Method of Correction for Instrumental Astigmatism to Determine the Center-to-Limb Variation of the Solar Ultraviolet Continuum*

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Various methods for obtaining the solar ultraviolet limb-darkening profiles are investigated in terms of their accuracy and adaptability for use on electronic computers. The experimental data consist of a series of spectra in the wavelengths shorter than 3000 Å taken with a somewhat astigmatic spectrograph flown in an Aerobee rocket and a similar set of carbon-arc spectra obtained with the same spectrograph shortly before the flight. For our particular purpose, the densitometer tracings of these spectra are made at right angles to the direction of dispersion. Special attention is directed to the possibility of determining the instrumental astigmatism profiles from the carbon-arc spectra on the basis of assuming a rectangular form for the carbon-arc profile unbroadened by astigmatism. It is found that this method is quite satisfactory. In order to correct the observed limb-darkening profiles for instrumental broadening, the Fourier inversion method appears to be superior to the other methods considered, although the direct numerical evaluation can also be satisfactorily applied.

INDEX HEADINGS: Astronomy; Sun; Spectra; Fourier transform; Ultraviolet.

INTRODUCTION

As is well known, solar limb-darkening observations provide the most direct and useful data for the empirical determination of the continuous opacity in the solar atmosphere.[1] Likewise, the center-to-limb variation of monochromatic radiance for the sun provides a powerful means to examine the correctness of an assumed solar model atmosphere. The method has been widely applied in the visible region of the spectrum, and, with the advance in spectrographic observations outside of the earth's atmosphere, it is now desirable to extend the same analysis to the ultraviolet and far ultraviolet regions.

One of the important questions in the study of stellar atmospheres is the incomplete knowledge of the source of continuous opacity in the ultraviolet regions. Because of the great opacity, continuous radiation in the far ultraviolet emerges from the surface of the photosphere or the bottom of the chromosphere. It has, in fact, been observed by Tousey[2] that there is little or no limb darkening in the region 2000–1525 Å, and that limb brightening of the continuum commences at $\lambda \lesssim 1525$ Å. This is consistent with the existence of a minimum temperature in the solar atmosphere. Thus, the empirical determination of the center-to-limb variation is especially important in the study of the transitional region between the photosphere and the chromosphere.

While direct photoelectric measurement would be the most desirable way of obtaining the solar limb-darkening profile, this has not, as yet, been done for wavelengths shorter than about 3000 Å. On the other hand, a number of spectra of the solar ultraviolet photographed from space probes are now available; it is the purpose of this paper to investigate means for deriving from these spectra the variation of radiance along the solar diameter.

Among the difficulties encountered is the problem of instrumental broadening due to astigmatism of the spectrograph. While it is possible to derive the instrumental astigmatism profile across the spectrum from the detailed construction of the instrument used, this approach is generally too complicated to be useful. In the present work, we utilize the continuous spectrum of carbon-arc source photographed for calibration purposes through the same spectrograph shortly before the flight. We may reasonably assume that the radiance of the crater of the carbon arc, as imaged on the slit of the spectrograph, is uniform along the slit, and that it is sharply reduced to zero outside of the crater image, giving a rectangular form for the true, or unbroadened, profile of the carbon-arc spectrum. Then, provided that the true profile is broader than the instrumental profile, it can be shown that the gradient of the observed carbon-arc profile completely determines the instrumental profile.

In the following, we first consider the method of obtaining the instrumental profiles. We then consider the various techniques of correcting the observed solar limb-darkening profiles for instrumental broadening. Although a number of different methods may be used to make the needed corrections, the availability of high speed computers has led us to adopt the more rigorous numerical methods.

OBSERVATIONAL DATA

The data used in this analysis consist of a series of spectra taken by Johnson, Malitson, Purcell, and Tousey from an Aerobee rocket that reached 115-km altitude.[3] The spectrograph used for obtaining the solar spectra was especially designed for high speed,

* Work supported by the National Science Foundation, in part by Grant, NSF GP4742, and in part by the E. O. Hulburt Center for Space Research of the U. S. Naval Research Laboratory.

