The Anelastic Approximation for Thermal Convection

D. O. GOUGH

Goddard Institute for Space Studies, New York, N. Y., and Courant Institute of Mathematical Sciences, New York, N. Y.

(Manuscript received 26 September 1968, in revised form 5 December 1968)

ABSTRACT

A formal scale analysis of the equations of motion in a plane parallel atmosphere is made, assuming conditions to be such that relative fluctuations in density and temperature are small. It is found that an energetically consistent set of approximate equations can be derived which preclude the existence of acoustic motions. Such equations can be used to describe subsonic convection or internal gravity waves. Under certain conditions the analysis can be generalized to include vertical pulsations of the atmosphere.

1. Introduction

In theoretical studies of thermal convection it is usual to apply some simplifying approximations to the equations of motion. Perhaps the most widely used are those attributed to Boussinesq (1903), and which Rayleigh (1916) used to study the onset of convective instability. Similar approximations had been used previously by Oberbeck (1879), and their applicability to flow in a thin layer of compressible fluid has been discussed by Jeffreys (1930), for infinitesimal steady motions, and by Spiegel and Veronis (1960). However, since the convective reigons in, for example, stars or the earth's atmosphere are generally not thin, a set of less restrictive assumptions must be sought. To this end Ogura and Charney (1962) approximated the equations governing adiabatic motions of an inviscid fluid by filtering out sound waves, which are not of great meteorological interest and whose presence would require the use of very small time steps in a numerical integration. The resulting equations, which Charney has named "anelastic," had previously been obtained by Batchelor (1953) in a discussion of dynamical similarity. Ogura and Phillips (1962) have derived them by a formal scale analysis and have shown that they result from assuming that the relative range of potential temperature in the convective region is small, and that the time scale of the motions is solely that associated with gravity-driven advection.

In order to study a convective region in its entirety, one must also consider the molecular and radiative transport processes which necessarily occur and which are important at least at the edges of the region. It is the purpose of this investigation to derive equations in anelastic approximation which take these processes into account, and to ascertain under what conditions they are valid. It will be found that the approximate equations can be used to describe convection or internal gravity waves for which the Mach number is small. The strict elimination of acoustic motions to describe circumstances in which their effect is small considerably simplifies both analytical and numerical studies, and thus provides some of the justification for this discussion. In addition, it is hoped that the analysis sheds some light on the balance of the dominant physical processes in a convecting fluid.

The equations are first derived for a plane parallel atmosphere which is static in the mean by a formal scale analysis modeled on that by Ogura and Phillips and a derivation of the Boussinesq equations by Malkus (1964), who draws on techniques used by Mihaljan (1962) for Boussinesq liquids. The analysis is then extended to treat cases in which the mean properties of the atmosphere vary in time, with a view to describing the interaction between convection and radial pulsations in stars.

2. Basic equations

For simplicity, we shall first consider a plane stratified atmosphere, infinite in horizontal extent, with no mean shear, and for which all horizontally averaged quantities are independent of time. It will be assumed that deviations from local thermodynamic equilibrium are sufficiently small so that a temperature T can be defined which is a function of pressure p and density ρ alone. The equation of state can thus take the form

$$\rho=\rho(p,T), \tag{2.1}$$

and the values of all thermodynamic functions are determined by any two state variables. Fluctuations in the gravitational acceleration g arising from the motions will be neglected, so that g is a function only of height. With respect to inertial rectangular coordinates (x_1,x_2,x_3) with x_3 vertical, the equations of motion are:

$$\frac{\partial m_i}{\partial t}+\frac{\partial}{\partial x_k}\left(\frac{m_i m_k}{\rho}\right)=-\frac{\partial p}{\partial x_i}-g\rho\delta_{i3}+\frac{\partial \tau_{ik}}{\partial x_k}, \tag{2.2}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial m_k}{\partial x_k}=0, \quad (2.3)$$

$$\rho\frac{DU}{Dt}-\frac{p}{\rho}\frac{D\rho}{Dt}=\tau_{ik}\frac{\partial}{\partial x_k}\left(\frac{m_i}{\rho}\right)+Q-\frac{\partial F_k}{\partial x_k}, \quad (2.4)$$

where

$$m_i=\rho u_i \quad (2.5)$$

is the momentum density, u_i being the velocity, U is the internal energy per unit mass, Q an internal heat source, and F_i the combined heat flux by conduction and radiation. The operator D/Dt is the material time derivative and δ_{ik} is the Kronecker delta. The viscous stress tensor is given by

$$\tau_{ik}=\mu\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}-\tfrac{2}{3}\delta_{ik}\frac{\partial u_l}{\partial x_l}\right), \quad (2.6)$$

where μ is the coefficient of shear viscosity. It will be assumed that the coordinates are chosen so that there is no net mass flux across any stationary plane, i.e.,

$$\bar{m}_i=0, \quad (2.7)$$

where the overbar denotes a horizontal average. With the help of the thermodynamic relations

$$\rho dU-\frac{p}{\rho}d\rho=\rho dh-dp=\rho C_p dT-\delta dp, \quad (2.8)$$

where h is the specific enthalpy, Eq. (2.4) may be rewritten as

$$\rho C_p\frac{\partial T}{\partial t}-\delta\frac{\partial p}{\partial t}+m_k\left(\frac{\partial h}{\partial x_k}-\frac{1}{\rho}\frac{\partial p}{\partial x_k}\right)=\tau_{ik}\frac{\partial}{\partial x_k}\left(\frac{m_i}{\rho}\right)+Q-\frac{\partial F_k}{\partial x_k}. \quad (2.9)$$

Different substitutions in the time derivatives and the advection terms are made merely for convenience. In these equations C_p is the specific heat at constant pressure and

$$\delta(p,T)=-\left(\frac{\partial \ln\rho}{\partial \ln T}\right)_p. \quad (2.10)$$

It will also be convenient to define

$$\alpha(p,T)=\left(\frac{\partial \ln\rho}{\partial \ln p}\right)_T. \quad (2.11)$$

3. Anelastic scaling

To obtain the anelastic approximation the equations of motion will first be expressed in nondimensional form. An expansion to first order in a small dimensionless parameter then yields the desired equations. Pedagogically, it would be preferable to express this parameter directly in terms of imposed constraints on the system (the boundary conditions, for example) and to expand about a known reference atmosphere. Ogura and Phillips (1962) and Malkus (1964) have shown that this can be done either if the fluid layer is thin or if motions are adiabatic (using the adiabatic atmosphere in hydrostatic equilibrium as the reference atmosphere), but if both these restrictions are simultaneously relaxed it no longer seems possible. Instead, each variable f will be expressed as the sum of a horizontal average and a fluctuation, i.e.,

$$f(x_i,t)=\bar{f}(x_3)+f_1(x_i,t). \quad (3.1)$$

The anelastic approximation results when relative fluctuations in the thermodynamic variables are small. The mean (horizontally averaged) variables can then serve to define the reference atmosphere. The obvious disadvantage of this approach is that one may not know whether the approximation is consistent, for any particular case, until the equations have been solved.

