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ASTRONOMY AND ASTROPHYSICS

Turbulent viscosity

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Summary. A key quantity needed to model the physics of accretion disks is the turbulent viscosity ν_T. Most disks calculations have thus far used an expression for ν_T that contains an unknown quantity, the Shakura-Sunyaev α-parameter. This precludes the possibility of making theoretical predictions. Astrophysical data are often used to fix α, which should be calculated by a model of turbulence.

A successful model to treat fully developed turbulence (the Direct Interaction Approximation, DIA) was derived from first principles by Kraichnan in the early 60's and yet it has not been used in astrophysical problems like accretion disks or turbulent convection in stars.

This paradoxical situation may perhaps be explained by the fact that the DIA equations are rather complex in structure and time consuming to solve, a difficulty which becomes all the more serious ,when turbulence is just one component of a larger problem.

To bridge the gap between the fully predictive but hard to use DIA and the phenomenological, easy to use, but non-predictive α-model, we propose a model for fully developed turbulence whose predictions compare favorably with those of the DIA and whose main equations are easy to handle.

Using this model, we derive *four different expressions* for ν_T, Eqs. (50) and (56). The four expressions contain no free parameters. Two of the expressions are given in terms of properties of the turbulent fluid itself; the other two are given in terms of the instability that generated the turbulence and of the properties of the mean flow (shear). The numerical coefficients entering these relations are evaluated and found to be in good agreement with previous theoretical estimates based on a) Kraichnan's DIA, b) the Renormalization Group Method, and c) Turbulence Modeling. In the case of shear in the mean flow, we show that $\alpha < 10^{-2}$.

The four expressions can be generalized to include the effect of rotation and/or magnetic fields.

Key words: turbulence – hydrodynamics

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1. Introduction

The existence of gaseous structures surrounding massive objects has by now permeated the field of astrophysics and planetary physics. Although the presence of a gaseous, dusty disk around the young Sun was postulated long ago by Laplace, it was not until this century that quantitative attempts were made to describe in detail the structure and evolution of such disks structures. More recently, the probable existence of disk-like structures surrounding compact collapsed objects and binary stars has also significantly contributed to current interest in the physics of disks (Pringle, 1981).

In almost all types of disks of astrophysical interest, the main problem is that of removing the main component, i.e., the gas, by causing it to drift outward as well as inward toward the central object, may that be a black hole or a central star. Since in the economy of the problem, it is usually assumed that the disk is not acted upon by external forces, one is forced to search for an internally generated mechanism capable of initiating the drifting process. To break the otherwise stable Keplerian motion, "viscous forces" are often invoked. Since kinematic viscosity is far too weak to cause a timely dispersal of the gas, it has become customary to call upon the presence of dynamical processes, like turbulence, to obtain an "enhanced" or "turbulent" viscosity. The effect comes about in the following way. If $\boldsymbol{v}$ represents the total velocity of the fluid under consideration, the Navier-Stokes equations read

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial}{\partial x_j} v_i = -\frac{1}{\varrho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu \sigma_{ij}), \tag{1}$$

$$\sigma_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}, \tag{2}$$

where σ_{ij} is the stress tensor. Suppose now that the total velocity $\boldsymbol{v}$ is decomposed into the sum of a mean flow velocity, $\boldsymbol{U}$, plus a fluctuating or turbulent velocity, $\boldsymbol{u}$,

$$\boldsymbol{v} = \boldsymbol{U} + \boldsymbol{u}. \tag{3}$$

Substituting (3) into (1) and (2), one obtains two coupled equations for $\boldsymbol{U}$ and $\boldsymbol{u}$. The equation for $\boldsymbol{U}$ is known to be, (see for example, Hinze, 1975)

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial}{\partial x_j} U_i = -\frac{1}{\varrho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\nu S_{ij} + \tau_{ij}), \tag{4}$$

where τ_{ij} is the Reynolds stress tensor defined as

$$\tau_{ij} = -\langle u_i u_j \rangle . \tag{5}$$

The structure of the mean flow thus depends on the presence of turbulence, i.e., in order to solve Eq. (4), one needs to have first solved the turbulence problem so as to be able to evaluate the Reynolds stress tensor.

Until a satisfactory model of turbulence became available, one had to make educated guesses about the tensor τ_{ij}. This usually meant that one proposed empirical formulae containing one or more free parameters to be determined later using experimental data. This approach was proposed long ago to deal primarily with engineering types of flows, where the main interest is that of describing rather than predicting phenomena. Furthermore, the success of the approach was assured by the availability of many laboratory data that allowed the use of increasingly complex phenomenological expressions for τ_{ij}. This "engineering" approach, dating back to Boussinesq (1877, 1897), Taylor (1915), and Prandtl (1925), consists of writing τ_{ij} in terms of the shear S_{ij} as

$$\tau_{ij} = \nu_T S_{ij} , \tag{6}$$

where ν_T is a "turbulent" or "enhanced" viscosity. By inserting (6) into (4), one notices that the effect of turbulence is that of renormalizing the kinematic viscosity ν to

$$\nu \rightarrow \nu + \nu_T . \tag{7}$$

While (7) is physically appealing, the problem remains of how to compute ν_T. For example, in the case of channel flow, i.e. a two dimensional flow in the x-direction, limited in the y-direction between two planes at $y = \pm 1$, one can solve Eqs. (4) and (6) and show that (Hussain and Reynolds, 1975)

$$U(y) = A \int_0^y \frac{1-y}{1+\nu_T(y)/\nu} \, dy . \tag{8}$$

Since in the laboratory the mean flow, $U(y)$, can be measured at different positions in the channel, Eq. (8) is often used to "derive" the turbulent viscosity from the experimental data.

In the case of astrophysical disks, the spirit of the above approach seems intrinsically inadequate since the available data are limited and because the main goal is to predict rather than describe the phenomena. For example, Eq. (8) is of little use since one cannot measure the mean field velocity U, which must be computed once a model of turbulence has provided either τ_{ij} or ν_T.

In their pioneering work on the structure of disks, Shakura and Sunyaev (1973) adopted Eq. (6) but did not attempt to derive ν_T from a model of turbulence. Rather, they proposed on dimensional grounds that ν_T be written as

$$\nu_T = \alpha c_s H , \tag{9}$$

where c_s is the local speed of sound, H is a scale height (usually taken to be the pressure scale height), and α is an unknown dimensionless parameter adjusted to fit the data. The disk structures so constructed are known as α-disks.

It seems to us that the original spirit of the SS paper was to adopt (9) as a tool to analyze the "qualitative" effects of an "enhanced eddy viscosity", rather than as a "quantitative" tool. In fact, Eq. (9) does not provide an α and cannot therefore be used to make theoretical predictions. The use of astrophysical data to determine α is also not a very satisfactory procedure. First, it should not be astrophysics that fixes the parameters of turbulence but rather a model of turbulence. Secondly, the value of α derived from one type of disk need not be the same as the one derived from another type of disk. In fact, enshrined in the parameter α is the information characterizing possibly very different types of disks.