[1] V. A. Ambartsumyan, *Theoretical Astrophysics* (Pergamon Press, New York, 1958), Chap. 7.

[2] R. Tousey, Space Sci. Rev. 2, 3 (1963).

[3] F. S. Johnson, H. H. Malitson, J. D. Purcell, and R. Tousey, Astrophys. J. 127, 80 (1958).

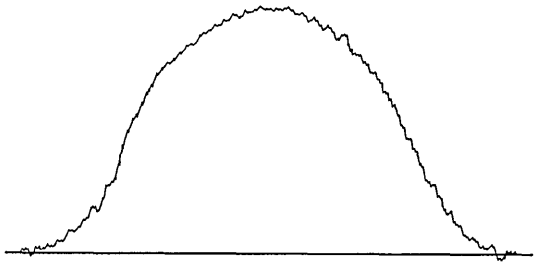

FIG. 1. Experimental solar limb-darkening profile. The logarithmic densitometer tracing has been made at right angles to the direction of dispersion at a wavelength of 2300 Å.

freedom of stray light, resolution on the order of 1 Å, and partial correction for astigmatism. The approximate position of the slit was along the solar north–south direction with maximum deviations being no more than $\pm 5'$ from the center of the sun. However, for most exposures the pointing accuracy was believed to be within $\pm 2'$. The solar image was focused on the spectrograph slit by an astigmatism-compensating collector mirror. As a result, the radiance along the slit corresponded directly, without geometrical transformation, to the radiance along the solar diameter. Also, the speed of the spectrograph was fast in comparison with the rocket movements, so that smearing was not a problem.

For our purpose, the densitometer tracings were made at right angles to the direction of dispersion. The tracings were made at various wavelengths between 2000 and 3000 Å along the continuum selected to lie between absorption lines in order that the tracings not include portions of absorption lines and thus give rise to false radiance readings in the limb regions. The densitometer tracings of the carbon-arc spectra were also made at right angles to the direction of dispersion at wavelengths corresponding as closely as possible to those of the solar tracings. Typical logarithmic densitometer tracings across the solar and carbon-arc spectra are shown in Fig. 1 and Fig. 2, respectively.

INSTRUMENTAL BROADENING

The mathematical formulation of the observed radiance distribution resulting from instrumental broadening due to various aberrations inherent in optical instruments has been discussed by different authors.[4] Accordingly, the observed radiance distribution, as derived by photographic photometry from a densitometer tracing, is given by the well-known convolution integral

$$g(x)=\int_{-\infty}^{\infty} f(x-y)h(y)dy, \qquad (1)$$

where $g(x)$ is the observed radiance distribution, $f(x)$ is the instrumental-broadening function due to all broadening effects in the optical system, and $h(y)$ is the true radiance distribution at the source. We, henceforth, refer to $g(x)$, $f(x)$, and $h(y)$ as the observed profile, the instrumental profile, and the true profile, respectively.

As shown in Fig. 3, the radiance distribution in the observed profile (solid line) as given by Eq. (1) can be considered to be a superposition of radiance elements of the true profile (dashed line), each broadened according to the functional form of the instrumental profile (dotted line). The parameter x and the integration variable y are distances measured at right angles to the dispersion from a common origin. Basic to the derivation of Eq. (1) is the requirement that the functional form of the instrumental profile is a slowly varying function of wavelength, i.e., the functional form of the instrumental profile can be assumed to be constant over the small wavelength interval accepted by the densitometer.

In the problem considered here, the instrumental profiles are derived on the basis of assuming a rectangular form for the true profiles of the carbon-arc spectrum. However, it should be noted that the rectangular form in itself does not fully determine the true carbon-arc profile; it is also necessary to know the relationship between the height and the width of the profile. If the width (here we mean the total interval over which the profile has nonzero values) of the true profile is greater than the width of the instrumental profile, the profile maxima of the true and the observed profiles are the same, thus fixing the height–width ratio. The fact that the maxima of the observed carbon-arc profiles are constant over an extended portion of the profile indicates that the true carbon-arc profiles are indeed wider than the instrumental profiles. It then follows that the gradient of the observed carbon-arc profile completely determines the instrumental profile.[5]

