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EXPANSION OF SCATTERING MATRIX IN GENERALIZED SPHERICAL HARMONICS FOR RADIALLY INHOMOGENEOUS SPHERICAL PARTICLES

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A method is described for calculating the coefficients of an expansion in generalized spherical harmonics of the elements of the scattering matrix for radially inhomogeneous polydisperse spheres. A numerical example is given.

The model of scattering particles as radially inhomogeneous spheres is widely used in the physics of planetary atmospheres and the interstellar medium. The single scattering of light by radially inhomogeneous polydisperse spherical particles has by now been studied quite thoroughly [4], but there has been no study of the corresponding aspects of the theory of multiple scattering.

In this paper we discuss a first step in the numerical solution of problems in the theory of the transport of polarized radiation. This first step is an expansion of the elements of the scattering matrix $\hat{F}(\theta)$ in a series in generalized spherical harmonics $p^s_{mn}(\cos\theta)$, where θ is the scattering angle. In the CP representation of the polarized light [7] this expansion is

$$F_{mn}(\theta) = \sum_{s=s_*}^{\infty} g^s_{mn} P^s_{mn}(\cos\theta), \quad m, n = 2, 0, -0, -2,$$

where $s_* = \max(|m|, |n|)$. An algorithm was proposed in [6] for calculating the coefficients g^s_{mn} for polydisperse homogeneous spheres. It should be noted, however, that the angular structure of the series for the elements of the amplitude scattering matrix,

$$S_1(\theta) = \sum_{n=1}^{\infty} (2n+1)\,[a_n \pi_n(\theta) + b_n \tau_n(\theta)]/[n(n+1)];$$

$$S_2(\theta) = \sum_{n=1}^{\infty} (2n+1)\,[b_n \pi_n(\theta) + a_n \tau_n(\theta)]/[n(n+1)]$$

is the same for both homogeneous and radially inhomogeneous spheres. Only the values of the coefficients a_n and b_n depend on the nature of the radial variation [4]. We can therefore derive the following expressions for the quantities g^s_{mn}:

$$g^s_{ll} = d_s \left\{ 2 \sum_{n=1}^{\infty} \sum_{m=m_*}^{s+n} V^2_{lsmn} (2n+1)(2m+1)\,\mathrm{Re}\,\langle (a^*_n + b^*_n)(a_m + b_m) \rangle + \right.$$
$$\left. + \sum_{n=1}^{\infty} V^2_{lsnn} (2n+1)^2 \langle |a_n + b_n|^2 \rangle \right\}; \tag{1}$$

$$g^s_{l-l} = d_s \left\{ 2 \sum_{n=1}^{\infty} \sum_{m=m_*}^{s+n} V^2_{lsmn} (-1)^{s+m+n} (2n+1)(2m+1)\,\mathrm{Re}\,\langle (a^*_n - b^*_n)(a_m - b_m) \rangle + \right. \tag{2}$$
$$\left. + (-1)^s \sum_{n=1}^{\infty} V^2_{lsnn} (2n+1)^2 \langle |a_n - b_n|^2 \rangle \right\}$$