Given the astrophysical importance of disks and the fact that successful theories of turbulence were available long before the Shakura and Sunyaev paper, it is at first surprising that no attempts were made to calculate ν_T from first principles.

A possible explanation may lie in the rather intimidating derivation and final structure of the most successful turbulence theory today, the Direct Interaction Approximation (DIA) (Kraichnan, 1964), whose equations for the turbulent energy spectral function are two coupled non-linear integral equations (see Sect. 4). The same reason may also explain why the DIA has not been employed in other astrophysical contexts where turbulence is equally important, e.g., in the convective interiors of stars (for an application of DIA to laboratory convection, see Hartke et al., 1987).

In the present paper, we shall present a model of turbulence that yields, among other things, a turbulent energy spectral function very similar to the one derived from DIA and yet, is easier to visualize and simpler to use. One of the assets of the new model lies in its easy inclusion of the effects of magnetic fields and rotation. Its weakness is that, unlike DIA, it is not a deterministic approach from first principles (although it must be stated that the DIA does contain approximations, see Martin et al., 1973).

The turbulence model to be presented here is based on a physical model for the non-linear interactions whose implications have been tested against a variety of experimental data on turbulence and found to yield satisfactory results. Moreover, Kraichnan (1987) has recently shown that the DIA model yields an expression for the turbulent viscosity of the same form as the one suggested by our model. This feature, together with the similarity of the resulting spectral functions as well as other comparisons discussed elsewhere (Hartke et al., 1987), leads us to believe that our model contains the main features of the DIA without the complexities that have made the latter impractical for most astrophysical applications.

Before presenting the model, it is important to stress that the turbulent viscosity derived from the DIA or our model, usually called ν_t, represents the action of a given group of eddies on all the larger ones. The concept, first introduced by Heisenberg, makes no reference to the existence of a mean flow and represents an intrinsic property of any fully developed turbulent flow. If in the calculations of ν_t one includes all the eddies smaller than the largest ones, in practice one computes the effect of turbulence on the mean flow since the largest eddies have dimensions comparable with that of the mean flow itself. Based on this qualitative physical argument, and following previous practice, we propose to use ν_t for ν_T in Eq. (6). Disk computations performed under this assumption were carried out by Cabot et al. (1987).

2. The new model for turbulence

A turbulent medium is characterized by eddies whose sizes range from that of the containing volume (largest eddies) to small enough sizes where kinematic viscosity dissipates into heat the energy originally injected into the fluid to make it become turbulent (we shall deal only with incompressible fluids). The eddies interact via non-linear forces that cause the energy fed into the system at the largest eddies to "cascade" into smaller eddies. Any given group of eddies may originate both from the source as well as from the break-up of larger eddies (into smaller ones). Clearly,

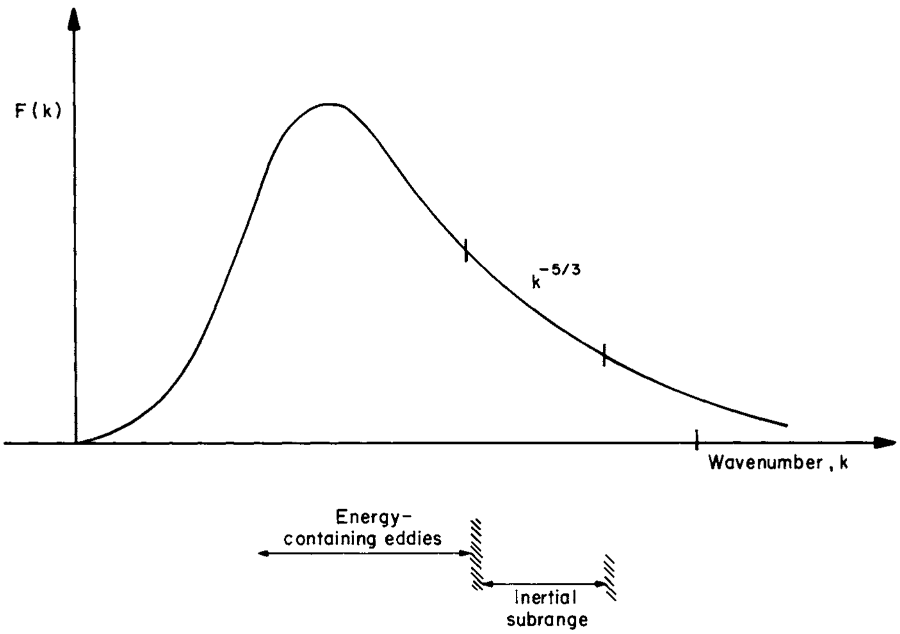

Fig. 1. The general shape of the energy spectral function $F(k)$ indicating the region of the energy-containing eddies, the Heisenberg-Kolmogoroff inertial region and the high-k viscosity dependent region

the largest eddies cannot originate from even larger ones and so they are primarily dependent on the type of stirring mechanism actually at work. By the same token, eddies that in k space are far removed from the source which acts primarily on the largest eddies (small k's), owe their existence mostly to the break-up processes. Much as the stirring (or input of) energy does not affect all the eddies equally, neither does the process of energy dissipation. Since molecular viscosity is the main agent in the latter process, the smallest eddies are the ones mostly involved in the dissipation while the largest eddies are largely immune to viscosity. In summary, the larger the eddies, the more they depend on the characteristics of the energy source, the less universal is their spectrum and the less they are affected by viscosity. Conversely, the medium size eddies, having lost memory of their origin because of the large number of break-ups that preceded them, acquire a universal spectrum. Finally, the smallest eddies fall under the influence of molecular viscosity. (As customary, we shall use the inverse of the wave number k to characterize the size of the eddies.)

Because of the non-linear nature of turbulence, the turbulent kinetic energy at a given k is the result of a complex transfer process caused by the non-linear interactions. One therefore defines the turbulent energy spectral function $F(k)$ as

$$u^2(k) = \int_k^\infty F(k')\,dk' \tag{10}$$

i.e. $1/2\,u^2(k)$, the turbulent kinetic energy (per unit mass) of the eddies of size k^{-1}, is contributed by all the eddies of sizes smaller than k^{-1}. From (10) it is clear that to obtain the total kinetic energy one must integrate from the smallest value of k allowed by the system.

