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ON THE HYDRODYNAMIC EXPANSION OF A RELATIVISTIC GAS

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ABSTRACT

Exact solutions to the relativistic Euler equations found by Landau that do not postulate a self-similar character are evaluated and presented graphically.

The temperature, energy density, entropy density, and total energy density when plotted against r (the distance from the center) exhibit maxima toward the end of the flow in sharp distinction to the self-similar case where such quantities monotonically decrease from the center outward.

Subject headings: galaxies: structure — hydrodynamics — radio sources: general — relativity

I. INTRODUCTION

Exact solutions in fluid mechanics are rare, and the ones that exist are usually of self-similar nature. When it can be proved that the physical system under consideration admits similarity solutions, it is found that either the partial differential equations describing the motion of the fluid can be converted to ordinary differential equations, or that the order of the equations can be reduced (Zel'dovich and Raizer 1966). At the least, this is the motivation for seeking such solutions. Stanyukovich (1960) and Landau and Lifshitz (1959) have given similarity solutions to the Euler equations in detail. Instanced by an interest in plasmas, Mathews (1971) has extended such similarity solutions to include an external magnetic field. Sedov's (1959) book is devoted to similarity methods, which have been applied to problems ranging from explosions to cosmological expansion.

However, not all problems admit self-similar solutions. Actually, the rather specialized conditions necessary to manufacture these solutions make them inapplicable in many physical situations. In this paper, we shall present an exact solution to the one-dimensional relativistic Euler equations that satisfies certain initial conditions and is not of self-similar nature. These initial conditions have been found to be appropriate, for example, in the description of high-energy cosmic-ray collisions (Belenkij and Landau 1956), and may also be of interest in models of radio galaxies, as well as cosmology. Landau expressed the solution to the relativistic inviscid Euler equations in an implicit integral form that unfortunately makes understanding difficult. Because of the importance of these solutions to a variety of astrophysical phenomena, we have decided to perform a full numerical analysis in order to exhibit the behavior of all the thermodynamic quantities versus time at any given point in the expansion.

The main result may be summarized as follows. At the very beginning of the free expansion, the motion is self-similar. This description ceases to be valid when the entire bulk of the fluid is in motion, because a length characterizing the size of the system is brought into play. With the passage of time, the nonsimilar solution prevails over more and more of the fluid, except for a small region near the leading edge where the fluid borders on the vacuum. The reason is that the leading edge of the self-similar solution moves into the vacuum at the speed of light, but the leading edge of the nonsimilar solution (which is just behind the self-similar solution) approaches c only asymptotically. The distinguishing feature of this problem is that the very bulk of the expansion becomes nonsimilar.

Application of Landau's solution to cosmic-ray high-energy collisions and to extragalactic radio sources is discussed. Since particles are not conserved in Landau's solution, it is necessary that, for radio sources, only particles with thermal energies greatly exceeding their rest mass dominate the flow. For then the exact number of particles involved is irrelevant, even though the chemical potential does not vanish.

II. INITIAL AND BOUNDARY CONDITIONS

The one-dimensional fluid equations simplify considerably when no length-scale can be formed out of the dimensional constants that characterize the problem. Mathews (1971) studied this situation, and as a result the integration of the relativistic one-dimensional magnetohydrodynamic equations yielded temperature, entropy density, and energy density profiles of the type schematically indicated in Figures 1 and 2. His results were subsequently applied to the problem of radio galaxies. However, since the choice of initial conditions did not allow any of the thermodynamic quantities to possess a maximum anywhere along the flow pattern, an external medium had to be postulated which would pile up the expanding material in the so-called hot spots.

It is the purpose of the present paper to indicate that one can obtain an entirely different set of profiles (Figs. 1 and 2) that clearly exhibit a maximum somewhere toward the end of the flow, without the need of an external medium. Such profiles are obtained by

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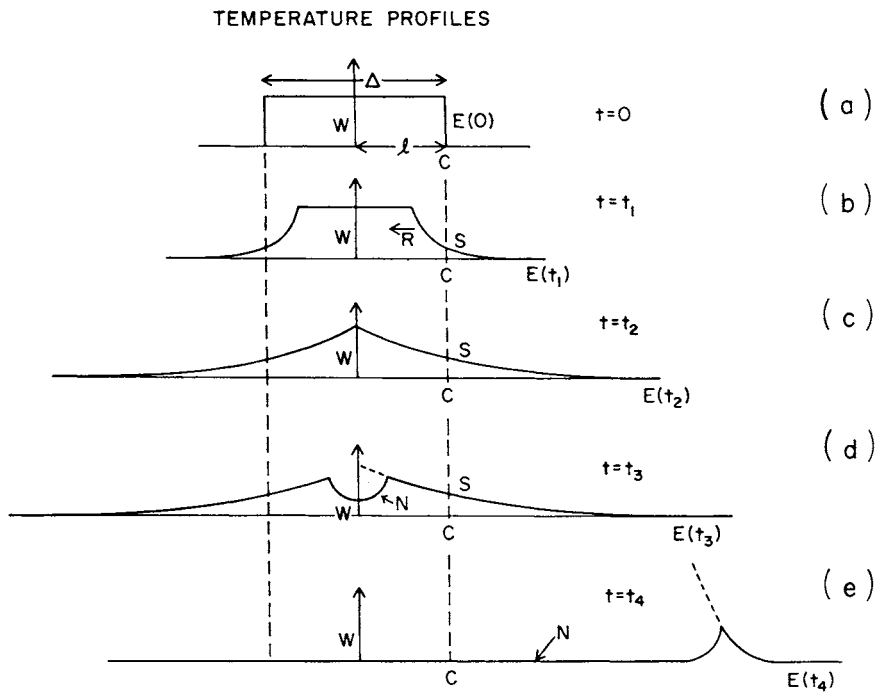

FIG. 1.—Temperature profiles at various times, from the time $t = 0$ when the expansion just begins. (*a*), (*b*) A simple wave S moves out from its center C. (*c*) Rarefaction wave R reaches W, the midpoint of the fluid mass. (*d*), (*e*) A new wave N moves away from W. The dotted line on the cusp of wave is the continuation of the simple wave.

imposing initial conditions different from the ones of Mathews, which we shall now describe in detail.