Before one can construct useful nondimensional equations it is necessary to discuss the physical processes one wishes to represent, and to estimate the magnitude of their effects. The situation of interest here is one in which buoyancy provides the main force to drive or inhibit eddy motions with a characteristic velocity w. Consider, therefore, a horizontal layer of the atmosphere of depth d, within which the motions are confined. We shall anticipate that since these motions are subsonic the deviation from hydrostatic equilibrium will not be great; the spatial scale of variation of the mean pressure will therefore be $H=\bar{p}/(g\bar{\rho})$, the pressure scale height, and $\bar{\rho}$ and $\bar{T}$ will vary over a similar distance.[1] Fluctuation quantities and gradients of $\bar{p}$, $\bar{\rho}$ and $\bar{T}$, on the other hand, are intimately associated with the motions, and their scale of variation is at most the shorter of d and H. Fluctuations change due to advection typically in the time τ_c taken to travel this length scale with vertical velocity w. It will be assumed that more rapid variations do not occur. If pressure fluctuations and viscous stresses are ignored, the characteristic velocity can be obtained by equating the associated kinetic energy to the work done by the buoyancy forces. Since, in general, the relative density and temperature fluctuations are similar, one would expect

$$w^2\approx\frac{gdT_1}{\bar{T}}. \quad (3.2)$$

The pressure fluctuations extract energy from the vertical motions to drive the horizontal flow; the work they do across the eddy, therefore, must balance the kinetic energy and

$$\frac{p_1}{\bar{p}}\approx\frac{\bar{\rho}w^2}{\bar{p}}, \quad (3.3)$$

[1] In the ionization zone of an abundant element the scale height of temperature can be somewhat smaller.

$$\approx\left(\frac{d}{H}\right)\frac{T_1}{\bar{T}}. \tag{3.4}$$

Provided d/H is not large these estimates are consistent, but when $d \gg H$ the pressure fluctuations are very efficient at inhibiting the flow and the estimate (3.2) is no longer valid. In general, the resulting relative pressure fluctuation is of the same order as the relative temperature and density fluctuations, the estimate (3.3) still stands, and so

$$w^2 \approx \frac{\bar{p}T_1}{\bar{\rho}\bar{T}} = \frac{gHT_1}{\bar{T}}. \tag{3.5}$$

If thermal diffusion arising directly from the temperature fluctuations is temporarily disregarded, the temperature fluctuations can be estimated by the relative range of potential temperature across the layer:

$$\theta = \int_0^d |\beta|\, dx_3 \approx \int_0^d \left|\frac{\bar{\delta}}{\bar{\rho}\bar{C}_p}\frac{d\bar{p}}{dx_3} - \frac{d\bar{T}}{dx_3}\right| dx_3, \tag{3.6}$$

where β is the superadiabatic temperature gradient of the mean atmosphere,

$$\beta = \frac{-1}{\bar{C}_p}\left(\frac{d\bar{h}}{dx_3} - \frac{1}{\bar{\rho}}\frac{d\bar{p}}{dx_3}\right), \tag{3.7}$$

and the convective heat flux would be

$$F_{c_{ad}} \approx \bar{\rho}\bar{C}_p w\theta. \tag{3.8}$$

This exceeds the actual heat flux F_c by the lateral flux of heat tending to smooth out the temperature fluctuations, so

$$\bar{K}|\nabla T_1| \approx \frac{\bar{K}T_1}{\lambda H} \approx F_{c_{ad}} - F_c \approx \bar{\rho}\bar{C}_p w(\theta - T_1), \tag{3.9}$$

where K is the thermal conductivity and

$$\lambda = \begin{cases} 1, & \text{if } d \geqslant H, \\ d/H, & \text{if } d \leqslant H. \end{cases} \tag{3.10}$$

This provides a second relation between T_1 and w. Finally, in view of Eqs. (2.5) and (2.7), we notice that the horizontally averaged velocity is of order $\rho_1/\bar{\rho}$ of the fluctuations; the same is true of the viscous stress tensor.

Guided by these remarks, we propose the introduction of the following dimensionless variables and operators in terms of a dimensionless parameter ϵ which measures the relative temperature and density fluctuations and which will be assumed small:

$$g = g_s\tilde{g}, \quad \beta = \theta d^{-1}\tilde{\beta}, \quad T = T_s(\tilde{T} + \epsilon T'), \quad \rho = \rho_s(\tilde{\rho} + \epsilon\rho'),$$
$$p = g_s\rho_s H_s(\tilde{p} + \epsilon\lambda p'), \quad C_p = C_{ps}(\tilde{C}_p + \epsilon C_p'),$$

(and similar expressions for h, α, δ, μ and Q), and

$$m_i = \rho_s\sqrt{\epsilon g_s \lambda H_s}\, m_i', \quad \tau_{ik} = \mu_s\sqrt{\frac{\epsilon g_s}{\lambda H_s}}(\epsilon\tilde{\tau}_{ik} + \tau_{ik}'),$$

$$F_i = \frac{K_s T_s}{\lambda H_s}(\lambda\tilde{F}_i + \epsilon F_i')$$

$$\frac{\partial}{\partial x_i} = \begin{cases} H_s^{-1}\dfrac{\partial}{\partial x_i'}, & \text{operating on } \tilde{p},\ \tilde{\rho},\ \tilde{T}, \\[2ex] \lambda^{-1}H_s^{-1}\dfrac{\partial}{\partial x_i'}, & \text{otherwise} \end{cases}$$

$$\frac{\partial}{\partial t} = \tau_c^{-1}\frac{\partial}{\partial t'} = \sqrt{\frac{\epsilon g_s}{\lambda H_s}}\frac{\partial}{\partial t'},$$

where the subscript s denotes a constant characteristic value, the tilde labels horizontally averaged dimensionless variables and the prime labels fluctuations. The parameter λ is now regarded as a constant, defined as in Eq. (3.10) but with H replaced by H_s. The flux F_i has been scaled assuming it to be proportional to the temperature gradient. From Eqs. (3.9), and (3.2) or (3.5), whichever applies, ϵ can be defined in terms of θ by

$$\epsilon = \psi(S)\frac{\theta}{T_s} \equiv \tfrac{1}{4}S^{-1}(\sqrt{4S+1} - 1)^2\,\frac{\theta}{T_s}, \tag{3.11}$$

where

$$S = \frac{g_s(1/T_s)\theta\lambda^3 H_s^3}{[K_s/(\rho_s C_{ps})]^2}. \tag{3.12}$$