The main goal of any theory of turbulence is the determination of the energy spectral function $F(k)$. Since the Navier-Stokes equations are highly non-linear, the equation satisfied by $F(k)$, usually an integral equation, is also highly non-linear. For reference purposes, we present in Fig. 1 the expected shape of $F(k)$. We have divided the k interval into two broad regions: the low wave number, energy containing part and the medium size eddies that may exhibit universal features, since they are somewhat equidistant between the regions influenced by the source and the regions acted on by molecular viscosity. To construct the non-linear equation satisfied by $F(k)$, we shall propose the following physical picture (Fig. 2). Consider the interval $k_0 - k$, where k_0 is the smallest k allowed by the geometry of the system. Let us call $\varepsilon(k)$ the energy (per gram per second) injected into that interval by the external source. Since a turbulent state is characterized by the break-up of the laminar flow and the onset of an instability characterized by a growth rate $n_s(k)$, it follows that (Canuto and Goldman, 1985)

$$\varepsilon(k) = \int_{k_0}^{k} [n_s(k') + \nu k'^2]\, F(k')\,dk'. \tag{11}$$

This input energy is partially dissipated by viscosity and partially transferred to higher k's by the non-linear interactions. The loss due to viscosity is simply given by

$$\nu \int_{k_0}^{k} k'^2\, F(k')\,dk'. \tag{12}$$

The most difficult problem lies in the determination of the transfer due to the non-linear interactions i.e., the closure problem. This process will be visualized as occurring in two steps: in the first process (A of Fig. 2), energy is "extracted" from the $k_0 - k$ interval, much as if it were due to molecular viscosity. However, since the non-linear interactions are exclusively a transfer process, the energy must be re-distributed to all the remaining eddies, i.e., process A must be followed by process B, the "redepositing" of the same amount of energy to all the eddies with wave numbers larger than the ones in the $k_0 - k$ interval. Alternatively, one may view the latter group of eddies as creating an "enhanced or turbulent eddy viscosity" on the eddies in the $k_0 - k$ interval. Thus, the whole non-linear transfer process may be written as the product of the A and B processes, i.e.,

$$\nu_t(k) \int_{k_0}^{k} k'^2\, F(k')\,dk', \tag{13}$$

where the turbulent viscosity must be of the form

$$\nu_t(k) = \int_k^\infty \Psi(k')\,dk', \tag{14}$$

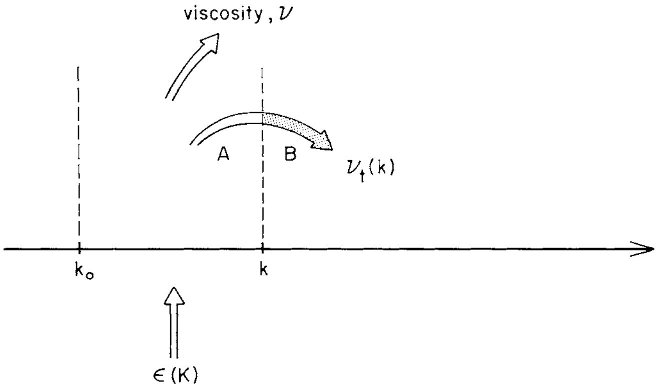

Fig. 2. The three main processes visualized in the construction of the present model of turbulence: the energy input $\varepsilon(k)$ is dissipated by molecular viscosity ν and redistributed by the non-linear interactions to all the eddies with wave number higher than k. The transfer process is visualized as a two step process, A and B. The first "extracts" energy from the k_0–k interval, like ordinary viscosity would, while the second, B, redistributes the same energy to all the remaining smaller eddies

i.e., it must be contributed by all the eddies from wave number k to infinity. Putting Eqs. (11), (12), and (13) together, we obtain the energy conservation equation

$$\varepsilon(k)=[\nu+\nu_t(k)]\int_{k_0}^{k} k'^2\, F(k')\,dk' \tag{15}$$

which becomes the non-linear equation for $F(k)$ once $\Psi(k)$ is specified in terms of $F(k)$. However, it is already clear from Eqs. (14) and (15) that by extending the integration to all k's, we obtain the expression for energy conservation,

$$\varepsilon=\nu\int_{k_0}^{\infty} k'^2\, F(k')\,dk', \tag{16}$$

where ε is the constant given by (11) when the upper limit k is extended to infinity. In accordance with the physical interpretation of the non-linear interactions as a purely transfer process that does not dissipate energy, the effect of the non-linear interactions has disappeared from Eq. (16). Let us now return to the determination of the turbulent viscosity ν_t or more specifically, of the function $\Psi(k)$. In technical jargon, this is known as the "closure problem". Since the non-linear interactions are characterized by a correlation time scale $n_c(k)$, we have in general

$$\Psi(k)=\Psi(k, F(k), n_c(k)).$$

On equally general grounds, it follows that

$$\nu_t(k)=\int_{k}^{\infty} \frac{F(k')}{n_c(k')}\,dk'. \tag{17}$$

The main problem is that of determining $n_c(k)$. If one wants to reproduce the universal or "inertial region" (see Fig. 1), where $F(k)\sim k^{-5/3}$, the choice of the correlation rate $n_c(k)$ is a simple matter. In fact, the "inertial region" is by construction independent of the features of both the source (low k's) and the sink (high k's) and can therefore depend only on local variables, i.e. k and $F(k)$. The only dimensionally correct expression is $n_c\sim k^{3/2}F^{1/2}$. This is the Heisenberg expression for n_c. Use of it in Eq. (17) and then in Eq. (15) with $\varepsilon(k)=\varepsilon$ constant, gives rise to the well-known Kolmogoroff spectrum in the inertial subrange,

$$F(k)\sim k^{-5/3},\quad \nu_t\sim k^{-4/3}.$$

The solution of Eq. (15) under these two assumptions can be found in Batchelor (1973). Since we are interested in constructing the spectral function $F(k)$ for the *whole* k-spectrum and not only for the inertial subrange, we *cannot* adopt either of the two assumptions, $\varepsilon(k)=$ const, or $n_c(k)\sim k^{3/2}F^{1/2}(k)$, since they are strictly valid only in a small portion of the k-spectrum. Clearly, the generalized form of $n_c(k)$ that we are about to propose must be proportional to $k^{3/2}F^{1/2}$ in the inertial subrange.

First, let us rewrite Eq. (15) using (17). With the definition of $\varepsilon(k)$ given by (11), we obtain the general non-linear integral equation for $F(k)$

$$\int_{k_0}^{k} n_s(k')\,F(k')\,dk'=\int_{k}^{\infty}\frac{F(k')}{n_c(k')}\,dk'\int_{k_0}^{k} k'^2\,F(k')\,dk'. \tag{18}$$

Introduce now the "mean squared vorticity" $Y(k)$

$$Y(k)=\int_{k_0}^{k} k'^2\,F(k')\,dk' \tag{19}$$

and differentiate (18) with respect to k. The result is

$$n_s(k)+n_c^{-1}(k)\;Y(k)=k^2\,\nu_t(k). \tag{20}$$

The left-hand side of (20) is the sum of two rates: $n_s(k)$, which represents the net rate of energy input from the source into the unit interval centered around k, and $n_c^{-1}(k)\;Y(k)$, which represents the rate of energy input into the same interval from the eddies in the interval k_0-k; the right hand side of (20) is the rate controlling the process of energy cascading from that unit interval into all wave numbers larger than k. Since the latter process is caused by the non-linear interactions, the right hand side of Eq. (20) must be related to n_c itself. This gives rise to the "closure equation", namely,

$$\nu_t(k)=\gamma k^{-2}\,n_c(k), \tag{21}$$

where γ is a numerical constant that can be shown to be equal to ~ 0.1 since in the inertial range $F(k)$ must reproduce the Heisenberg-Kolmogoroff spectrum.

