(n,m)P(R)} \ . \tag{15}$$

None of the factors in (15) depends on the direction from the point source at $\mathbf{r}'$. Because the velocity and diffusivity are assumed uniform, the probability of a trajectory of zero net displacement, $P(R)$, is actually independent of position. $B(n,m)$ is the number of trajectories that go $\mathbf{r}$ to $\mathbf{r}$, and depends on $|\mathbf{r}-\mathbf{r}'|$ through the step number m, but not on the direction. Therefore Z is symmetric, as are all its temporal moments, including the mean tracer age A.

Let us summarize the essence of the tracer age symmetry. Every trajectory from $\mathbf{r}'$ to $\mathbf{r}$ has a reflection to $-\mathbf{r}$, formed by reversing all the steps. If single step probabilities are spatially uniform (equivalent to uniform velocity and diffusivity) and the steps are statistically independent, then the sequence of steps in a trajectory can be reordered with no impact on the trajectory's total probability. One such reordering results in a recirculation sub-trajectory about $\mathbf{r}'$ of zero net displacement (same number of steps in all directions) followed by a direct flight to $\mathbf{r}$. But the recirculation is the same for the trajectory and its reflection. The difference in the trajectory probabilities comes only from the difference in probabilities of the direct flights. These direct flight probabilities do not depend on the overall trajectory duration, and thus the difference in probability of a trajectory and its reflection does not depend on the duration. In other words, the distributions by trajectory duration of trajectory probabilities to $\mathbf{r}$ and $-\mathbf{r}$ differ by a single scaling factor, the difference in direct flight probability to $\mathbf{r}$ and $-\mathbf{r}$. Upon dividing by the particle number to obtain Z, the distribution among the particles present, the distribution and its reflection become identical.

3.1 One-Dimensional Examples

We now illustrate these arguments in a 1-D example. Consider steps of equal magnitude δx every time step δt, with a probability p of a positive step and a probability $q = 1 - p$ of a negative step; that is, a single step probability distribution function (pdf) consisting of spikes of unequal magnitude at ± 1. (The macroscopic transport coefficients are related to the random-walk parameters by $u = (p - q)\delta x/\delta t$ and $k = pq\delta x^2/\delta t$.) To arrive at $x > 0$ there must be $m = x/\delta x$ more positive steps than negative. Because of the statistical independence of steps, the probability of any trajectory to x in time $\xi = n\delta t$, where n is the total step number, can be written

$$P(x, \xi) = p^m p^{\frac{1}{2}(n-m)} q^{\frac{1}{2}(n-m)} \tag{16}$$

The probability for the reflection trajectory is obtained by reversing all the steps; that is, by interchanging p and q in (16):

$$P(-x, \xi) = q^m q^{\frac{1}{2}(n-m)} p^{\frac{1}{2}(n-m)} \tag{17}$$

All other trajectories of duration ξ to x and $-x$ are permutations of (16) and (17). Note that the ratio

$$\frac{P(-x,\xi)}{P(x,\xi)} = \left(\frac{q}{p}\right)^m = \left(\frac{q}{p}\right)^{|x|/\delta x} \tag{18}$$

does not depend on ξ. Thus, the distributions by ξ of trajectory probabilities differ by the constant scaling factor $(q/p)^{|x|/\delta x}$, and Z is symmetric in x. That is,

$$Z(x, \xi) = \frac{S(t-\xi)B(n,m)p^{\frac{1}{2}(n-m)}q^{\frac{1}{2}(n-m)}}{\sum_{n=m}^{\infty} \delta t S(t-\xi)B(n,m)p^{\frac{1}{2}(n-m)}q^{\frac{1}{2}(n-m)}} \tag{19}$$

is invariant under exchange of p and q. (Here, $B(n,m) = n!/(\frac{1}{2}(n-m)!\frac{1}{2}(n+m)!)$ is the number of distinct permutations of n total steps with a net m either positive of negative.)

As an additional 1-D random walk example that relaxes the earlier restriction to quantized steps in x, consider the following: At each time step 1000 particles are given a random displacement according to a single step pdf that is equal to unity for $-0.45 < \delta x < +0.55$ and zero otherwise. Figures 1a and 1b show trajectories after 50 time steps. Also shown among all the trajectories in Figure 1a are the subset that reach $x = 1 \pm 0.25$. Figure 1b shows those that reach $x = -1 \pm 0.25$. More particles follow trajectories reaching +1 than -1, because of the preferred direction for single steps. We now form the tracer age distributions, $Z(x, \xi)$, at $x = +1$ and $x = -1$ by binning the number of particles at these positions according to their step number at arrival, then dividing by the total number reaching the locations. These Z, shown in Figure 1c, are symmetric, discounting statistical fluctuations.