In addition to ϵ, S and λ, we can construct the following independent dimensionless parameters which characterize the atmosphere:

$$C = \frac{\rho_s C_{ps} T_s}{\delta_s p_s}, \quad D = \frac{h_s\rho_s}{p_s}, \quad q = \frac{Q_s\tau_c}{\rho_s C_{ps} T_s}, \quad \sigma = \frac{\mu_s C_{ps}}{K_s}.$$

The first, C, is a dimensionless specific heat, D measures the resilience of the gas [and is equal to $\gamma/(\gamma-1)$ for a perfect single-phase gas], q measures the heat source and σ is the Prandtl number. The parameter S is the product of the Prandtl number and a Rayleigh number based on the shorter of d and H_s. We notice from Eqs. (3.2) and (3.5) that ϵ also estimates an upper bound to the square of the Mach number of the flow.

In terms of these dimensionless quantities, the equations of motion (2.2), (2.3), (2.9) become

$$\epsilon\left[\frac{\partial m_i'}{\partial t'} + \frac{\partial}{\partial x_k'}\left(\frac{m_i' m_k'}{\tilde{\rho} + \epsilon\rho'}\right)\right] = -\frac{\partial}{\partial x_i'}(\tilde{p} + \epsilon p') - \tilde{g}(\tilde{\rho} + \epsilon\rho')\delta_{i3}$$
$$+ \epsilon\sigma(\psi S)^{-\frac{1}{2}}\frac{\partial}{\partial x_k'}(\epsilon\tilde{\tau}_{ik} + \tau_{ik}'), \tag{3.13}$$

$$\epsilon\frac{\partial\rho'}{\partial t'}+\frac{\partial m_k'}{\partial x_k'}=0, \tag{3.14}$$

$$\epsilon\left[(\tilde{\rho}+\epsilon\rho')(\tilde{C}_p+\epsilon C_p')\frac{\partial T'}{\partial t'}-\lambda C^{-1}(\tilde{\delta}+\epsilon\delta')\frac{\partial p'}{\partial t'}\right]$$
$$+(\delta_s C)^{-1}m_k'\left[D\frac{\partial}{\partial x_k'}(\lambda\tilde{h}+\epsilon h')-\frac{\lambda}{\tilde{\rho}+\epsilon\rho'}\frac{\partial}{\partial x_k'}(\tilde{p}+\epsilon p')\right]$$
$$=\epsilon\sigma\lambda(\delta_s C)^{-1}(\psi S)^{-\frac{1}{2}}(\epsilon\tilde{\tau}_{ik}+\tau_{ik}')\frac{\partial}{\partial x_k'}\left(\frac{m_i'}{\tilde{\rho}+\epsilon\rho'}\right)$$
$$+q(\tilde{Q}+\epsilon Q')-(\psi S)^{-\frac{1}{2}}\frac{\partial}{\partial x_k'}(\lambda\tilde{F}_k+\epsilon F_k'). \tag{3.15}$$

The equation of state may be expanded about the mean pressure and temperature to yield

$$\tilde{\rho}+\epsilon\rho'=\rho(\tilde{p},\tilde{T})\left\{1+\epsilon\lambda\alpha(\tilde{p},\tilde{T})\frac{p'}{\tilde{p}}-\epsilon\delta(\tilde{p},\tilde{T})\frac{T'}{\tilde{T}}+O(\epsilon^2)\right\}. \tag{3.16}$$

Similar expansions can be made for the other thermodynamic functions. It follows from Eq. (3.13) that

$$\frac{1}{\tilde{\rho}+\epsilon\rho'}\frac{d\tilde{p}}{dx_3'}=\frac{1}{\tilde{\rho}}\left(\frac{d\tilde{p}}{dx_3'}+\epsilon\tilde{g}\rho'\right)+O(\epsilon^2).$$

Eq. (3.15) can thus be rewritten as

$$\epsilon\left[(\tilde{\rho}+\epsilon\rho')(\tilde{C}_p+\epsilon C_p')\frac{\partial T'}{\partial t'}-\lambda C^{-1}(\tilde{\delta}+\epsilon\delta')\frac{\partial p'}{\partial t'}\right]$$
$$+\epsilon\lambda\psi^{-1}(H_s/d)\tilde{C}_p\tilde{\beta}m_3'-\epsilon\lambda(\delta_s C)^{-1}\tilde{g}\frac{\rho'}{\tilde{\rho}}m_3'$$
$$+\epsilon(\delta_s C)^{-1}m_k'\left(D\frac{\partial h'}{\partial x_k'}-\frac{\lambda}{\tilde{\rho}}\frac{\partial p'}{\partial x_k'}\right)$$
$$=\epsilon\sigma\lambda(\delta_s C)^{-1}(\psi S)^{-\frac{1}{2}}(\epsilon\tilde{\tau}_{ik}+\tau_{ik}')\frac{\partial}{\partial x_k'}\left(\frac{m_i'}{\tilde{\rho}+\epsilon\rho'}\right)$$
$$+q(\tilde{Q}+\epsilon Q')-(\psi S)^{-\frac{1}{2}}\frac{\partial}{\partial x_k'}(\lambda\tilde{F}_k+\epsilon F_k')+O(\epsilon^2). \tag{3.17}$$