Equation (21) completely determines the spectral function $F(k)$. *In fact, differentiating Eq. (21) with respect to* k and using (17) and (20), one obtains the differential equation satisfied by the "mean squared vorticity" $Y(k)$, namely

$$\frac{d}{dk}[Y(k)+1/2\,\gamma\,n_c^2(k)]=2\gamma k^{-1}\,n_c^2(k). \tag{22}$$

The correlation rate $n_c(k)$ is obtained by solving Eqs. (20) and (21) with the result

$$2\gamma\,n_c(k)=n_s(k)+[n_s^2(k)+4\gamma\,Y(k)]^{1/2}. \tag{23}$$

Equation (23) is the general expression for the eddy correlation rate that we would like to propose. To understand its content, let us first consider the small k region, where the largest eddies reside. Since the latter are known to possess small vorticities, we can take $Y(k)\ll n_s^2$ and so Eq. (23) reduces to

$$n_c(k)\sim n_s(k), \tag{23a}$$

which implies that near the source the dominant time scale is the one that characterizes the source (Canuto and Goldman, 1985). As we consider larger values of k, i.e. smaller eddies, the values of the vorticity increases while the time scale n_s^{-1} characterizing the source becomes increasingly less important. In fact, one is approaching the universal or inertial subrange, where the most important time scale is the one due to the local break-up of the eddies. In this region, $n_s^2 \ll Y(k)$ and so

$$n_c \sim Y^{1/2}(k). \tag{23b}$$

Insertion of (23b) into (15) with $\varepsilon(k) = \text{const}$ yields the Heisenberg-Kolmogoroff spectrum, $F(k) \sim k^{-5/3}$ and (23b) then yields $n_c \sim k^{3/2} F^{1/2}(k)$, as expected.

Substituting (23) into (22), one obtains the desired equation for $Y(k)$. The equation is non-linear but easy to solve numerically. Once the vorticity is known, the energy spectral function $F(k)$ is easily obtained since from (19) ($' = d/dk$)

$$F(k) = k^{-2} Y'(k). \tag{24}$$

Therefore the procedure is as follows: for a given instability, one inserts the corresponding growth rate $n_s(k)$ into Eq. (23). The resulting $n_c(k)$ is then used in Eq. (22) to solve for $Y(k)$. Equations (24) then yields the corresponding spectral function $F(k)$.

It is of interest to note that by eliminating $n_c(k)$ between Eqs. (17) and (21), one obtains an integral equation for $\nu_t(k)$ of the form

$$\nu_t(k) = \gamma \int_k^\infty \frac{F(k')}{k'^2 \nu_t(k')} dk',$$

which can be rewritten as

$$\nu_t(k) = \left(2\gamma \int_k^\infty \frac{F(k')}{k'^2} dk'\right)^{1/2}, \tag{25}$$

i.e., the turbulent viscosity is completely expressed in terms of $F(k)$. A similar result was derived by Moffat (1981) for temperature fluctuations using phenomenological arguments, while Kraichnan (1987) has recently shown that the DIA theory also gives rise to an expression for $\nu_t(k)$ of the same form as Eq. (21). Considering that our treatment of turbulence is based on a physical model rather than on an a priori deterministic approach like DIA, the agreement between the two expressions for the turbulent viscosity is very reassuring.

The treatment presented above is quite general and can be applied to any problem once $n_s(k)$, the growth rate of the instability that generates the turbulence, is known. In the example given below, we shall use the form of $n_s(k)$ corresponding to a convective instability.

3. Convective instability

Since this type of instability is quite common in many astrophysical problems, we shall study it in detail although we expect that many of the results have a more general validity.

In this case, the form of the linear $n_s(k)$ is well known (Canuto and Goldman, 1985)

$$2n_s(k) = \chi(1+\sigma) \cdot \left[\sqrt{4\frac{S}{(1+\sigma)^2}\frac{x}{1+x}\frac{1}{\Delta^4} + (1-\mu)k^4} - k^2\right]. \tag{26}$$

Here, χ is the thermometric conductivity, σ the Prandtl number $= \nu/\chi$, where ν is the kinematic viscosity, $S = \sigma R$, where R is the Rayleigh number $R = g\alpha\beta\Delta^4/\nu\chi$, g is the local gravity, α the volume expansion coefficient and β the temperature gradient excess over the adiabatic gradient. Finally, Δ is the geometrical size of the convective region, x the degree of anisotropy of the eddy sizes, and the parameter $\mu = 4\sigma/(1+\sigma)^2$. Following previous authors (e.g., Spiegel, 1962) $x = x(k) = (k_x^2 + k_y^2)/k_z^2$, will be taken to be equal to $x = (k\Delta/\pi)^2 - 1$.

For the case of very low Prandtl numbers (as in most cases of astrophysical interest), the function $F(k)$ derived from (22) using (26) is presented in Fig. 3a and b for two different values of S. As one can see, $F(k)$ goes naturally into the Kolmogoroff spectrum in the inertial region. Once $F(k)$ is known, one can compute the "convective flux" defined as (Canuto and Goldman, 1985).

$$F_c = c_p \varrho \beta \chi_T \equiv c_p \varrho \beta \chi \Phi, \tag{27}$$

where the "turbulent conductivity" χ_T can be derived to be (Canuto and Hartke, 1986, Appendix A)

$$\chi_T = (g\alpha\beta)^{-1} \int_{k_0}^\infty [n_s(k') + \nu k'^2] F(k') dk'. \tag{28}$$

Once χ_T is known, the quantity ε, Eq. (16), is easily derived to be

$$\varepsilon = g\alpha\beta\chi_T = S\chi_T\chi^2\Delta^{-4}. \tag{29}$$

Finally, the turbulent kinetic energy (per unit mass) K, defined as

$$K = 1/2\langle u^2\rangle = 1/2\int_{k_0}^\infty F(k') dk' \tag{30}$$

is obtained by setting $k = k_0$ in Eq. (10). The turbulent viscosity can be obtained by means of Eq. (25). Actually, the largest value of ν_t, i.e., $\nu_t(k_0)$ can be obtained by noticing that since at $k = k_0$ the vorticity vanishes [see Eq. (19)], we obtain from Eq. (20)

$$\nu_t \equiv \nu_t(k_0) = k_0^{-2} n_s(k_0), \tag{31}$$

a result first derived in Canuto et al. (1984). Dividing Eq. (20) by k^2 and differentiating, it is easy to show that the wave number k_0 defined so that

$$F(k_0) = 0, \tag{32}$$

corresponding to the point where ν_t is maximum, can be found by solving

$$\frac{d}{dk}(n_s k^{-2}) = 0 \tag{33}$$

Using Eq. (26), we may solve Eq. (33) for k_0. We find

$$k_0\Delta = \pi\sqrt{3/2}. \tag{34}$$

4. Comparison with DIA results

As stated earlier, Kraichnan's DIA (for a detailed presentation, see Leslie, 1973), represents to date the most successful approach to the problem of describing fully developed turbulence starting from the full non-linear Navier-Stokes equations. The approximations contained in arriving at the final DIA equations have been elucidated by Martin et al. (1973). Using the formalism of quantum field theory, they showed that the DIA equations are equivalent to taking the lowest order in the vertex corrections. With that proviso, the equations are (Leslie, 1973)

$$\left(\frac{d}{d\tau} - n_s(k)\right) Q(k, \tau-\sigma) = 2\pi \iint^{\Delta} dp\, dr\, kprb(k,p,r) \left[\int_{-\infty}^{\sigma} ds'\, G(k, \sigma-s')\, Q(p, \tau-s') \times Q(r, \tau-s') - \int_{-\infty}^{\tau} ds'\, G(p, \sigma-s')\, Q(r, \tau-s')\, Q(k, \sigma-s')\right], \tag{35}$$