Also for the above case, a number of useful properties and relationships between the observed, true, and instrumental profiles may be obtained. For example: (1) the total width of the instrumental profile is equal to the width of the interval within which the observed profile increases from zero to maximum; (2) the width

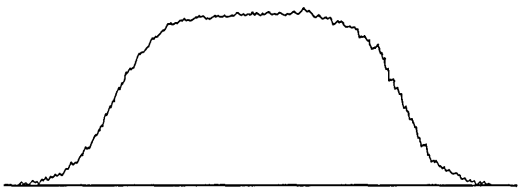

FIG. 2. Experimental carbon-arc profile in logarithmic scale. Also traced at right angles to the direction of dispersion, the carbon-arc profile provides the means of obtaining the instrumental profile.

[4] F. W. Jones, Proc. Roy. Soc. (London) 166, 16 (1938).

[5] This point has been noted by Fellgett and Schmeidler while obtaining the form of the atmospheric scattering function for the purpose of sharpening solar limb-darkening data taken during a partial eclipse of the sun. P. B. Fellgett and F. B. Schmeidler, Roy. Astron. Soc. M. N. 112, 445 (1952).

of the true profile is equal to the total area of the observed profile divided by the maximum of the observed profile; (3) the right-hand side of the observed carbon-arc profile is found to be the inverted image of the left-hand side independent of the functional form of the instrumental profile, provided only that the functional form remains constant within the width of the true profile. This inverted-symmetry relationship is of great importance to the present problem because of the fact that the foot portions of the observed carbon-arc profiles are not fully reliable due to the unfavorable nature of the correction for the characteristic curve at low radiances.

Figure 4 shows the foot portions of the observed profile of the carbon-arc spectrum for the wavelength of 2300 Å after corrections for the characteristic curve. By explicit use of the inverted-symmetry relation, the foot portions of the carbon-arc profile may be reconstructed by graphical or numerical fitting of the upper right-hand side values in place of the left-side foot portion. The maximum error introduced in fitting the slopes of the two curves is of the order of $\frac{1}{2}\%$ of the width of the carbon-arc profile and is negligible compared to other errors involved in the problem. We have thus, in effect, reconstructed the observed profile of the carbon spectrum from only those portions of the profile for which the characteristic curve is most reliable. Then, by taking the gradient of the reconstructed carbon-arc profile, the instrumental profile was determined.

It is also possible to calculate the instrumental profiles by means of the Fourier inversion method discussed in the following sections. With the assumed rectangular form for the true carbon-arc profile, the integral equation may be inverted and the instrumental profile obtained. Mainly for the purpose of checking the computer programs, calculations of this type were made; the profiles obtained were the same as those determined by the simpler gradient method.

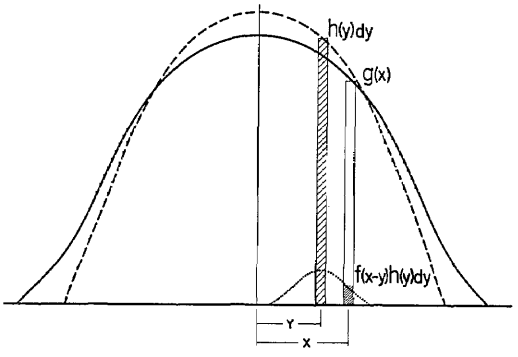

FIG. 3. Superposition principle of instrumental broadening. The observed profile (solid line), $g(x)$, is the sum of radiance contributions from the elements, $h(y)dy$, of the true profile (dashed line). The contribution at the point x from the radiance element at the point y is determined by the instrumental profile, $f(x-y)$ (dotted line).