4. The anelastic approximation

The anelastic approximation results from expanding the mean and fluctuating parts of Eqs. (3.13), (3.14), (3.16) and (3.17) in powers of ϵ and retaining terms up to the first order. For the mean equations we set for each variable f

$$\tilde{f}=\tilde{f}_0+\epsilon\tilde{f}_1+\epsilon^2\tilde{f}_2+\cdots, \tag{4.1}$$

$$f'=f_0'+\epsilon f_1'+\epsilon^2 f_2'+\cdots. \tag{4.2}$$

The zero- and first-order momentum and energy equations are then

$$\frac{d\tilde{p}_0}{dx_3'}=-\tilde{g}\tilde{\rho}_0, \tag{4.3}$$

$$q\tilde{Q}_0+\lambda(\psi S)^{-\frac{1}{2}}\frac{d\tilde{F}_{03}}{dx_3'}=0, \tag{4.4}$$

$$\frac{d}{dx_3'}\left(\frac{\overline{m_{03}'m_{0i}'}}{\tilde{\rho}_0}\right)=0, \tag{4.5}$$

$$\frac{d}{dx_3'}\left(\tilde{p}_1+\frac{\overline{m_{03}'^2}}{\tilde{\rho}_0}\right)=-\tilde{g}\tilde{\rho}_1, \tag{4.6}$$

$$-\lambda(\delta_s C)^{-1}\tilde{g}\frac{\overline{\rho_0'm_{03}'}}{\tilde{\rho}_0}+\overline{(\delta_s C)^{-1}m_{0k}'\left(D\frac{\partial h_0'}{\partial x_k'}-\frac{\lambda}{\tilde{\rho}_0}\frac{\partial p_0'}{\partial x_k'}\right)}$$
$$=\sigma\lambda(\delta_s C)^{-1}(\psi S)^{-\frac{1}{2}}\overline{\tau_{0ik}'\frac{\partial}{\partial x_k'}\left(\frac{m_{0i}'}{\tilde{\rho}_0}\right)}+q\tilde{Q}_1-\lambda(\psi S)^{-\frac{1}{2}}\frac{d\tilde{F}_{13}}{dx_3'}. \tag{4.7}$$

The equation of state may be written

$$\tilde{\rho}=\rho(\tilde{p},\tilde{T}), \tag{4.8}$$

to this order, and the other thermodynamic functions may be similarly evaluated. The zero-order viscous stress tensor is

$$\tau_{0ik}'=\tilde{\mu}_0\left[\frac{\partial}{\partial x_i'}\left(\frac{m_{0k}'}{\tilde{\rho}_0}\right)+\frac{\partial}{\partial x_k'}\left(\frac{m_{0i}'}{\tilde{\rho}_0}\right)-\tfrac{2}{3}\delta_{ik}\frac{\partial}{\partial x_l'}\left(\frac{m_{0l}'}{\tilde{\rho}_0}\right)\right]. \tag{4.9}$$

Fluctuating quantities appear in these equations only in zero order. Equations determining them can be obtained by subtracting from the full nondimensional equations their horizontal averages, introducing the expansion (4.2) for the fluctuations and retaining only terms of leading order in ϵ. There results

$$\frac{\partial m_{0i}'}{\partial t'}+\frac{\partial}{\partial x_k'}\left(\frac{m_{0i}'m_{0k}'}{\tilde{\rho}}\right)-\frac{d}{dx_3'}\left(\frac{\overline{m_{03}'^2}}{\tilde{\rho}}\right)\delta_{i3}$$
$$=-\frac{\partial p_0'}{\partial x_i'}-\tilde{g}\rho_0'\delta_{i3}+\sigma(\psi S)^{-\frac{1}{2}}\frac{\partial\tau_{0ik}'}{\partial x_k'}, \tag{4.10}$$

$$\frac{\partial m_{0k}'}{\partial x_k'}=0, \tag{4.11}$$

$$\begin{aligned}\tilde{\rho}\tilde{C}_p\frac{\partial T_0'}{\partial t'}-\lambda C^{-1}\tilde{\delta}\frac{\partial p_0'}{\partial t'}-\frac{\lambda H_s}{\psi d}\tilde{C}_p\tilde{\beta}m_{03}' \\ -\lambda(\delta_s C)^{-1}\frac{\tilde{g}}{\tilde{\rho}}(\rho_0'm_{03}'-\overline{\rho_0'm_{03}'}) \\ +(\delta_s C)^{-1}\Big[D\Big(m_{0k}'\frac{\partial h_0'}{\partial x_k'}-\overline{m_{0k}'\frac{\partial h_0'}{\partial x_k'}}\Big) \\ -\frac{\lambda}{\tilde{\rho}}\Big(m_{0k}'\frac{\partial p_0'}{\partial x_k'}-\overline{m_{0k}'\frac{\partial p_0'}{\partial x_k'}}\Big)\Big] \\ =\sigma\lambda(\delta_s C)^{-1}(\psi S)^{-\frac{1}{2}}\Big[\tau_{0ik}'\frac{\partial}{\partial x_k'}\Big(\frac{m_{0i}'}{\tilde{\rho}}\Big)-\overline{\tau_{0ik}'\frac{\partial}{\partial x_k'}\Big(\frac{m_{0i}'}{\tilde{\rho}}\Big)}\Big] \\ +qQ_0'-(\psi S)^{-\frac{1}{2}}\frac{\partial F_{0k}'}{\partial x_k'}.\end{aligned} \tag{4.12}$$

The equation of state is

$$\frac{\rho_0'}{\tilde{\rho}}=\lambda\alpha(\tilde{p},\tilde{T})\frac{p_0'}{\tilde{p}}-\delta(\tilde{p},\tilde{T})\frac{T_0'}{\tilde{T}}, \tag{4.13}$$

and fluctuations in the other thermodynamic variables satisfy similar equations. The zero-order velocity is given by

$$u_{0i}'=\tilde{\rho}^{-1}m_{0i}' \tag{4.14}$$

in obvious units.

Restoring dimensions and combining the two orders of the mean equations and the fluctuating equations yields the desired anelastic equations:

$$\frac{\partial m_i}{\partial t}+\frac{\partial}{\partial x_k}\Big(\frac{m_i m_k}{\bar{\rho}}\Big)=-\frac{\partial}{\partial x_i}(\bar{p}+p_1)-g(\bar{\rho}+\rho_1)\delta_{i3}+\frac{\partial \tau_{ik}}{\partial x_k}, \tag{4.15}$$

$$\frac{\partial m_k}{\partial x_k}=0, \tag{4.16}$$

$$\bar{\rho}\bar{C}_p\frac{\partial T_1}{\partial t}-\bar{\delta}\frac{\partial p_1}{\partial t}-\bar{C}_p\beta m_3+m_k\Big(\frac{\partial h_1}{\partial x_k}-\frac{1}{\bar{\rho}}\frac{\partial p_1}{\partial x_k}\Big)-\frac{g}{\bar{\rho}}\rho_1 m_3 =\tau_{ik}\frac{\partial}{\partial x_k}\Big(\frac{m_i}{\bar{\rho}}\Big)+\bar{Q}+Q_1-\frac{\partial}{\partial x_k}(\bar{F}_k+F_{1k}), \tag{4.17}$$