Let us suppose that the fluid is initially a disk at rest, and let us follow the development in time of the temperature profile (Fig. 1*a*). The face of the disk is perpendicular to the x coordinate, and the thickness is $\Delta = 2l$. We shall investigate the one-dimensional motion along the x-axis. The system is symmetrical about the half-thickness of the disk, and we shall for convenience locate the plane of symmetry at $x = -l$. In the next stage, the edge of the blob, $E(t)$, that abuts the vacuum moves out. The subsequent motion is that of a simple wave centered at the leading edge of the blob, Figure 1*b*. The leading edge $E(t)$ advances at the speed of light c, whereas the trailing edge of the simple wave (rarefaction wave, R), traveling at the speed of sound, soon reaches the midpoint of the blob, or wall, W (Fig. 1*c*) at a time $t_2 = l3^{1/2}/c$, after which time the condition that the gas is at rest at this point becomes important. A new wave N, propagating away from the wall, satisfies two important conditions (Fig. 1*d*). First, the gas is at rest at the wall at all times, and second, the motion merges into the simple wave on the vacuum side.

The simple wave is, of course, Mathews's solution, and we see that it is not the whole story. There is the exact solution N which beginning at the time t_2 when the rarefaction wave reaches the wall, makes its presence felt more and more, until at later times, when the leading edge of the N wave has advanced far beyond the original length of the blob, it becomes the predominant wave over the extent of the fluid mass (Fig. 1*e*). At these times, only a thin slice at the leading edge is described by the simple wave. Figures 2*a*–2*e* describe the corresponding velocity profiles. Our results (Figs. 4–10) show that the exact solution gives a vivid picture of hot spots, which are terminal enhancements in density and temperature. Throughout this motion, the energy and entropy should be exact integrals of the motion.

It is interesting to consider the circumstances under which such an initial condition might obtain. One such way is to imagine two identical masses colliding head-on and face-to-face. Since the masses and temperatures of the fluids are the same, it is only necessary to consider one such blob, let us say the one at the right. This blob runs into the other and the fluid is brought to rest immediately at the plane of symmetry, which can be thought of as a wall. What happens next is well known in the theory of fluids. A shock wave propagates away from the wall into the advancing blob (Fig. 3*a*) raising the temperature of the gas behind the shock front and eventually reducing the entire fluid to rest, by which time the shock wave will have reached the tail-end of the advancing blob (Fig. 3*b*). This would be the initial condition that we have prescribed. Even though this example does not exhaust all possibilities, it is certainly of great interest since it occurs in cosmic-ray physics. In fact, where two high-energy protons collide, it is thought that the clouds of pions forming them, contracted along the motion by a factor M/E, give rise to a hot gas of strongly interacting pions that, after colliding, expand under their own pressure.

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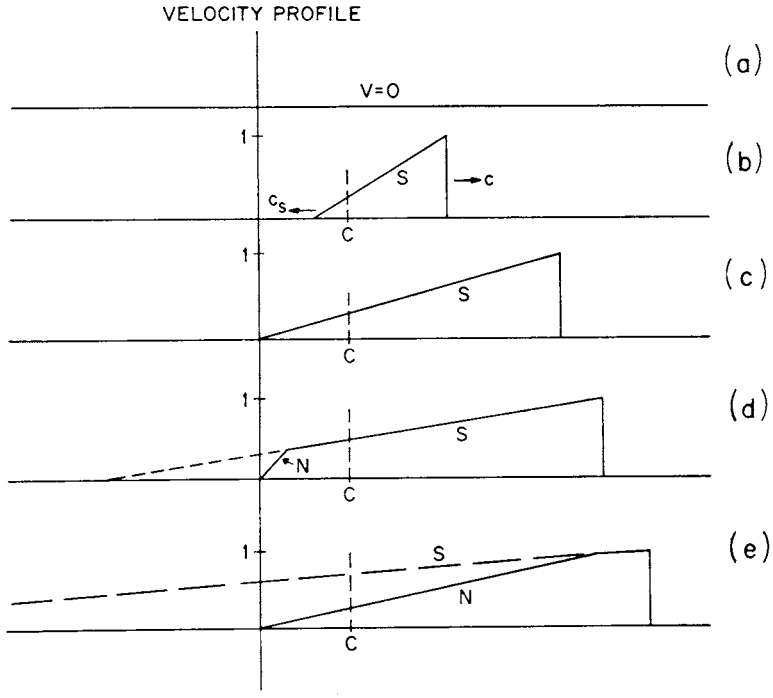

FIG. 2.—Same as Fig. 1, but for velocity profiles. The quantities c and c_s are velocities of light and sound, respectively. Dotted lines indicate continuation of single curve.