Table 1

Expansion Coefficients for a Water-Silicate H Haze

s	$g^s_{0,0}$	$g^s_{0,-0}$	$g^s_{2,2}$	$g^s_{2,-2}$	Re $g^s_{0,2}$	Im $g^s_{0,2}$
0	0.954 40	0.045 60	0.0	0.0	0.0	0.0
1	2.040 69	—0.039 79	0.0	0.0	0.0	0.0
2	2.111 82	0.026 43	3.643 22	0.113 56	0.041 42	0.058 47
3	1.720 15	—0.039 98	2.325 93	—0.002 04	0.059 79	0.135 76
4	1.156 08	0.012 80	1.645 11	0.058 26	0.022 13	0.095 41
5	0.740 06	—0.013 97	0.883 74	0.008 44	0.043 00	0.089 09
6	0.419 94	0.007 28	0.569 36	0.024 56	0.007 44	0.052 57
7	0.231 19	—0.003 00	0.252 74	0.005 06	0.018 93	0.033 28
8	0.123 05	0.003 44	0.164 50	0.008 58	0.001 76	0.020 18
9	0.058 76	—0.000 59	0.059 25	0.001 68	0.006 13	0.008 90
10	0.031 73	0.001 36	0.042 32	0.002 71	0.000 33	0.006 45
11	0.012 85	—0.000 18	0.011 89	0.000 39	0.001 59	0.001 84
12	0.007 54	0.000 48	0.010 05	0.000 82	0.000 06	0.001 85
13	0.002 50	—0.000 07	0.002 09	0.000 07	0.000 35	0.000 30
14	0.001 69	0.000 16	0.002 25	0.000 24	0.000 01	0.000 49
15	0.000 44	—0.000 03	0.000 33	0.000 01	0.000 07	0.000 04
16	0.000 36	0.000 05	0.000 48	0.000 07	0.000 00	0.000 12
17	0.000 07	—0.000 01	0.000 05	0.000 00	0.000 01	0.000 00
18	0.000 07	0.000 01	0.000 10	0.000 02	0.000 00	0.000 03
19	0.000 01	—0.000 00	0.000 01	—0.000 00	0.000 00	—0.000 00
20	0.000 01	0.000 00	0.000 02	0.000 00	0.000 00	0.000 00
21	0.000 00	—0.000 00	0.000 00	—0.000 00	0.000 00	—0.000 00

for j = 0, 2;

$$g^s_{02} = d_s\Big\{\sum_{n=1}^{\infty}\sum_{m=m_s}^{s+n} V_{0smn}V_{2smn}\,(2n+1)\,(2m+1)\,[\langle(a^*_m+b^*_m)\,(a_n-b_n)\rangle + \\ +(-1)^{s+m+n}\,\langle(a^*_n+b^*_n)\,(a_m-b_m)\rangle] + \\ +\sum_{n=1}^{\infty} V_{0snn}V_{2snn}\,(2n+1)^2\,\langle(a^*_n+b^*_n)\,(a_n-b_n)\rangle\Big\}, \tag{3}$$

These expressions were given (with some misprints) in [5,6]. The only exception is that the coefficients a_n and b_n should be calculated not from the Mie theory but from the corresponding theory for radially inhomogeneous spheres [4]. The angle brackets in (1)-(3) mean an average over the ensemble of particles; $d_s = (2s + 1)\,\pi[k^2 \langle C_{sca}\rangle]^{-1}$, where $k = 2\pi/\lambda$, and λ is the wavelength of the light; $m_s = \max(s - n, n + 1)$; the scattering cross section is $\langle C_{sca}\rangle = 2\pi k^{-2}\times\sum_{n=1}^{\infty}(2n+1)\;\langle|a_n|^2+|b_n|^2\rangle$; and $V_{jsmn} = \begin{pmatrix} s & m & n \\ j & -1 & (1-j)\end{pmatrix}$ are the Wigner 3jm symbols.

An algorithm and a corresponding set of programs for calculating the coefficients g^s_{mn} for polydisperse two-layer spheres on a computer have been constructed on the basis of the discussion above. A modified version of the BHCOAT subprogram is used to calculate a_n and b_n [1]. The Wigner 3jm symbols are calculated with the help of the recurrence relations from [3,5].

A numerical example of the use of the algorithm is shown in Table 1. These calculations were carried out for a Deirmendjian H haze [2] (λ = 0.7 μm). It was assumed that the ratio of radii of the core and the shell is 0.9, regardless of the size of the particles. For the refractive indices of the core and the shell we used the values 1.54 (silicate) and 1.33 (water), respectively. The calculations took 1 min of processor time on an ES-1060 computer.

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