$$\bar{\rho}=\rho(\bar{p},\bar{T}), \text{ etc.,} \tag{4.18}$$

$$\frac{\rho_1}{\bar{\rho}}=\alpha(\bar{p},\bar{T})\frac{p_1}{\bar{p}}-\delta(\bar{p},\bar{T})\frac{T_1}{\bar{T}}, \text{ etc.,} \tag{4.19}$$

where

$$\beta=-\frac{1}{\bar{C}_p}\Big(\frac{d\bar{h}}{dx_3}-\frac{1}{\bar{\rho}}\frac{d\bar{p}}{dx_3}\Big)\approx-\Big(\frac{d\bar{T}}{dx_3}-\frac{\bar{\delta}}{\bar{\rho}\bar{C}_p}\frac{d\bar{p}}{dx_3}\Big), \tag{4.20}$$

$$\tau_{ik}=\bar{\mu}\Big[\frac{\partial}{\partial x_k}\Big(\frac{m_i}{\bar{\rho}}\Big)+\frac{\partial}{\partial x_i}\Big(\frac{m_k}{\bar{\rho}}\Big)-\tfrac{2}{3}\delta_{ik}\frac{\partial}{\partial x_l}\Big(\frac{m_l}{\bar{\rho}}\Big)\Big], \tag{4.21}$$

$$m_i=\bar{\rho}u_i. \tag{4.22}$$

The approximation may be summarized as follows:

(i) All thermodynamic quantities are expressed as the sum of a mean and a fluctuating part, and the equations are linearized in the fluctuating parts; in particular, the mean of thermodynamic functions can be replaced by the corresponding function of the mean state variables. The momentum density is not regarded as small, and nonlinearities involving it are retained.

(ii) The fluctuating continuity equation is replaced by $\mathrm{div}(\bar{\rho}\mathbf{u})=0$.

It is of interest to note that the equations in anelastic approximation governing marginal stability of an atmosphere in hydrostatic equilibrium are identical to the exact linear stability equations. Their solutions for a polytrope are discussed by Spiegel (1965). Acoustic modes are never present; this results from part (ii) of the approximation. It will be seen later, however, that anelastic motions can coexist with acoustic oscillations on a larger scale.

If the layer is thin so that $\lambda\ll 1$, one can set

$$\tilde{f}_n=\tilde{f}_{n0}+\lambda\tilde{f}_{n1}+\lambda^2\tilde{f}_{n2}+\cdots,$$
$$f_n'=f_{n0}'+\lambda f_{n1}'+\lambda^2 f_{n2}'+\cdots,$$

and expand Eqs. (4.3)–(4.13) in powers of λ. It has been shown by Malkus (1964) that if one retains terms up to $O(\lambda)$ but neglects terms which are $O(\epsilon\lambda)$, remembering that all terms in the fluctuating equations (4.10)–(4.13) are already $O(\epsilon)$, the Boussinesq equations result.

It should be emphasized that this is a formal expansion in powers of a single dimensionless parameter ϵ, which is assumed small, and no statement has been made about the magnitudes of the other parameters. But to assess the balance of the dominant physical processes in any particular case the values of all the parameters must be considered. Normally in stellar and planetary conditions C and D are of order unity (though C may rise to about 10 in an ionization zone of an abundant element), q cannot be large if the mean atmosphere is to remain in equilibrium, and σ is about unity for air and is usually very small ($\approx 10^{-9}$) in stars so that viscous stresses have little effect on motions whose length scale is great enough for buoyancy to be important. Had σ been very large, it would have been necessary to take the viscous stresses into account when estimating the magnitude of the convective velocities, and a different scaling would have resulted. The parameter S can take a wide range of values; when it is small or of order unity Eq. (4.4) reflects the principal thermal energy balance, but when it is large the terms in this equation must be individually small and the terms of Eq. (4.7) are of comparable or greater importance. Eqs. (4.3) and (4.6) contain no parameters and ϵ thus measures the small departure from hydrostatic equilibrium, whereas it is S, whose magnitude is not specified, which measures the departure from radiative (or conductive) equilibrium. It is for this reason that a known reference atmosphere cannot be established for a one-parameter expansion.

5. Energetics of the anelastic equations

In this section it will be demonstrated that the anelastic equations are at least energetically consistent in the sense that the energy balance does not depend on contributions from higher terms in the expansion presented above. We first construct the exact total energy equation by adding the kinetic energy equation, obtained by taking the scalar product of Eq. (2.2) with m_i/ρ, to the thermal energy equation (2.4) and using Eq. (2.3); thus,

$$\frac{\partial}{\partial t}\left(\rho U+\frac{m_i m_i}{2\rho}\right)+g m_3 =Q-\frac{\partial}{\partial x_k}\left(F_k+m_k h-\frac{m_i \tau_{ik}}{\rho}+\frac{m_i m_i m_k}{2\rho^2}\right), \tag{5.1}$$

which, with the use of the relations (2.8), may be rewritten as

$$\rho C_p\frac{\partial T}{\partial t}-\delta\frac{\partial p}{\partial t}+h\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial t}\left(\frac{m_i m_i}{2\rho}\right)+g m_3 =Q-\frac{\partial}{\partial x_k}\left(F_k+m_k h-\frac{m_i \tau_{ik}}{\rho}+\frac{m_i m_i m_k}{2\rho^2}\right). \tag{5.2}$$

It will be shown that if this equation is approximated in the same way as those in the previous section, the result is the same as the total energy equation derived from the anelastic equations (4.15)–(4.22).

In terms of the dimensionless variables introduced in Section 3, Eq. (5.2) is

$$\epsilon\left[(\bar{\rho}+\epsilon\rho')(\bar{C}_p+\epsilon C_p')\frac{\partial T'}{\partial t'}-\lambda C^{-1}(\tilde{\delta}+\epsilon\delta')\frac{\partial p'}{\partial t'}+D(\delta_s C)^{-1}(\tilde{h}+\epsilon h')\frac{\partial \rho'}{\partial t'}\right]+\epsilon\lambda(\delta_s C)^{-1}\frac{\partial}{\partial t'}\left[\frac{m_i' m_i'}{2(\tilde{\rho}+\epsilon\rho')}\right]+\lambda(\delta_s C)^{-1}\tilde{g} m_3'$$
$$=q(\tilde{Q}+\epsilon Q')-\frac{\partial}{\partial x_k'}\left[(\psi S)^{-\frac{1}{2}}(\lambda\tilde{F}_k+\epsilon F_k')+D(\delta_s C)^{-1} m_k'(\tilde{h}+\epsilon h')\right.$$
$$\left.+\epsilon\sigma\lambda(\delta_s C)^{-1}(\psi S)^{-\frac{1}{2}}\frac{m_i'}{\tilde{\rho}+\epsilon\rho'}(\epsilon\tilde{\tau}_{ik}+\tau_{ik}')-\epsilon\lambda(\delta_s C)^{-1}\frac{m_i' m_i' m_k'}{2(\tilde{\rho}+\epsilon\rho')^2}\right]. \tag{5.3}$$