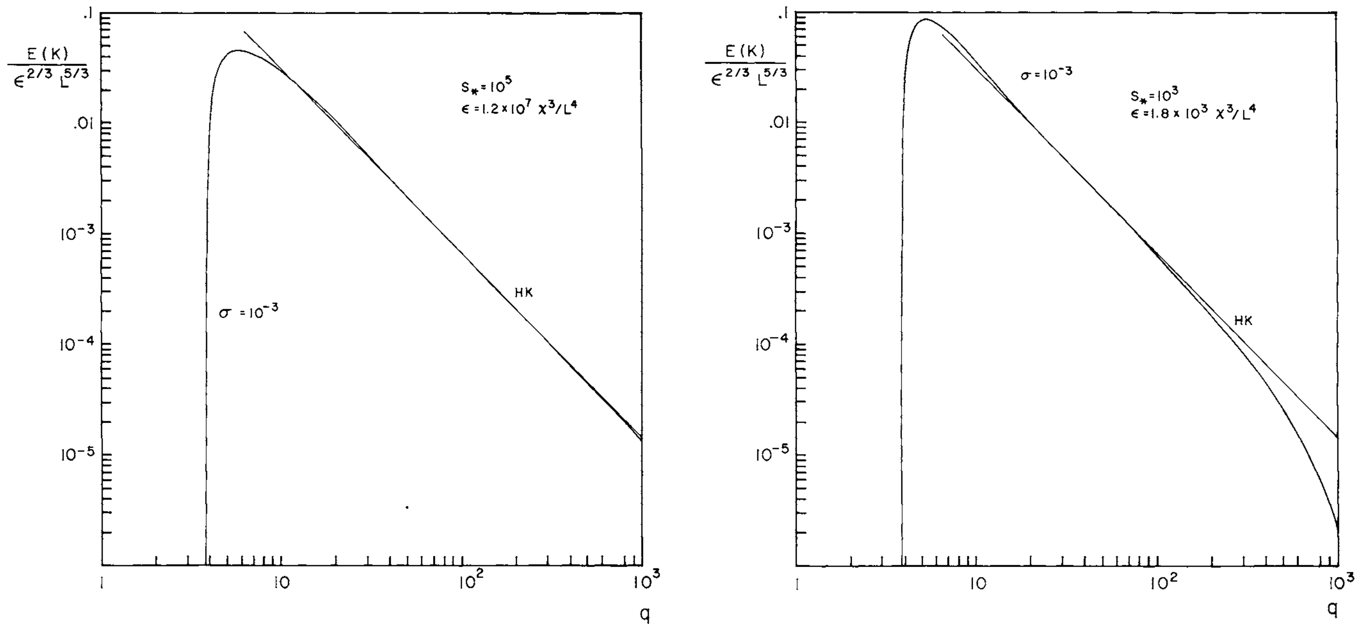

Fig. 3a and b. The energy spectral function $F(k)$ vs. wave number $k\,(k\Delta = q)$ for Prandtl number $\sigma = 10^{-3}$. The quantity S_* is defined as $S_* = (1+\sigma)^{-2} S$. The quantity ε is defined in Eqs. (16). As one can see, the HK (Heisenberg-Kolmogoroff) region is reproduced by the present model

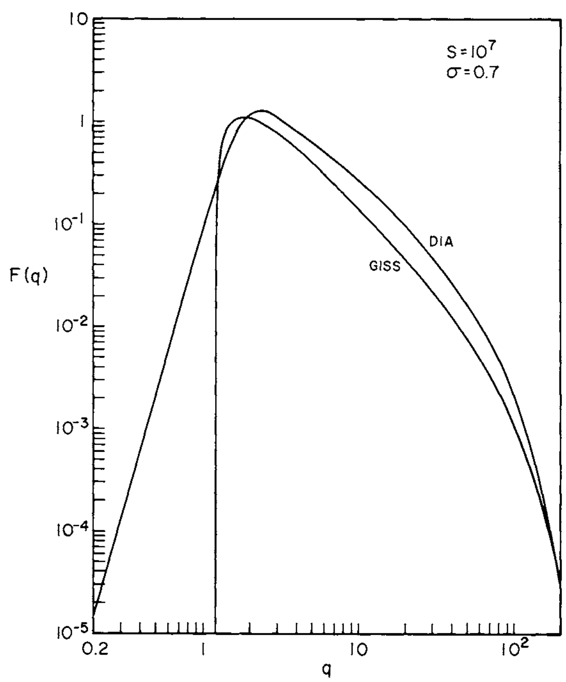

Fig. 4. The energy spectral function derived from the DIA model, Eqs. (35)–(37) and from the present model for the case of a convective instability. The units of $F(k)$ are $2g\alpha\beta D^3\pi^{-3}$, those of k are π/D

$$\left(\frac{d}{d\tau} - n_s(k)\right) Q(k, \tau-\sigma)$$
$$= -2\pi \int\int^{\Delta} dp\, dr\, kprb(k,p,r) \int_{\sigma}^{\tau} ds'\, G(p, \tau-s')\, Q(p, \tau-s')$$
$$\times G(k, \sigma-s') + \delta(\tau-\sigma), \tag{36}$$

where $b(k,p,r)$ is a geometrical coefficient,

$$\frac{1}{2}\langle u^2 \rangle = 4\pi \int_0^{\infty} k^2\, Q(k,0)\, dk \equiv \frac{1}{2} \int_0^{\infty} F(k)\, dk \tag{37}$$

and $G(k,\tau)$ is the response function. We have solved the DIA equations for the growth rate $n_s(k)$ corresponding to a thermal instability, Eq. (26). The resulting $F(k)$ is presented in Fig. 4 together with the $F(k)$ obtained from our model. The agreement is satisfactory, implying that our model must contain the main ingredients of the DIA theory. The fact that at low k's our $F(k)$ is skinnier than the DIA is most likely due to the absence in our model of back-scatter from the small eddies to the large ones. In principle, our "pure cascade" model can be made to accommodate the latter effect without undue complications. However, it must be noted that in the region affected by the back-scatter, $F(k)$ is rather small and therefore the back-scatter may be expected to have little effect on the bulk properties of interest here.