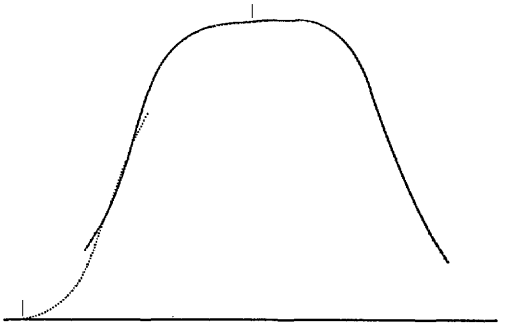

FIG. 4. Reconstruction of carbon-arc profile foot portions. The instrumental profile for 2300 Å is obtained from the gradient of the reconstructed profile.

CORRECTION PROCEDURE

Once the instrumental profile is reliably known, we can proceed to solve the integral equation for the true solar limb-darkening profile. The exact method to be used, however, depends both on the form and the quality of the data as well as on the computational facilities available. Sometimes, as in the case of correcting spectral line breadths for instrumental broadening, it is possible to obtain solutions to the integral equation in analytical form, so that the corrections may be made from tabulated values.[6] Also in common use is the method of successive approximation whereby an initial solution is successively improved until the solution is consistent with the integral equation.[7] In the more general case, when it is difficult to fit analytical functions to the profiles, or when computer facilities are available, it is more desirable to use the Fourier inversion or direct numerical methods to calculate the true profiles. In the following analysis we have considered several numerical approaches and the Fourier inversion method in terms of their accuracy and applicability on an IBM 7040 computer.

1. Fourier Inversion Method

The Fourier inversion method has been considered in detail by a number of authors.[8] When the Fourier transform of Eq. (1) is taken, the integral equation breaks down into a product of Fourier transforms given by

$$G(t)=F(t)H(t), \tag{2}$$

where $G(t)$, $F(t)$, and $H(t)$ are all given in the form

$$G(t)=\int_{-\infty}^{\infty} g(x)e^{ixt}dx, \tag{3}$$

respectively. The true profile, normalized for numerical convenience, is then found by taking the Fourier in-

[6] H. C. van de Hulst and J. J. Reesinck, Astrophys. J. **106**, 121 (1947).
[7] P. H. van Cittert, Z. Phys. **69**, 298 (1931).
[8] A. R. Stokes, Proc. Phys. Soc. (London) **61**, 382 (1948).

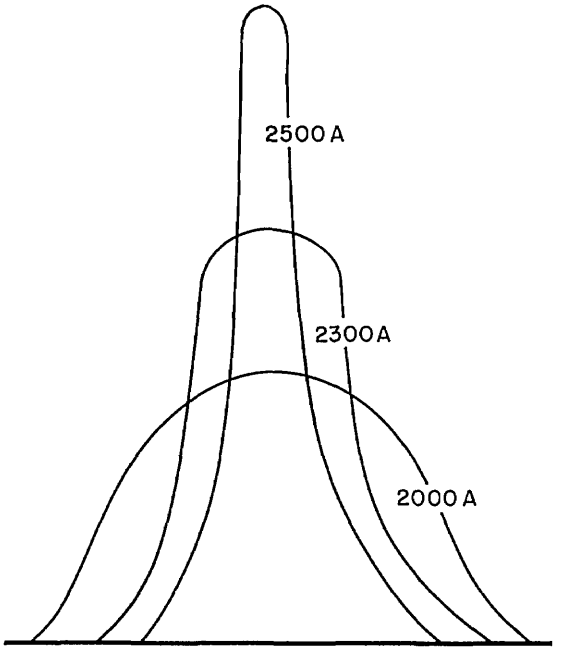

FIG. 5. Samples of instrumental profiles obtained from reconstructed carbon-arc profiles.

verse of $H(t)$

$$h(x)=\int_{-\infty}^{\infty}\frac{G(t)}{F(t)}e^{-ixt}dt. \tag{4}$$

Since Eq. (4) is a general result, it is, at least in principle, always possible to obtain the true profile once the instrumental and observed profiles are known. In practice, however, good results can be obtained only when the instrumental profile is narrow compared to the observed profile, in which case the ratio $G(t)/F(t)$ rapidly approaches zero as t becomes large. Replaced by a summation, Eq. (4) can then be readily evaluated for each value of x to obtain the true profile.