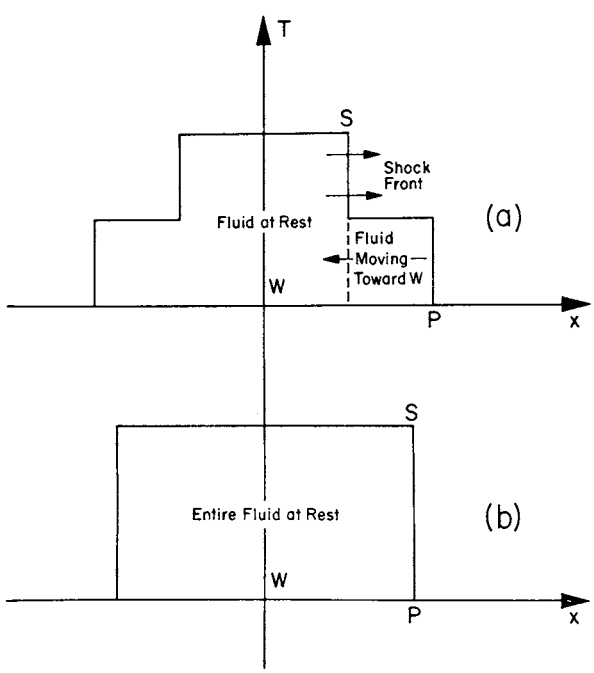

FIG. 3.—Temperature profiles at different times after two blobs of fluid run into each other. The blobs meet in the plane W. (a) The shock front S propagates into the incoming fluid. The fluid between S and W is brought to rest. (b) S reaches P. The entire fluid mass is brought to rest.

The present solution also suggests a model for double radio sources which does not require the action of an intergalactic medium to produce the so-called hot spots. In all the models presented so far, the existence of an intergalactic medium was indeed a condition sine qua non to produce the braking action of an otherwise freely expanding gas. In the present analysis, the crucial role is played by the initial conditions which demand the existence of a hot stationary blob of gas confined to a finite region of space. Such an initial condition can be achieved, for example, but certainly not uniquely, by two colliding blobs of gas brought to rest at the moment of the collision. The central galaxy plays no part in the subsequent dynamics of the expanding gas since there is no continuous supply of energy to the components from this source. The energy of the gas at the initial instant is assumed to be sufficient for the subsequent evolution of the source.

III. THE EQUATIONS

Landau's solution to the inviscid relativistic fluid dynamic equations is valid when the number of particles forming the fluid is not a constant of motion. This is certainly true for the specific example quoted above, but more generally it is valid for any system of particles when the temperature is higher than their rest masses so that copious particle-antiparticle creation and annihilation processes are continuously going on. An example of this is encountered in the early phases of a big bang cosmology when nucleosynthesis occurred. The thermodynamic relations used in what follows will therefore be valid for zero chemical potential.

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We relegate the details of the calculation to the Appendix, and content ourselves with quoting essential steps. Starting from the conservation equation

$$\frac{\partial T^{\nu}{}_{\mu}}{\partial x^{\nu}} = 0\,, \tag{3.1}$$

where

$$T_{\mu\nu} = wu_{\mu}u_{\nu} + pg_{\mu\nu} \tag{3.2}$$

is the energy-momentum density, $w = \epsilon + p$ is the enthalpy per unit volume, and p is the pressure, we get Euler's equation and the internal energy equation.

$$wu^{\mu}\frac{\partial u_{\nu}}{\partial x^{\mu}} = -\frac{\partial p}{\partial x^{\nu}} - u_{\nu}u^{\mu}\frac{\partial p}{\partial x^{\mu}} \tag{3.3}$$

and

$$\frac{\partial(su^{\mu})}{\partial x^{\mu}} = 0\,. \tag{3.4}$$

To derive equation (3.4), we have used the relation

$$Ts = \epsilon + p\,, \tag{3.5}$$

where s is the entropy per unit volume.

In one-dimension, the flow is describable by a velocity potential φ, and equation (3.3) and (3.4) combine to give, after a hodographic transformation (Belenkij and Landau 1956),

$$\frac{\partial^2\chi}{\partial\alpha^2} - c_0{}^2\frac{\partial^2\chi}{\partial y^2} + (c_0{}^2 - 1)\frac{\partial\chi}{\partial y} = 0\,, \tag{3.6}$$

where

$\alpha = \tanh v\,,\quad y = \ln T\,,\quad c_0 = 3^{-1/2}$ (speed of sound),

v being the ordinary velocity and χ the Legendre-transformed velocity potential φ. To these we must add boundary conditions and the set chosen are:

i) $$\frac{\partial\chi}{\partial\alpha} = le^{y} \quad \text{at } \alpha = 0\,, \tag{3.7}$$

ii) $$\chi = 0 \quad \text{at } \alpha = -y/c_0\,, \tag{3.8}$$

iii) a transformation admitted by equation (3.6), i.e.,

$$\chi(\alpha, y) = \chi_1 e^{-y}\,, \tag{3.9}$$

where

$$\chi_1 = \chi_1(z)\,, \qquad z = \alpha + y/c_0\,.$$

The solutions satisfying all these conditions are

$$\frac{T}{l\sqrt{3}} = p_1\cosh\alpha + p_2\sinh\alpha\,,$$

$$\frac{R}{l\sqrt{3}} = p_1\sinh\alpha + p_2\cosh\alpha\,, \tag{3.10}$$

where

$$p_1 = I_0\exp(-2y) - \int_0^{\zeta}\frac{\zeta' I_0(\zeta')\exp[2(\zeta'^2 + \lambda^2)^{1/2}]d\zeta'}{(\zeta'^2 + \lambda^2)^{1/2}}\,,$$