5. The results

Using the $F(k)$ calculated in Sect. 2, we then calculated the convective flux F_c, Eq. (27), the turbulent energy K, Eq. (30), and turbulent viscosity ν_t, Eq. (31) for the case of $\sigma = 0$, as appropriate for most astrophysical scenarios. The results are presented in Table 1 for different values of S. (see also Figs. 5–7). Also listed in parenthesis are the values of χ_T/χ from the Mixing Length Theory [1] (Gough and Weiss, 1976), i.e.

$$\chi_T/\chi = (729/16)\ S^{-1}\,[\sqrt{1+(2/81)\,S} - 1]^3\,. \tag{38}$$

6. Relations between turbulent kinetic energy, turbulent viscosity and energy dissipation rate

Since the results just described have been obtained using a specific, although common, form for the growth rate corresponding to a

[1] In Cox and Giuli (1968) $S \equiv 160\, A^2 (\nabla - \nabla_{ad})$, where A is defined in Eq. (14.99). Equation (38) above, once multiplied by $c_p \varrho \beta \chi$ coincides with Eq. (14.108) of Cox and Giuli (1968)

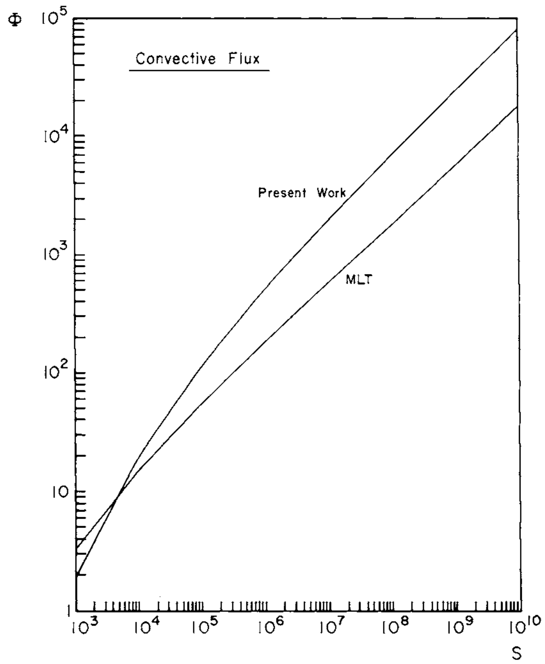

Fig. 5. The dimensionless convective flux, $\Phi = \chi_T/\chi$ Eq. (27), vs. S, for zero Prandtl number. Also shown is the result from the Mixing Length Theory, MLT, Eq. (38) of the text

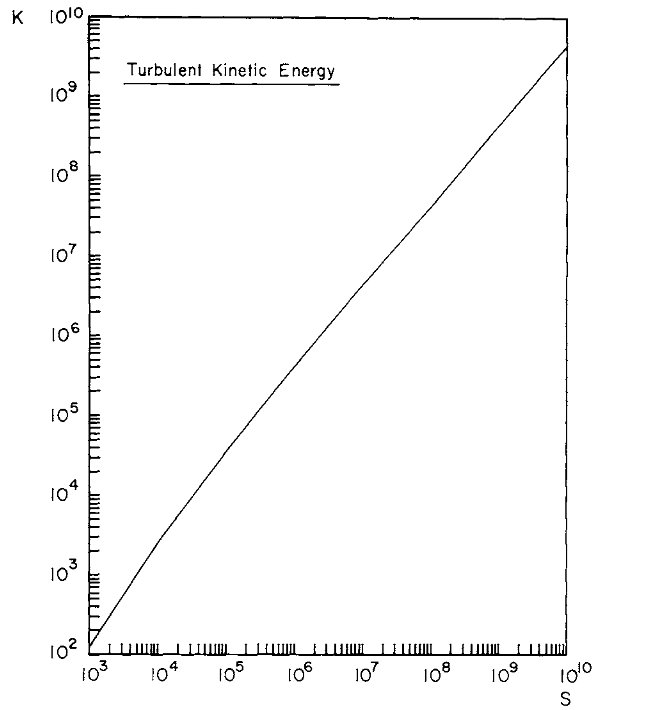

Fig. 6. The turbulent kinetic energy, K, Eq. (30), in units of $(\chi/\Delta)^2$. The result is for zero Prandtl number

convective instability, it might appear that these results are only valid for that case. However, since global turbulent properties are obtained by integrating $F(k)$ and $n_s(k)$ over all k's and since only a few k's may contribute most of the global values (the partial success of the "one mode analysis" like the MLT is an example), one may try to obtain *a set of relations that are independent of the value of the growth rate and therefore of more general validity*. One may expect that a different functional dependence of $n_s(k)$ will yield a similar set of relations with coefficients that are of the same order of magnitude as those corresponding to a convective instability. If the dominant k is called k_* and the corresponding value of the growth rate $n_* \equiv n_s(k_*)$, one may write

$$K = K(n_*), \qquad \varepsilon = \varepsilon(n_*), \qquad \nu_t = \nu_t(n_*). \tag{39}$$

Eliminating n_* among these relations, one may obtain K vs. ε or ν_t vs. K relations. We have found that the best choice for n_* is the value of n_s computed at $k = k_0$. We shall therefore parameterize the turbulent velocity $\langle u^2 \rangle^{1/2} \equiv u$, the energy dissipation rate ε and the turbulent viscosity ν_t as (A and B dimensionless coefficients)

$$u = A \Delta n_*, \tag{40}$$

$$\varepsilon = B \Delta^2 n_*^3, \tag{41}$$

$$\nu_t = k_0^{-2} n_* = \frac{2}{3\pi^2} \Delta^2 n_*, \tag{42}$$

where we have used Eqs. (31) and (34) in Eq. (42). The left hand sides of Eqs. (40)–(42) were computed using the model of turbulence described in Sect. 2 (the results are presented in Table 1) and the right hand sides were computed using Eq. (26). This permitted us to estimate the coefficients A and B. Only if they are reasonably constant over the wide range of values of S of physical interest, would the parameterization (40)–(41) for u and ε be meaningful. The values of A and B are listed in Table 2.

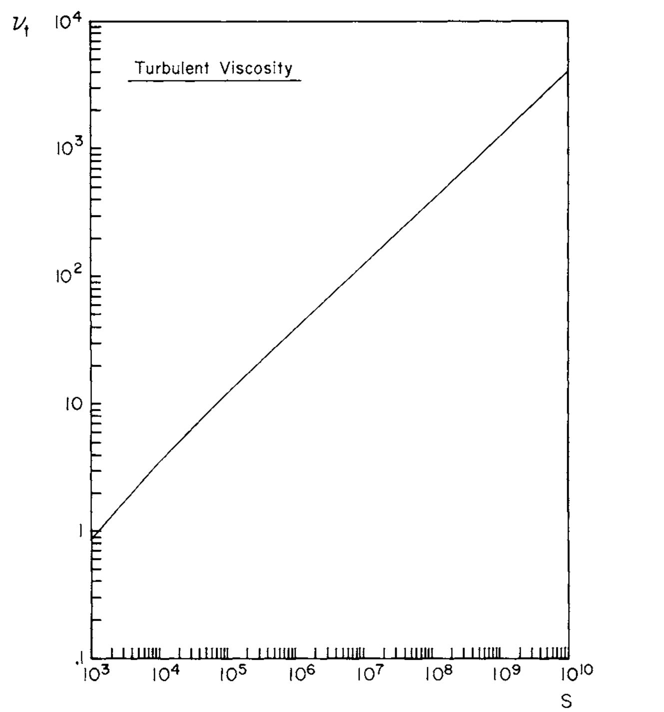

Fig. 7. The turbulent viscosity ν_t, Eq. (31), in units of χ, for the growth rate given in Eq. (26)