The zero- and first-order mean equations are

$$q\tilde{Q}_0-\lambda(\psi S)^{-\frac{1}{2}}\frac{d\tilde{F}_{03}}{dx_3'}=0, \tag{5.4}$$

$$q\tilde{Q}_1-\frac{d}{dx_3'}\left[\lambda(\psi S)^{-\frac{1}{2}}\tilde{F}_{13}+D(\delta_s C)^{-1}\overline{m_{03}' h_0'}-\sigma\lambda(\delta_s C)^{-1}(\psi S)^{-\frac{1}{2}}\tilde{\rho}_0^{-1}\overline{m_{0i}'\tau_{0i3}}+\lambda(\delta_s C)^{-1}\frac{1}{2\tilde{\rho}_0^2}\overline{m_{0i}' m_{0i}' m_{03}'}\right]=0. \tag{5.5}$$

With the help of the mean of the momentum equation (3.13) and Eq. (3.14), we can write

$$\lambda(\delta_s C)^{-1}\tilde{g} m_3'+D(\delta_s C)^{-1}\frac{\partial}{\partial x_k'}(m_k'\tilde{h}) =-\frac{\epsilon\lambda H_s}{\psi d}\tilde{\beta}\tilde{C}_p m_3'-\epsilon\lambda(\delta_s C)^{-1}\frac{d}{dx_3'}\left(\frac{\overline{m_3'^2}}{\tilde{\rho}}\right)-\epsilon D(\delta_s C)^{-1}\tilde{h}\frac{\partial \rho'}{\partial t'}+O(\epsilon^2). \tag{5.6}$$

If this relation is substituted into the fluctuating part of Eq. (5.3), the resulting equation, to leading order in ϵ, is

$$\tilde{\rho}\tilde{C}_p\frac{\partial T_0'}{\partial t'}-\lambda C^{-1}\tilde{\delta}\frac{\partial p_0'}{\partial t'}+\lambda(\delta_s C)^{-1}\frac{\partial}{\partial t'}\left(\frac{m_{0i}' m_{0i}'}{2\tilde{\rho}}\right)-\lambda\left[\frac{H_s}{\psi d}\tilde{C}_p\tilde{\beta}+(\delta_s C)^{-1}\frac{1}{\tilde{\rho}}\frac{d}{dx_3'}\left(\frac{\overline{m_{03}'^2}}{\tilde{\rho}}\right)\right]m_{03}$$
$$=qQ_0'+\frac{\partial}{\partial x_k'}\left[(\psi S)^{-\frac{1}{2}}F_{0k}'+D(\delta_s C)^{-1}(m_{0k}' h_0'-\overline{m_{0k}' h_0'})-\sigma\lambda(\delta_s C)^{-1}(\psi S)^{-\frac{1}{2}}\tilde{\rho}^{-1}(m_{0i}'\tau_{0ik}'-\overline{m_{0i}'\tau_{0ik}'})\right.$$
$$\left.+\lambda(\delta_s C)^{-1}\frac{1}{2\tilde{\rho}^2}(m_{0i}' m_{0i}' m_{0k}'-\overline{m_{0i}' m_{0i}' m_{0k}'})\right]. \tag{5.7}$$

To obtain the total energy equation implied by the anelastic equations, we first construct the kinetic energy equation from Eqs. (4.15) and (4.16):

$$\frac{\partial}{\partial t}\left(\frac{\overline{m_i m_i}}{2\bar{\rho}}\right)+\frac{\partial}{\partial x_k}\left(\frac{\overline{m_i m_i m_k}}{2\bar{\rho}^2}\right)-\frac{m_3}{\bar{\rho}}\frac{d}{dx_3}\left(\frac{\overline{m_3^2}}{\bar{\rho}}\right)$$
$$=-\frac{m_i}{\bar{\rho}}\frac{\partial p_1}{\partial x_i}-g\frac{\rho_1}{\bar{\rho}}m_3+\frac{m_i}{\bar{\rho}}\frac{\partial}{\partial x_k}\tau_{ik}. \quad (5.8)$$

The total energy equation results from adding this to the anelastic thermal energy equation (4.17):

$$\bar{\rho}\bar{C}_p\frac{\partial T_1}{\partial t}-\bar{\delta}\frac{\partial p_1}{\partial t}+\frac{\partial}{\partial t}\left(\frac{\overline{m_i m_i}}{2\bar{\rho}}\right)-m_3\left[\bar{C}_p\beta-\frac{1}{\bar{\rho}}\frac{d}{dx_3}\left(\frac{\overline{m_3^2}}{\bar{\rho}}\right)\right]$$
$$=\bar{Q}+Q_1-\frac{\partial}{\partial x_k}\left(\bar{F}_k+F_{1k}+m_k h_1-\frac{1}{\bar{\rho}}m_i\tau_{ik}+\frac{m_i m_i m_k}{2\bar{\rho}^2}\right). \quad (5.9)$$

This equation is the dimensional form of Eqs. (5.4), (5.5) and (5.7) combined.

6. Convection in a moving atmosphere

We shall now consider briefly the equations governing the flow when horizontally averaged quantities depend also on time. This is necessary in order to study, for example, the coupling between convection and atmospheric pulsations. It will be shown that under certain conditions a simple separation of the flow into acoustic and anelastic parts is possible. The discussion will be limited to cases where the mean flow is vertical and independent of horizontal coordinates, so that the total velocity v_i may be written

$$v_i=V\delta_{i3}+u_i, \quad (6.1)$$

where

$$\overline{\rho v_i}=\bar{\rho}V\delta_{i3}, \quad (6.2)$$

$$\overline{\rho u_i}=\bar{m}_i=0. \quad (6.3)$$

So as not to complicate the algebra unnecessarily, the atmosphere will still be regarded as being plane parallel. It is convenient to introduce a new coordinate system (q_i) defined by

$$\frac{\partial}{\partial x_i}=\bar{\rho}\frac{\partial}{\partial q_i}, \quad (6.4)$$

which has the property that there is no mean mass flux across any surface of constant q_i, i.e.,

$$\left(\frac{\partial x_i}{\partial t}\right)_{q_k}=V\delta_{i3}. \quad (6.5)$$

In terms of these mixed Lagrangian-Eulerian coordinates the continuity equation becomes

$$\frac{\partial\rho}{\partial t}-\bar{\rho}V\frac{\partial\rho}{\partial q_3}+\bar{\rho}\frac{\partial}{\partial q_k}[\rho(V\delta_{k3}+u_k)]=0, \quad (6.6)$$

whose horizontal average is

$$\bar{\rho}\frac{\partial V}{\partial q_3}=-\frac{\partial \ln\bar{\rho}}{\partial t}. \quad (6.7)$$