The symmetry of $Z(x, t)$ in the example above reflects the uniformity of the transport coefficients, expressed as velocity u and diffusivity κ macroscopically and by the single step pdf microscopically. More generally, u and κ (and the single step pdfs) need not be uniform, but merely symmetric, to result in symmetric $Z(x, \xi)$. Figure 2a shows Z at $x = \pm 1$ resulting from a random walk with the single step pdf of Figure 1, except that now the pdf width increases symmetrically with distance from the origin according to $1+3(1-e^{-|x|})$. The symmetry of Z in x is preserved. By contrast, when the width increases upstream but remains uniform downstream, Z is asymmetric, as shown in Figure 2b.

4 Summary and Discussion

Deleersnijder et al. [2001a] and *Beckers et al.* [2001] noted a counterintuitive symmetry in the "age" of a tracer released from a point source in an advective-diffusive flow with uniform coefficients. We have explained this symmetry physically by analyzing random walks with statistically independent steps, a description that underlies advective-diffusive transport. Every trajectory from a source $\mathbf{r}'$ to $\mathbf{r}$ has a reflection to $-\mathbf{r}$. The step sequence in a trajectory and its reflection can be reordered with no effect on the probability of being sampled by a particle. One such reordering results in a recirculation about $\mathbf{r}'$ of zero net displacement followed by a direct flight from $\mathbf{r}'$ to $\mathbf{r}$. But the recirculation is the same for the trajectory and its reflection. The difference in the trajectory probabilities comes only from the difference in probability of the direct flights. These direct flight probabilities do not depend on the overall trajectory duration (transit time), and thus the difference in probability of a trajectory and its reflection does not depend on the transit time. Therefore, the normalized distributions of transit times to $\mathbf{r}$ and $-\mathbf{r}$ are identical.

It is worthwhile contrasting the symmetry properties of two different definitions of "age." The age symmetry of *Deleersnijder et al.* [2001a] and *Beckers et al.* [2001] arises in the case where age is considered to be a property of the tracer itself—what we have called "tracer age". In an alternate use of the term "age," the symmetry does not arise. It is common in ocean tracer studies to consider the age to be a property of a water mass. One speaks of the elapsed time (or distribution of times) since a water mass was last at the ocean surface [e.g., *England*, 1995; *Beining and Roether*, 1996]. (For clarity, we have referred to the "transit times" for irreducible fluid elements to travel from a specified boundary region to the interior, although simply "age" is common.) Observable tracers allow an estimation of the transit time distribution and its moments to varying degrees, depending on the tracer and the flow conditions [*Waugh et al.*, 2002].

To make explicit the different symmetry properties of these timescales consider an unbounded 1-D advective-diffusive system with uniform coefficients, the system analyzed by *Deleersnijder et al.* [2001a]. The transit time of an irreducible fluid element is the time since it was last at the origin. Note the distinction: transit time is always zero at the origin, whereas tracer age is generally nonzero at the origin. In the simplest case of a tracer having a constant source, the mean tracer age $A(x)$ downstream is given by expression (10), whereas the mean transit time (also known as the "mean age" and the "ideal age") is $\Gamma(x) = x/u$. In this idealized case the two timescales are related simply: $\Gamma(x) = A(x) - A(0)$. (Contrast this with the relationship noted in Section 2.1 for a conservative tracer with constant source in a *bounded* domain.)

We now ask what is the mean transit time upstream? One could attempt to construct the transit time distribution $\mathcal{G}$ (also known as the age spectrum) following *Hall and Plumb* [1994] by computing the response to a $\delta(t)$ boundary condition at $x = 0$ and looking at positive x with

$u < 0$; i.e., the fluid flow running toward the origin from the parcel location. This yields

$$\mathcal{G}(x,\xi) = \frac{x}{\sqrt{4\pi k \xi^3}}\, e^{-(x+u\xi)^2/4k\xi} \tag{20}$$

However, one finds that $\int_0^\infty \mathcal{G} d\xi = e^{-2xu/k} < 1$. Unlike the case downstream, where $\int_0^\infty \mathcal{G} d\xi = 1$, the upstream transit time distribution is not normalized. A fraction of the fluid parcel that increases exponentially with x has never been at the origin. The mean transit time solely among the fluid fraction that has been at the origin is $\int_0^\infty \xi \mathcal{G} d\xi \,/\, \int_0^\infty \mathcal{G} d\xi = x/u$, identical to the downstream solution. But *over the entire fluid parcel* the mean transit time since last contact with the origin is infinite, since much of the parcel has *never* been at the origin. It is therefore not symmetric. A second approach is to consider the steady-state solution to the equation for the ideal age, τ_{id}, which downstream is equivalent to the mean transit time [e.g., *Khatiwala et al.*, 2001]:

$$\frac{\partial \tau_{id}}{\partial t} - u\frac{\partial \tau_{id}}{\partial x} - k\frac{\partial^2 \tau_{id}}{\partial x^2} = 1 \tag{21}$$

with $\tau_{id}(0,t) = 0$. In steady-state, one finds $\tau_{id} = -x/u$, again not symmetric about $x = 0$. A negative timescale upstream to describe the elapsed time since the fluid made contact with $x = 0$ is as plausible as an infinite timescale: most of the upstream fluid has never been at the origin but will make contact with the origin at a future time; that is, a negative elapsed time. We conclude that upstream in an unbounded domain the mean transit time is either infinite or negative, depending on definition, but in any case is not equal to the downstream value. Eric Deleersnijder (personal communications) has recently confirmed the asymmetry of the ideal age (also known as the "water age"), extending the analysis to include the transient solution.