$$3^{1/2}p_2 = \lambda\int_0^{\zeta}\frac{I_1(\zeta')\exp[2(\zeta'^2 + \lambda^2)^{1/2}]d\zeta'}{(\zeta'^2 + \lambda^2)^{1/2}} + \exp(2\lambda)\,, \tag{3.11}$$

and where

$$\lambda = \alpha 3^{-1/2}\,, \tag{3.12}$$

$$y = -(\zeta^2 + \lambda^2)^{1/2}\,, \tag{3.13}$$

I_0, I_1 being the modified Bessel functions of the first kind.

IV. RESULTS AND CONCLUSIONS

In Figures 4–10 we display the velocity v/c, temperature T/T_0, energy density ϵ/ϵ_0, entropy density s/s_0, and total energy density E/E_0, where

$$\frac{\epsilon}{\epsilon_0} = \left(\frac{T}{T_0}\right)^4(1 - \beta^2)^{-1}\,,$$

$$\frac{s}{s_0} = \left(\frac{T}{T_0}\right)^3(1 - \beta^2)^{-1/2}\,,$$

$$\frac{E}{E_0} = \left(\frac{T}{T_0}\right)^4(+\tfrac{1}{3}\beta^2)(1 - \beta^2)^{-1}\,,$$

$$\beta = v/c\,,$$

versus distance from the wall for different times. The unspecified initial values are here indicated by T_0, ϵ_0, s_0, E_0. The position R is measured in units of $3^{1/2}l$ and the time in units of $3^{1/2}l/c$.

The features that we have qualitatively discussed in § II concerning the hot spots are borne out by these profiles. The integrals of the total energy density E/E_0 and entropy density s/s_0 over the fluid, that is, the total energy and entropy, are conserved quantities. Their values are

$$\int EdR = 0.9873E_0\,, \quad \int sdR = 1.0045s_0\,.$$

On the basis of our calculations and results, we enact the following development: as the gas expands into the vacuum and cools adiabatically, the temperature, energy density, and entropy density all become peaked about the wave front. This has been achieved without the aid of a confining medium or magnetic field.

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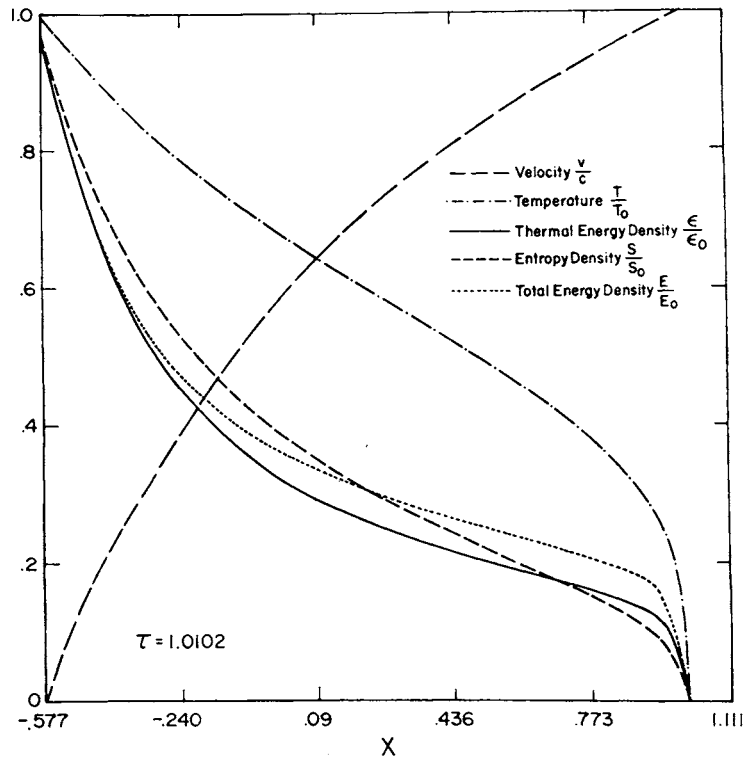

FIG. 4.—Velocity, temperature, thermal energy density, entropy density, and total energy density plotted against the dimensionless coordinate x ($x = R(t)/3^{1/3}l$) at dimensionless time $\tau = 1.0102$ ($\tau = ct/3^{1/2}l$).