It can be safely stated that in spite of the wide range of values of S, the values of A *and* B do remain constant within a very

Table 1. Turbulent kinetic energy, turbulent eddy viscosity and turbulent conductivity [K is in units of $(\chi/\Delta)^2$] vs. S

S	K	ν_t/χ	χ_T/χ
10^3	120	0.83	1.8 (3.1)
10^4	$2.6\ 10^3$	3.4	20 (15)
10^5	$3.5\ 10^4$	12	115 (53)
10^6	$4.0\ 10^5$	39	513 (173)
10^7	$4.3\ 10^6$	123	$2.0\ 10^3$ (560)
10^8	$4.4\ 10^7$	390	$7.1\ 10^3$ ($1.8\ 10^3$)
10^9	$4.5\ 10^8$	$1.2\ 10^3$	$2.4\ 10^4$ ($5.6\ 10^3$)
10^{10}	$4.5\ 10^9$	$3.9\ 10^3$	$8.1\ 10^4$ ($1.8\ 10^4$)

Table 2. The coefficients A and B

S	A	B
10^4	1.4	1.5
10^5	1.5	2.1
10^6	1.6	2.8
10^7	1.6	3.3
10^8	1.6	3.7
10^9	1.6	4.0
10^{10}	1.6	4.2

acceptable margin. We can therefore conclude that the parameterizations (40)–(42) may be meaningfully used.

Let us now elimintate the growth rate $n_* = n_s(k_0)$ from any two of the relations (40)–(42). We obtain the following results:
a) From (41) and (42), we obtain

$$\nu_t = \frac{2}{3\pi^2} B^{-1/3} \varepsilon^{1/3} \Delta^{4/3} \equiv \xi_1 \varepsilon^{1/3} \Delta^{4/3}, \tag{43}$$

where

$$0.042 \leqq \xi_1 \leqq 0.059. \tag{44}$$

b) A second type of relation can be obtained by eliminating the growth rate between (40) and (41), with the result

$$K = \tfrac{1}{2} A^2 B^{-2/3} \Delta^{2/3} \varepsilon^{2/3} \equiv \xi_2 \Delta^{2/3} \varepsilon^{2/3} \tag{45}$$

where

$$0.49 \leqq \xi_2 \leqq 0.75. \tag{46}$$

c) Finally, eliminating Δ between relations (43) and (45) yields the $K-\varepsilon$ relation

$$\nu_t = \xi_3 K^2 \varepsilon^{-1} \tag{47}$$

with $\xi_3 \equiv \xi_1 \xi_2^{-2}$ given by

$$0.087 \leqq \xi_3 \leqq 0.17. \tag{48}$$

7. Comparison with DIA, RNG and Turbulence Modelling

The numerical coefficients ξ_1, ξ_2, and ξ_3 have in the past been evaluated using DIA, (Yoshizawa, 1982), RNG (Renormalization Group, Yakhot and Orszag, 1986) and Turbulence Modeling (Launder and Spalding, 1972; Launder et al., 1975; Reynolds, 1976). It is therefore important that we compare these previous results with the ones obtained from our model.

Table 3. Comparison of the values of ξ_1, ξ_2, and ξ_3 from different models

	Present work	DIA	RNG	Turb. modeling
ξ_1	0.042–0.059	0.053	0.0424	
ξ_2	0.49 –0.75	0.66	0.7116	
ξ_3	0.087–0.17	0.12	0.0837	0.09

Upon inspecting Table 3, two considerations are in order. First, the agreement of our results with those of the other methods is very reassuring. Second, while the evaluation of ξ_1, ξ_2, and ξ_3 using DIA and RNG is rather complex, in our method it is a simple matter.

Furthermore, our method can be easily generalized to include additional physics such as magnetic fields and rotation by appropriately modifying the growth rate. One would then determine the functions ($i = 1, 2, 3$)

$$\xi_i = \xi_i(B, \Omega), \tag{49}$$

where B is the magnetic field and Ω the rotation. The inclusion of additional physics within the DIA and RNG formalisms is a more complicated matter.

8. Four "representations" of the turbulent viscosity

Thus far, we have provided three alternative representations of the turbulent viscosity ν_t, namely

$$\nu_t = k_0^{-2} n_s(k_0), \qquad \nu_t = \xi_1 \varepsilon^{1/3} \Delta^{4/3}, \qquad \nu_t = \xi_3 K^2 \varepsilon^{-1}, \tag{50}$$

where k_0 and Δ are related by Eq. (34).

The first expression yields ν_t in terms of the physical parameters contained in $n_s(k_0)$. From this form, one can tell whether turbulence occurs, and if it does, how it depends on the physics of the problem. Such an expression is useful when one knows precisely the type of instability that causes the turbulent state; alternatively, it may serve the purpose of deciding, among possible candidate instabilities, the one that contributes the most to the turbulent viscosity. For example, consider the case of the primitive solar nebula, where turbulence is suspected to have played a major role but where one cannot be certain of which mechanism dominated. This representation was recently used to "quantify" the importance of a particular instability, convective instability (Cabot et al., 1987). This expression for ν_t is also useful if one wants to understand the effect of external forces like rotation and magnetic fields; their presence can in fact be incorporated into the form of the growth rate $n_s(k)$. In this respect, it is important to recall the work of Chandrasekhar (1961), where the form of the linear $n_s(k)$ is given for different types of instabilities with and without rotation and magnetic fields (convective instabilities, Rayleigh-Taylor instabilities, Kelvin-Helmholtz instabilities, etc.).

The second representation may be useful to evaluate ν_t once one knows the total amount of energy ε (per second per gram) that goes into stirring the medium, without necessarily having to know its wave number dependence. (The length Δ, in both the first and the second expressions, need not be known a priori; it will in fact be the result of the self-consistent solution of the disk structure equations, pressure balance and energy flux). One circumstance of particular interest for solar nebula type disks has direct bearing on this representation for ν_t. After the original molecular cloud collapsed into a rapidly rotating disk with most of the gas following "Keplerian grooves", one may be left with an "excess" energy ε that may disturb the orderly Keplerian motion, thus leading to a turbulent state (Terebey et al., 1984). In this case, the second representation may be useful in quantifying the amount of turbulent viscosity generated by this excess energy.

The third representation may be useful if one knows the turbulent kinetic energy; the latter may be calculated, for example, from mearurements of spectral line broadening.