It is perhaps not surprising that transport timescales should have peculiar properties in an open domain (unbounded in some direction), given the continuous and unlimited source of new fluid from upstream. Although an open approximation may be useful in certain instances, all geophysical domains are ultimately closed; that is, the fluid has finite mass. This has considerable bearing on the transport timescales discussed here. Given a constant source applied on some boundary region and no sink, the tracer concentration and mean tracer age will eventually increase everywhere linearly in time if the domain is closed. This is in contrast to the steady-state tracer age in the open 1-D domain, where escape from the domain acts as an effective sink for any finite sub-domain. On the other hand, the mean transit time reaches a finite steady state even in a bounded domain, if the circulation is stationary. Sufficiently far enough back in time, all fluid elements have made boundary contact.

Finally, we note that in a closed domain streamlines of the flow are closed. While "upstream" and "downstream" may be meaningful locally, the mean transit time upstream is not determined locally, but is rather set by the remote boundaries that cause streamlines to close. In closed domains the mean transit time is not symmetric. Downstream, parcels are dominated by fluid that made recent boundary contact and are therefore young. In contrast, upstream parcels may have had only weak diffusive contact locally with the boundary region. The majority of their fluid elements have circulated about streamlines that may span much of the domain, and the parcels are therefore much older.

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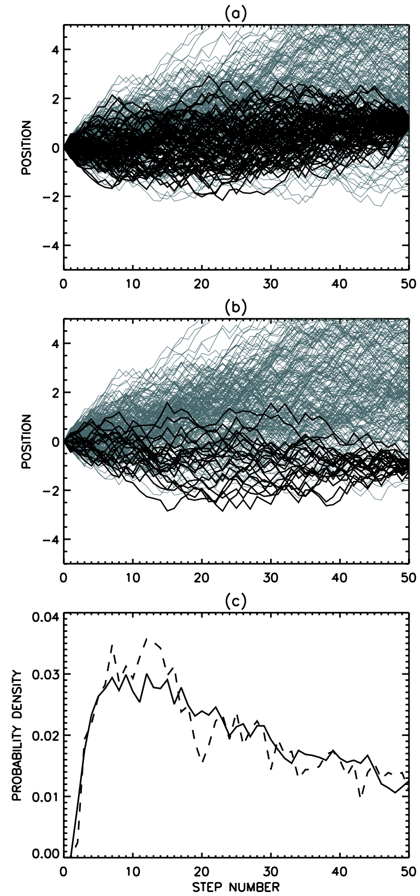

Figure 1: Position versus step number of 1000 trajectories starting at the origin. (a) All trajectories (gray) and those that end at $x = +1 \pm 0.25$ (black). (b) All trajectories (gray) and those that end at $x = -1 \pm 0.25$ (black). (c) Tracer age distributions at $x = +1$ (solid)

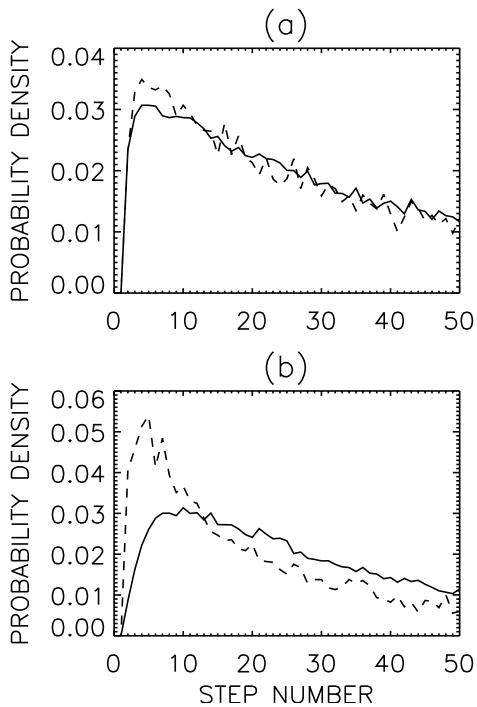

Figure 2: Tracer age distributions at $x = +1$ (solid) and $x = -1$ (dashed) for 5000 trajectories. (a) The width of the single step pdf increases symmetrically with distance from the origin. (b) The width of the single step pdf is constant downstream but increases upstream with distance from the origin.