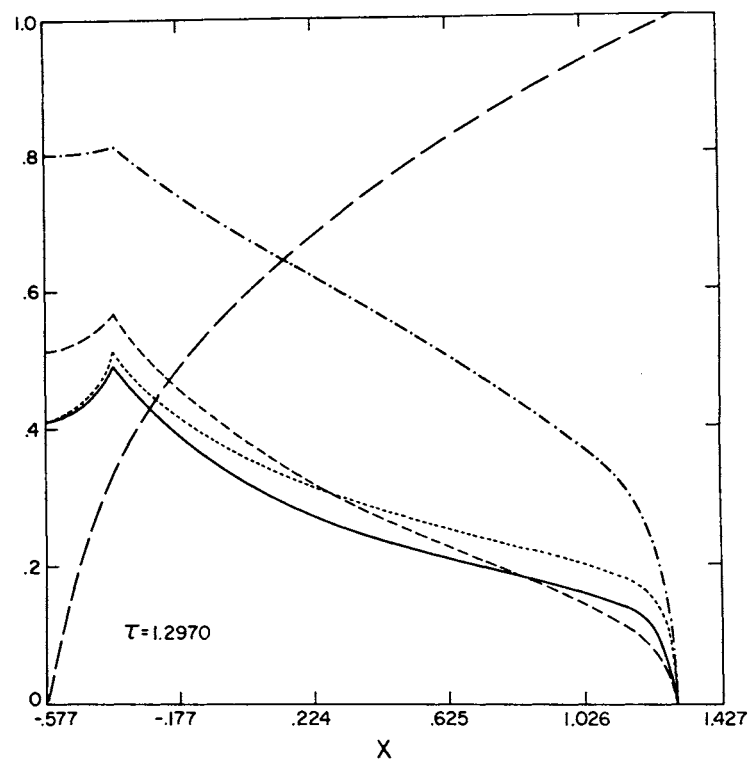

FIG. 5.—Same as Fig. 4, at time $\tau = 1.2970$

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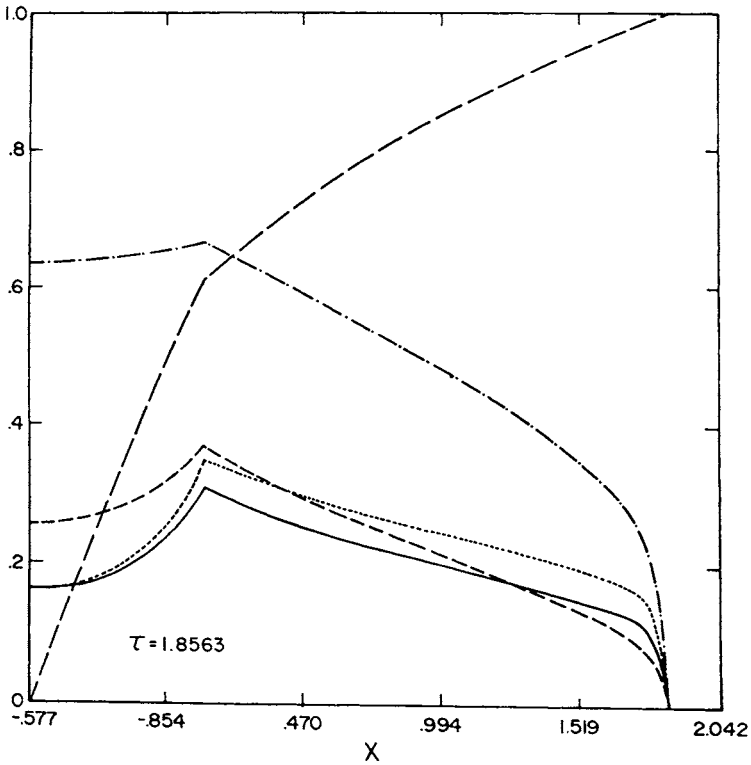