With this equation to define V, the fluctuating part of the continuity equation and the remaining equations of motion become

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\bar{\rho}}\right)+\frac{\partial m_k}{\partial q_k}=0, \quad (6.8)$$

$$\frac{\partial m_i}{\partial t}+\bar{\rho}\frac{\partial}{\partial q_k}\left(\frac{m_i m_k}{\rho}\right)$$
$$=(m_i+m_3\delta_{i3})\frac{\partial \ln\bar{\rho}}{\partial t}-\bar{\rho}\frac{\partial p}{\partial q_i}-g\rho\delta_{i3}+\bar{\rho}\frac{\partial\tau_{vik}}{\partial q_k}, \quad (6.9)$$

$$\rho C_p\frac{\partial T}{\partial t}-\delta\frac{\partial p}{\partial t}+\bar{\rho}m_k\left(\frac{\partial h}{\partial q_k}-\frac{1}{\rho}\frac{\partial p}{\partial q_k}\right)$$
$$=\bar{\rho}\tau_{vik}\frac{\partial}{\partial q_k}\left(\frac{m_i}{\rho}\right)-\tau_{v33}\frac{\partial \ln\bar{\rho}}{\partial t}+Q-\bar{\rho}\frac{\partial F_k}{\partial q_k}, \quad (6.10)$$

where g is now the apparent gravitational acceleration in the q coordinate system, i.e.,

$$g(q_3)=g(x_3)+\frac{\partial V}{\partial t}, \quad (6.11)$$

and

$$\tau_{vik}=-\tfrac{4}{3}\mu\frac{\partial \ln\bar{\rho}}{\partial t}\delta_{ik}+\tau_{ik}, \quad (6.12)$$

with τ_{ik} defined by Eq. (2.6).

The procedure to follow is similar to that employed above, except that now there are two time scales associated with the system: the convective time τ_c and a time τ_p characterizing the motion of the atmosphere as a whole. We shall scale time on the smaller of these two quantities. All other variables can be scaled as before, and the resulting nondimensional equations expanded in powers of ϵ; the anelastic equations result simply from retaining terms up to first order in ϵ in the mean equations and terms of only leading order in the fluctuating equations. If τ_p/τ_c is not small, these are, after combining the mean and fluctuating equations,

$$\epsilon\left[\phi\frac{\partial m_i'}{\partial t'}+\tilde{\rho}\frac{\partial}{\partial q_k'}\left(\frac{m_i'm_k'}{\tilde{\rho}}\right)\right]$$
$$=\epsilon\phi(m_i'+m_3'\delta_{i3})\frac{\partial \ln\tilde{\rho}}{\partial t'}-\tilde{\rho}\frac{\partial}{\partial q_i'}(\tilde{p}+\epsilon p')-\tilde{g}(\tilde{\rho}+\epsilon\rho')\delta_{i3}$$
$$+\epsilon\sigma(\psi S)^{-\frac{1}{2}}\tilde{\rho}\left[-\frac{4}{3}\lambda\phi\frac{\partial}{\partial q_3'}\left(\tilde{\mu}\frac{\partial \ln\tilde{\rho}}{\partial t'}\right)\delta_{i3}+\frac{\partial\tau_{ik}'}{\partial q_k'}\right], \quad (6.13)$$

$$\frac{\partial m_k'}{\partial q_k'}=0, \tag{6.14}$$

$$\begin{aligned}
&\phi\tilde{\rho}\tilde{C}_p\left[\frac{\partial}{\partial t'}(\tilde{T}+\epsilon T')+\epsilon\left(\frac{\rho'}{\tilde{\rho}}+\frac{C_p'}{\tilde{C}_p}\right)\frac{\partial\tilde{T}}{\partial t'}\right]\\
&\quad-\phi C^{-1}\tilde{\delta}\left[\frac{\partial}{\partial t'}(\tilde{p}+\epsilon\lambda p')+\epsilon\frac{\delta'}{\tilde{\delta}}\frac{\partial\tilde{p}}{\partial t'}\right]\\
&\quad-\frac{\lambda H_s}{\psi d}\tilde{C}_p\tilde{\beta}m_3'+(\delta_sC)^{-1}m_k'\left(D\tilde{\rho}\frac{\partial h'}{\partial q_k'}-\lambda\frac{\partial p'}{\partial q_k'}\right)\\
&\quad-\lambda(\delta_sC)^{-1}\tilde{g}\frac{\rho' m_3'}{\tilde{\rho}}\\
&=\epsilon\lambda\sigma(\delta_sC)^{-1}(\psi S)^{-\frac{1}{2}}(\tilde{\Phi}+\Phi')+q(\tilde{Q}+\epsilon Q')\\
&\quad-(\psi S)^{-\frac{1}{2}}\tilde{\rho}\frac{\partial}{\partial q_k'}(\lambda\tilde{F}_k+\epsilon F_k'),
\end{aligned} \tag{6.15}$$

where

$$\tilde{\Phi}+\Phi'=\tfrac{4}{3}\tilde{\mu}\left(\frac{\partial\ln\tilde{\rho}}{\partial t'}\right)^2+2\tilde{\mu}\frac{\partial\ln\tilde{\rho}}{\partial t'}\left(\frac{4}{3}m_3'\frac{\partial\ln\tilde{\rho}}{\partial q_3'}-\frac{\partial m_3'}{\partial q_3'}\right)+\tilde{\rho}\tau_{ik}'\frac{\partial}{\partial q_k'}\left(\frac{m_i'}{\tilde{\rho}}\right). \tag{6.16}$$

The stress tensor τ_{ik} is defined as in Eq. (4.9), and

$$\phi=\begin{cases}\dfrac{\tau_c}{\tau_p}, & \text{if} \quad \tau_c\geqslant\tau_p,\\ 1, & \text{if} \quad \tau_c\leqslant\tau_p.\end{cases} \tag{6.17}$$

The approximated thermodynamic equations are as before.