There is, however, one more representation that we could like to derive. Suppose one wants to express the turbulent viscosity not in terms of the properties of the turbulent fluid itself, as the last two representations in (50) do, but in the same spirit of the first representation, i.e., in terms of the properties of the mean flow (*if* there is one), an instability of which may be the cause of turbulence. In the case of a shear in the "mean fow", one may want to employ the strain-rate rate tensor

$$S_{ij} = \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \tag{51}$$

or perhaps more conveniently the scalar

$$\bar{S} = (S_{ij} S_{ij})^{1/2}. \tag{52}$$

In 1963, Smagorinsky suggested the following representation for ν_T,

$$\nu_T = C \Delta^2 \bar{S}, \tag{53}$$

where C, known as Smagorinsky's constant, has been traditionally fitted to the data. In what follows, we shall derive (53) and calculate the value of C.

The derivation is based on two steps. First, one employs the relation

$$\varepsilon = 1/2\, \tau_{ij} S_{ij}. \tag{54}$$

This is an exact relation that has been derived several times in the literature (Lilly, 1967; Hinze, 1975; Monin and Yaglom, 1971; Deissler, 1984; Stewart, 1976). The ingredients of the derivation are as follows: consider the Navier-Stokes equation for the total velocity field $\boldsymbol{u}$; divide $\boldsymbol{u}$ into mean flow $\boldsymbol{U}$ and turbulent flow $\boldsymbol{v}$; derive the equation for $\boldsymbol{v}$ and then that for v^2. *Integrate the latter over the volume of the system.* The final result is Eq. (54), where the tensor τ_{ij} is defined in Eq. (5).

Next, one needs an expression relating τ_{ij} to ε and S_{ij}. Such a relation is given by Eq. (6). Putting together Eqs. (5), (6) and (54), we obtain

$$\varepsilon = 1/2\, \nu_T \bar{S}^2. \tag{55}$$

Eliminating ε between Eqs. (55) and the second of (50), one obtains

$$\nu_T = C \Delta^2 \bar{S} \tag{56}$$

with the Smagorinsky's constant C, given by

$$C = [\tfrac{1}{2} \xi_1^3]^{1/2}. \tag{57}$$

Using the previously derived values of ξ_1, our model predicts

$$6.1\ 10^{-3} \leqq C \leqq 1.01\ 10^{-2}. \tag{58}$$

Table 4. Values of the constant $C\,(\times 10^2)$

Present work	0.61–1.01	
DIA	1.21	(Yoshizawa, 1982)
RNG	0.6	(Yakhot and Orszag, 1986)
Turb. modeling	0.3–0.7	(Deardorff, 1971; Moin and Kim, 1982)

For completeness, we quote in Table 4 the values of C that have been published thus far in the literature.

Like in the case of the constants ξ_1, ξ_2, and ξ_3, the agreement among the DIA, RNG, Turbulence Modeling and our model is reassuring. In the presence of external fields, use of Eq. (49) and (57) would thus give the dependence of C on those fields, i.e.,

$$C = C(B, \Omega). \tag{59}$$

9. The Shakura-Sunyaev α parameter

We have presented four representations for the turbulent viscosity, Eqs. (50) and (56); three of them contain a length Δ that may not be calculable from within the model of turbulence; it must be calculated self-consistently when solving the disk structure equations (the thickness of a disk is in fact determined by the point $u=0$ where the total flux vanishes). In spite of not knowing the value of Δ a priori, we can nevertheless carry out an estimate of the disk parameter α, Eq. (9), for the case when there is a shear in the mean flow,, i.e., when Eq. (56) holds.

Equating Eqs. (9) and (56) yields

$$\alpha = \frac{C \Delta^2 \bar{S}}{c_s H}. \tag{60}$$

Since in disks the largest component of the tensor S_{ij} is $S_{r\varphi}$, where for Keplerian motion

$$S_{r\varphi} = \Omega \frac{d(\ln \Omega)}{d(\ln r)} = -3/2\, \Omega \tag{61}$$

and since the speed of sound c_s is related to Ω and H by $c_s = H\Omega$, we have

$$\alpha = 3/2\, C \left(\frac{\Delta}{H}\right)^2. \tag{62}$$

Since Δ is not expected to be larger than H, we have the upper limit

$$\alpha < 3C/2 \tag{63}$$

or using Table 4,

$$\alpha < 10^{-2} \tag{64}$$

in agreement with results recently obtained by Cabot et al. (1987).

The above calculation of α serves hopefully another purpose, namely that of clarifying the problem of whether ν_t must be computed locally (z-dependent) or globally (z-averaged). Cabot et al. (1987) used the first of the representations (50), which depends on "local values" through the local gravity $g = z\Omega^2$. These authors, however, pointed out that since only the average value of ν_t should be used in the disks equations [see Eq. (65) below], the local value of ν_t was first averaged over the height of the disk and then employed in the disk structure equations. Other authors (Lin et al., 1980), have instead used local values of ν_t. The above analysis, and especially the derivation of (56), show that it is incorrect to use local values of ν_t if, at the same time, one uses Eq. (54). In fact, we have already pointed out that (54) follows only after the turbulence equations have been integrated over the physical volume. Once that process has been carried out, it is clearly inconsistent to try and resurrect local properties. As shown in detail in Cabot et al. (1987), an integration over the volume is a basic ingredient in the derivation of the relation satisfied by the total flux F (Pringle, 1981),

$$\frac{dF}{dz} = \varrho\, \tau_{r\varphi} S_{r\varphi}, \tag{65}$$

used in disks calculations.

In conclusion, whenever one uses (65), one has already committed oneself to an averaging process and therefore, for consistency reasons, one must also use an average ν_t. The derivation of ν_t given above clearly shows that the same averaging process that is implicit in (65) is also at the very basis of Eq. (56) for ν_t.

10. Conclusions

In 1973, Shakura and Sunyaev presented the first detailed analysis of the physics of accretion disks. Their paper influenced and spurred much detailed work on one of the most interesting phenomena in astrophysics.

Regrettably, however, little progress has been made to improve the physics of perhaps the most important parameter in the description of disks: *the turbulent viscosity*. More than a decade after the Shakura-Sunyaev paper, most disk calculations still employ Eq. (9) that contains the unknown parameter α, thus precluding the achievement of the major objective of theoretical computations, that of predicting new phenomena.

Even before the Shakura-Sunyaev paper appeared, a well known model for fully developed turbulence had been worked out (Kraichnan, 1964; Leslie, 1973). Kraichnan's DIA model is however rather complex and that might explain why it has not been used in astrophysics and in particular in accretion disk problems.

The present paper tries to fill the gap between the phenomenological α-model approach which is easy to use but has no predictive power and the DIA model which is fully predictive but hard to use. Clearly, any model that tries to fill the gap must share the advantages of both approaches while avoiding their shortcomings, i.e. it must be simple to use while it must yield results of comparable quality to those of DIA. The model we have proposed here satisfies these criteria.

Using it, we have derived *four* theoretical expressions for the turbulent viscosity. The four representations reflect different physical situations that may give rise to turbulence. The first is given in terms of the physical parameters at a given radius on the disk via the growth rate of the instabilities that generate the turbulence. This representation may also be used to determine if turbulence is indeed generated. The other three representations are valid once turbulence is known to exist and are useful depending on the available data for a given problem. Contrary to Eq. (9), these representations contain no free parameters. Extension of the model to encompass the effects of rotation and/or magnetic fields can also be carried out.

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