FIG. 6.—Same as Fig. 4, at time $\tau = 1.8563$

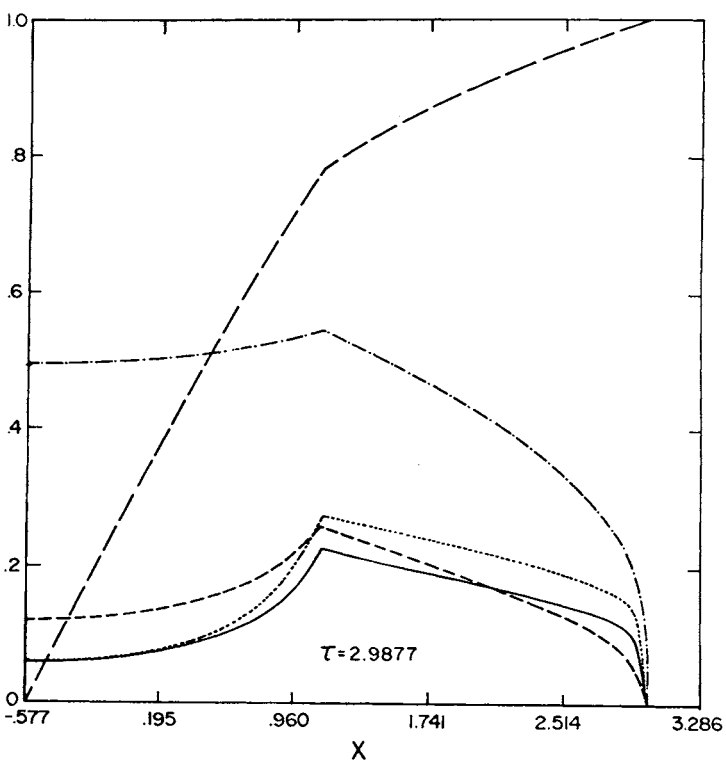

FIG. 7.—Same as Fig. 4, at time $\tau = 2.9877$

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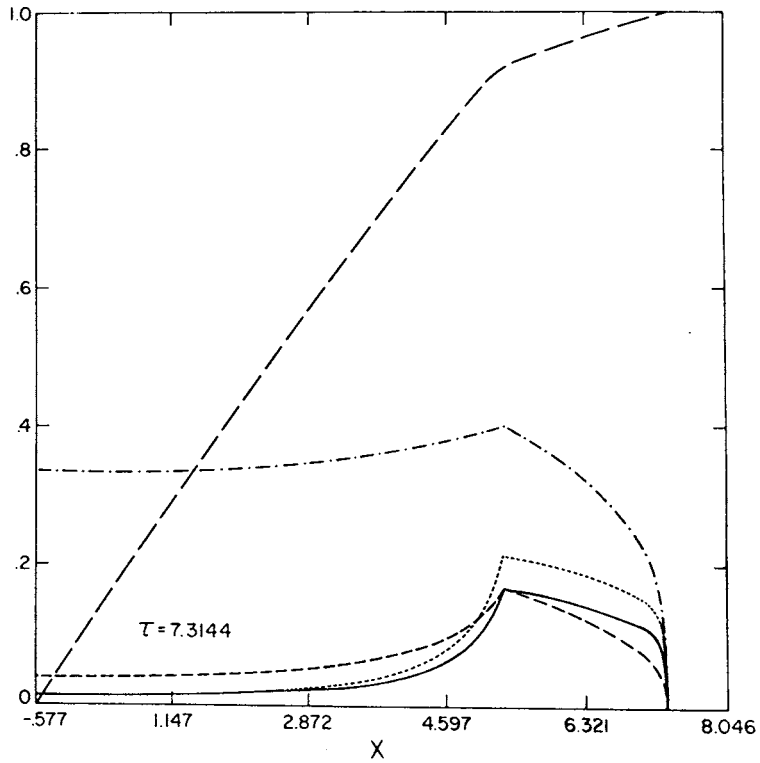

FIG. 8.—Same as Fig. 4, at time $\tau = 7.3144$

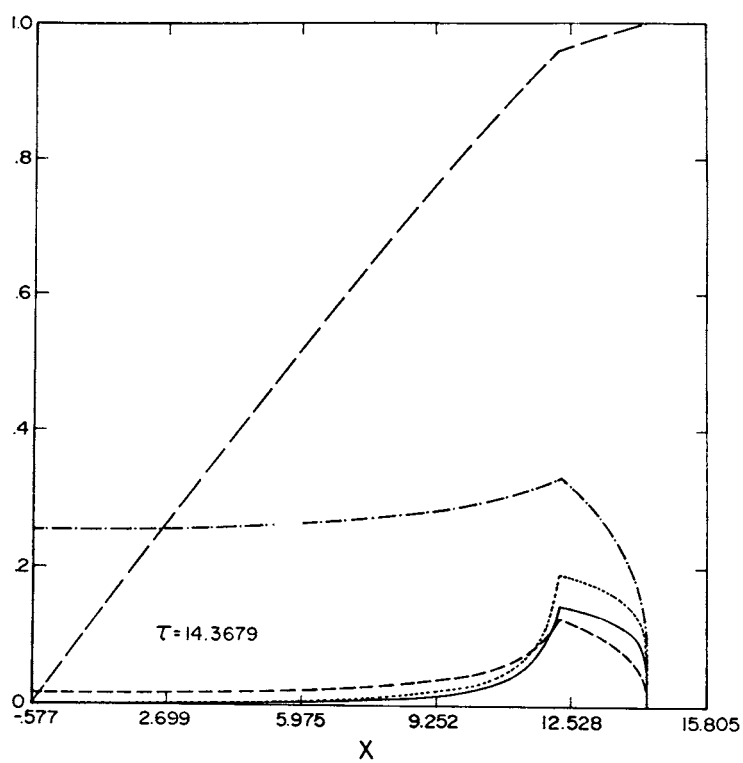

FIG. 9.—Same as Fig. 4, at time $\tau = 14.3679$

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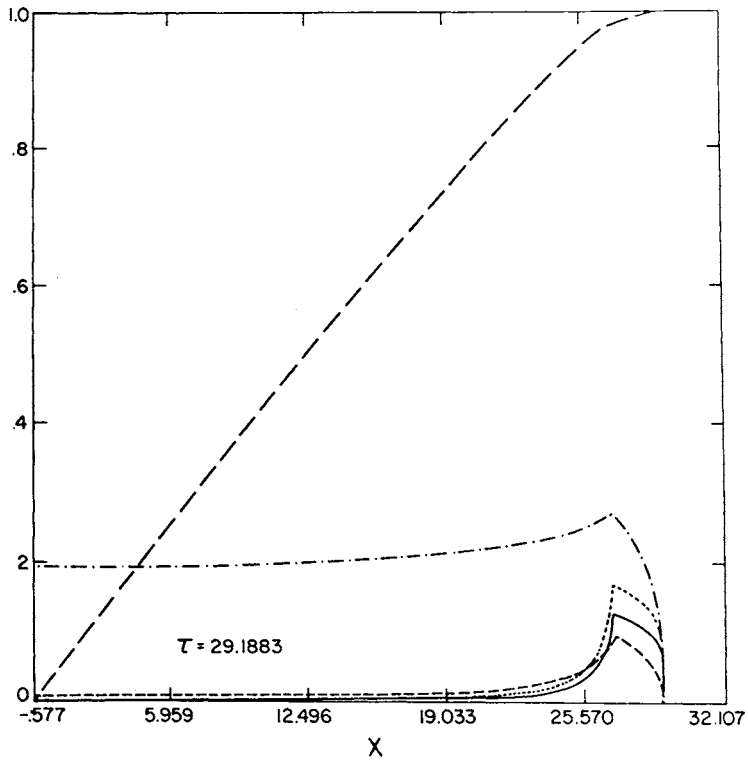

FIG. 10.—Same as Fig. 4, at time $\tau = 29.1883$

APPENDIX

Here we shall consider the relativistic expansion of a gaseous disk, by first writing the equations of motion in their general form

$$\frac{\partial T\mu^{\nu}}{\partial x^{\nu}} = 0\,, \tag{A1}$$

where

$$T_{\mu\nu} = wu_{\mu}u_{\nu} + pg_{\mu\nu} \tag{A2}$$

is the energy-momentum tensor and

$$w = \epsilon + p \tag{A3}$$

is the enthalpy per unit volume.