It is worth recording the Boussinesq approximation in this case. From Eq. (4.13) we can see that pressure fluctuations do not contribute to the fluctuating part of the equation of state. The same is true in all the other thermodynamic relations; in particular, the fluctuating specific enthalpy satisfies

$$h'=\delta_sCD^{-1}\tilde{C}_pT'+O(\lambda)+O(\epsilon), \tag{6.18}$$

and similar equations relate C_p' and δ' to the temperature fluctuation. Provided again that τ_p/τ_c is not small, the equations in Boussinesq approximation are, with dimensions restored,

$$\begin{aligned}
\frac{\partial m_i}{\partial t}+\frac{\partial}{\partial q_k}(m_im_k)=(m_i+m_3\delta_{i3})\frac{\partial\ln\bar{\rho}}{\partial t}-\bar{\rho}\frac{\partial}{\partial q_i}(\bar{p}+p_1)\\
-g\bar{\rho}\left(1-\frac{\bar{\delta}}{\bar{T}}T_1\right)\delta_{i3}+\bar{\rho}\frac{\partial\tau_{ik}}{\partial q_k},
\end{aligned} \tag{6.19}$$

$$\frac{\partial m_k}{\partial q_k}=0, \quad \text{or} \quad \frac{\partial u_k}{\partial q_k}=0, \tag{6.20}$$

$$\begin{aligned}
\bar{\rho}\bar{C}_p\left\{\frac{\partial}{\partial t}(\bar{T}+T_1)+\left[\overline{\left(\frac{\partial\ln C_p}{\partial\ln T}\right)_p}-\bar{\delta}\right]\frac{\partial\ln\bar{T}}{\partial t}T_1\right.\\
\left.-\frac{\beta m_3}{\bar{\rho}}+m_k\frac{\partial T_1}{\partial q_k}\right\}-\bar{\delta}\left[1+\overline{\left(\frac{\partial\ln\delta}{\partial\ln T}\right)_p}\frac{T_1}{\bar{T}}\right]\frac{\partial\bar{p}}{\partial t}\\
=\bar{Q}+Q_1-\bar{\rho}\frac{\partial}{\partial q_k}(\bar{F}_k+F_{1k}),
\end{aligned} \tag{6.21}$$

and Eqs. (4.20)–(4.22) still correctly define β, τ_{ik} and u_i.

It behooves us finally to inquire under what circumstances we might expect these approximations to be valid. If the time variations of the mean variables arise from a pulsation in the fundamental mode or a low overtone, the period of pulsation is approximately the free fall time through the depth D of the atmosphere, i.e., $\tau_p\approx(D/g_s)^{\frac{1}{2}}$, and so

$$\frac{\tau_c}{\tau_p}\approx\left(\frac{\lambda H_s}{\epsilon D}\right)^{\frac{1}{2}}.$$

For this ratio not to be large, it is necessary either that the atmosphere be at least as deep as the average scale height of potential temperature or that the convection be confined to a thin layer so that $\lambda=O(\epsilon)$. In the latter case the Boussinesq approximation applies. If neither of these conditions is satisfied, $\phi=O(\epsilon^{-\frac{1}{2}})$ and a different expansion of the equations must be made. In particular, the divergence of the momentum density is no longer zero, and mixing of acoustic motions of finite horizontal length scale with the convection seems inevitable.

7. Summary

A scale analysis has been performed to derive approximate equations governing the motion of anelastic convection or internal gravity waves by a one-parameter expansion. The analysis is valid if the relative density and temperature fluctuations produced by the motion are small. This condition can be restated in terms of the mean properties of the atmosphere alone, and amounts to demanding that a dimensionless quantity ϵ, defined in terms of the total range of potential temperature across the region within which the convective motion is confined and the product of the Prandtl number and a Rayleigh number, be small. If the convective region extends over a pressure scale height or more, this is equivalent to saying that the square of the Mach number of the convective motions is small; but if the region is thin, the condition is even more restrictive. The analysis shows that advection is the dominant nonlinearity in the equations, and suggests that if the horizontal mean properties of the atmosphere are static or vary in a time which is not very much shorter than the convective time scale, the generation of acoustic motions with a horizontal length scale comparable to that of the convection is slight and may be suppressed entirely by removing the time derivative of the density fluctuation from the continuity equation. It is indicated how, for thin regions, a two-parameter expansion yields the Boussinesq approximation.

Acknowledgments. The author is very grateful to Prof. E. A. Spiegel for many useful discussions. The work was done while the author held a NAS-NRC Senior Postdoctoral Resident Research Associateship.

REFERENCES

Batchelor, G. K., 1953: The conditions for dynamical similarity of motions of a frictionless perfect-gas atmosphere. *Quart. J. Roy. Meteor. Soc.*, **79**, 224–235.

Boussinesq, J., 1903: *Théorie analytique de la chaleur*, Tome II. Paris, Gauthier-Villars, 157–176.

Jeffreys, H., 1930: The instability of a compressible fluid heated below. *Proc. Cambridge Phil. Soc.*, **26**, 170–172.

Malkus, W. V. R., 1964: Boussinesq equations. *Geophysical Fluid Dynamics*, Vol. 1. Woods Hole Oceanographic Institution, 1–12.

Mihaljan, J. M., 1962: A rigorous exposition of the Boussinesq approximations applicable to a thin layer of fluid. *Astrophys. J.*, **136**, 1126–1133.

Oberbeck, A., 1879: Ueber die Wärmeleitung der Flüssigkeiten bei Berücksichtigung der Strömungen infolge von Temperaturdifferenzen. *Ann. Phys. Chem.*, *Neue Folge*, **7**, 271–292.

Ogura, Y., and J. G. Charney, 1962: A numerical model of thermal convection in the atmosphere. *Proc. Intern. Symp. Num. Wea. Prediction*, Tokyo, 431–451.

——, and N. A. Phillips, 1962: Scale analysis of deep and shallow convection in the atmosphere. *J. Atmos. Sci.*, **19**, 173–179.

Rayleigh, Lord, 1916: On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side. *Phil. Mag., Ser. 6*, **32**, 529–546; see also *Sci. Papers, Cambridge Univ. Press*, **6**, 432–446.

Spiegel, E. A., 1965: Convective instability in a compressible atmosphere I. *Astrophys. J.*, **141**, 1068.

——, and G. Veronis, 1960: On the Boussinesq approximation for a compressible fluid. *Astrophys. J.*, **131**, 442–447. (Correction: **135**, 655–656.)