Our treatment follows closely that of Belenkij and Landau (1956). The relativistic internal energy ϵ and entropy ϵ are assumed to satisfy the following Gibbs-Duhem relation when the chemical potentials of the particles are zero (nonconservation of particles)

$$w = \epsilon + p = Ts\,. \tag{A4}$$

By the thermodynamic law $dE = TdS - pdV$, where $E = \epsilon V$, $S = sV$, we get

$$dp = sdT\,, \qquad d\epsilon = Tds\,. \tag{A5}$$

The speed of sound is by definition

$$c_0{}^2 = \frac{dp}{d\epsilon} = \frac{sdT}{Tds}\,. \tag{A6}$$

The projection of (A1) in a direction perpendicular to u_{μ} is

$$u_{\mu}u^{\lambda}\frac{\partial T_{\lambda}{}^{\nu}}{\partial x^{\nu}} + \frac{\partial T_{\mu}{}^{\nu}}{\partial x^{\nu}} = u_{\mu}u^{\lambda}\frac{\partial T_{\lambda}{}^{\nu}}{\partial x^{\nu}} + \frac{\partial T_{\mu}{}^{\nu}}{\partial x^{\nu}} = 0\,, \tag{A7}$$

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which becomes Euler's equations after substituting (A2),

$$wu^{\mu}\frac{\partial u_{\nu}}{\partial x^{\mu}} = -\frac{\partial p}{\partial x^{\nu}} - u_{\nu}u^{\mu}\frac{\partial p}{\partial x^{\mu}}\,. \tag{A8}$$

The projection of (A1) along u^{μ} is

$$u^{\mu}\frac{\partial T_{\mu}{}^{\nu}}{\partial x^{\nu}} = 0 \tag{A9}$$

or

$$\frac{\partial}{\partial x^{\nu}}(wu^{\nu}) - u^{\nu}\frac{\partial p}{\partial x^{\nu}} = 0\,. \tag{A10}$$

By the use of (A4), (A10) simplifies to

$$\frac{\partial(su^{\mu})}{\partial x^{\mu}} = 0\,, \tag{A11}$$

which is the equation for adiabatic motion.

Let us add a null term $wu^{\mu}\,\partial u_{\mu}/\partial x^{\nu}$ to equation (A8) which then becomes

$$u^{\mu}u_{\nu}\frac{\partial p}{\partial x^{\mu}} + u^{\mu}w\frac{\partial u_{\nu}}{\partial x^{\mu}} + \frac{\partial p}{\partial x^{\nu}} - wu^{\mu}\frac{\partial u_{\mu}}{\partial x^{\nu}} = 0\,. \tag{A12}$$

By using $w = Ts$ (eq. [A4]), we arrive at

$$u^{\mu}\frac{\partial}{\partial x^{\mu}}(Tu_{\nu}) + \frac{\partial T}{\partial x^{\nu}} - u^{\mu}T\frac{\partial u_{\mu}}{\partial x^{\nu}} = u^{\mu}\frac{\partial}{\partial x^{\mu}}(Tu_{\nu}) - u^{\mu}\frac{\partial}{\partial x^{\nu}}(Tu_{\mu}) = u^{\mu}\left[\frac{\partial}{\partial x^{\mu}}(Tu_{\nu}) - \frac{\partial(Tu_{\mu})}{\partial x^{\nu}}\right] = 0\,. \tag{A13}$$

The relativistic version of potential flow is just

$$\frac{\partial(Tu_{\nu})}{\partial x^{\mu}} = \frac{\partial(Tu_{\mu})}{\partial x^{\nu}}\,, \tag{A14}$$

and so Tu_{μ} is just $\partial\varphi/\partial x^{\mu}$, where φ is the velocity potential.

To proceed further, we specialize our equations to one dimension, which is not a bad approximation if the disk is thin. We expect three-dimensional motion to be important at later times (Belenkij and Landau 1956). At any rate, the one-dimensional motion admits of an analytic solution. We have

$$Tu_1 = \frac{\partial\varphi}{\partial x}\,, \qquad Tu_0 = -\frac{\partial\varphi}{\partial t}\,, \tag{A15}$$

or

$$d\varphi = -Tu_0 dt + Tu_1 dx\,. \tag{A16}$$

By a standard Legendre transformation

$$\chi = \varphi - Tu_1 + Tu_0\,, \tag{A17}$$

it follows that

$$t = \frac{\partial\chi}{\partial T}\cosh\alpha - \frac{1}{T}\frac{\partial\chi}{\partial\alpha}\sinh\alpha\,,$$

$$x = \frac{\partial\chi}{\partial T}\sinh\alpha - \frac{1}{T}\frac{\partial\chi}{\partial\alpha}\cosh\alpha\,, \tag{A18}$$

where

$$u_1 \equiv \sinh\alpha\,, \qquad u_0 \equiv \cosh\alpha\,, \tag{A19}$$

and α is the rapidity variable. The adiabatic condition in one dimension is

$$\frac{\partial(su_0)}{\partial t} + \frac{\partial(su_1)}{\partial x} = 0\,. \tag{A20}$$

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By changing variables from t and x to T and α, equation (A20) becomes

$$\frac{\partial(t, x)}{\partial(T, \alpha)}\left[\frac{\partial(s \cosh \alpha, x)}{\partial(t, x)} + \frac{\partial(t, s \sinh \alpha)}{\partial(t, x)}\right] = \frac{\partial(s \sinh \alpha, x)}{\partial(T, \alpha)} - \frac{\partial(s \sinh \alpha, t)}{\partial(T, \alpha)}$$

$$= \frac{1}{s}\left(\frac{\partial s}{\partial T}\right)\left[\frac{\partial}{\partial \alpha}(x \cosh \alpha - t \sinh \alpha) - (x \sinh \alpha - t \cosh \alpha)\right] + \frac{\partial}{\partial T}(t \cosh \alpha - x \sinh \alpha) = 0 . \quad \text{(A21)}$$

By the aid of equations (A18), (A19), and (A6), we get

$$\frac{\partial^2 \chi}{\partial \alpha^2} - c_0{}^2 \frac{\partial^2 \chi}{\partial y^2} + (c_0{}^2 - 1)\frac{\partial \chi}{\partial y} = 0 , \quad \text{(A22)}$$

where

$$y = \ln T . \quad \text{(A23)}$$

Equation (A22) is a linear equation with constants, to which must be added the proper boundary conditions, taken as follows:

i)

$$\alpha = 0 \quad \text{at} \quad x_1 = -l \quad \text{(A24)}$$

or

$$le^y = \frac{\partial \chi}{\partial \alpha} \quad \text{at} \quad \alpha = 0 ; \quad \text{(A25)}$$

i.e., the velocity is zero at the plane of symmetry, taken as $x_1 = -l$.

ii) A progressive wave on the vacuum side,

$$x = t\frac{v - c_0}{1 - vc_0} \quad \text{and} \quad \alpha = -\frac{y}{c_0} , \quad \text{(A26)}$$

where

$$v = u_1/u_0 = \tanh \alpha \quad \text{(A27)}$$

is the ordinary velocity.

In α, y variables, (A26) is

$$\chi = 0 \quad \text{and} \quad \alpha = -y/c_0 . \quad \text{(A28)}$$

Defining $\chi = \chi_1 e^{-y}$ and letting $\chi_1 = \chi_1(z)$, where

$$z = -\alpha - y/c_0 , \quad \text{(A29)}$$

the order of equation (A22) is lowered by one. In what follows, we shall use

$$c_0 = 3^{-1/2} . \quad \text{(A30)}$$

The transformed equation (A22) is

$$\left(6\frac{\partial^2}{\partial \alpha \partial z} - 3\frac{\partial^2}{\partial \alpha^2} - 1\right)\chi_1 = 0 \quad \text{(A31)}$$

with boundary conditions

$$\chi_1 = 0 \quad \text{for} \quad z = 0 , \quad \text{(A32)}$$

$$\frac{\partial \chi_1}{\partial \alpha} - \frac{\partial \chi_1}{\partial z} = l \exp\left(-\frac{2z}{\sqrt{3}}\right) , \quad \text{for} \quad \alpha = 0 . \quad \text{(A33)}$$

By Laplace transformation techniques, it may be shown that

$$\chi = -l3^{1/2}e^y \int_{3^{-1/2}x}^{-y} e^{2y'} I_0[(y'^2 - \tfrac{1}{3}\alpha^2)^{1/2}]dy' . \quad \text{(A34)}$$

The expressions (A18) and (A19), that is,

$$t = e^{-y}\left(\frac{\partial \chi}{\partial y}\cosh \alpha - \frac{\partial \chi}{\partial \alpha}\sinh \alpha\right) , \qquad x = e^{-y}\left(\frac{\partial \chi}{\partial y}\sinh \alpha - \frac{\partial \chi}{\partial \alpha}\cosh \alpha\right) , \quad \text{(A35)}$$

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give the following values for t and x:

$$\frac{t}{l\sqrt{3}} = p_1 \cosh \alpha + p_2 \sinh \alpha \,, \tag{A36}$$

$$x \equiv \frac{R}{l\sqrt{3}} = p_2 \sinh \alpha + p_2 \cosh \alpha \,, \tag{A37}$$

where

$$p_1 = I_0 \exp(-2y) - \int_0^{\zeta} \frac{\zeta' I_0(\zeta') \exp[2(\zeta'^2 + \lambda^2)^{1/2}] d\zeta'}{(\zeta'^2 + \lambda^2)^{1/2}} \,, \tag{A38}$$

$$p_2 = \frac{1}{\sqrt{3}} \left\{ \lambda \int_0^{\zeta} \frac{I_1(\zeta') \exp[2(\zeta'^2 + \lambda^2)^{1/2}] d\zeta'}{(\zeta'^2 + \lambda^2)^{1/2}} + \exp(2\lambda) \right\} , \tag{A39}$$

$$\lambda = 3^{-1/2}\alpha \,, \tag{A40}$$

$$y = -(\zeta^2 + \lambda^2)^{1/2} = \ln(T/T_0) \,. \tag{A41}$$

T_0 is the arbitrary initial temperature, and I_0, I_1 are the modified Bessel functions of the first kind.

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