2. Numerical Evaluation

The instrumental broadening problem may also be solved by the direct numerical evaluation of the integral equation. This approach has been demonstrated by Paterson[9] and involves replacing Eq. (1) directly by a summation and obtaining a set of linear overdetermined simultaneous equations

$$g(x_i)=\sum_j f(x_i-y_j)h(y_j). \tag{5}$$

The above set of equations may then be solved by the least-squares method. However, with this method, care must be taken to avoid rounding-off errors when a large number of points are used.

To minimize rounding-off errors it is also possible to use an iteration approach. In this case, however, the number of equations must be equal to the number of unknowns, i.e., a number of the over-determined equations in expression (5) corresponding to the foot-portion values of the observed profile are dropped, and the remaining equations matched with their corresponding observed-profile values. If the width of the true solar image with respect to the observed solar image is not independently known, the exact number of equations needed to determine the true profile must be found by trial and error.

This problem does not arise in the Fourier method, although for the Fourier method, it is instead necessary to extrapolate the foot portions of the observed solar profiles. Since the foot portions are small in comparison with the total profile, errors introduced by extrapolation in the Fourier method and the elimination of some equations in the numerical method are also small.

3. Graphical Method

Another method that may be applied to obtain a first-order correction for instrumental broadening is a graphical method discussed by Bracewell[10,11] in connection with aerial smoothing in radio astronomy. Based on a finite central-differences principle, the first-order corrections are obtained by means of chord construction on the observed profiles. While this method is relatively simple as a first approximation, it does not seem to be as accurate as the other methods considered, especially in the limb regions where the radiance changes most rapidly. For this reason, and also because of the availability of computers, we find the numerical methods more suitable for this problem.

RESULTS

The instrumental profiles have been determined for a number of wavelengths between 2000 and 2500 Å from the carbon-arc profiles by reconstructing the foot portions and then taking the gradient of the profile as outlined in the previous section. Samples of the instrumental profiles thus obtained are normalized to unit area and are shown in Fig. 5. Instrumental profiles for wavelengths greater than 2500 Å could not be obtained because of incomplete characteristic curves. However, in the region investigated, the instrumental profile for 2500 Å was found to be the narrowest, with the breadth of the profiles increasing toward the shorter wavelengths.

Calculations were made on an IBM 7040 with the Fourier inversion method as well as the direct numerical methods to correct the observed solar profiles for instrumental broadening. The calculated profiles, however, were found to be not entirely symmetrical as would be expected for limb darkening. This is probably due to regions of greater solar activity since a number of profiles show similar asymmetry. The asymmetry, being relatively small, is eliminated by averaging the two

[9] M. S. Paterson, Proc. Phys. Soc. (London) 63, 477 (1950).

[10] R. N. Bracewell, Australian J. Phys. 8, 200 (1955).

[11] R. N. Bracewell, J. Opt. Soc. Am. 45, 873 (1955).

sides of the profiles as is general practice in determining limb darkening in the visual regions of the spectrum. The limb-darkening profile is then plotted as a function of cos θ where θ is the angular position of a point on the sun's surface with respect to the center of the sun. The curve thus obtained for 2500 Å is shown in Fig. 6. Since the observed limb-darkening curves in the visible region become steeper toward the shorter wavelengths implying that the continuous opacity in the solar atmosphere increases toward the ultraviolet, the fact that the slope shown in Fig. 6 is still greater than that near 3000 Å, the shortest wavelength so far observed, indicates that the opacity still increases toward the 2500-Å region but is not yet large enough for chromosphere contributions to dominate.

The factors limiting the accuracy of the calculated solar limb-darkening profiles have been mainly due to the increased broadening of the instrumental profiles toward the shorter wavelengths and to the uncertainty of values in the foot portions of the observed profiles due to the unfavorable nature of the correction for the characteristic curve at low radiances as well as the possible errors caused by including parts of absorption lines in the foot portions of the traced profiles. With the carbon arc source especially designed to allow the calibration of absolute intensities of spectral lines, errors due to any astigmatism in the carbon-arc source were small.

While better pointing accuracy in photographing the solar spectrum would have been desirable, errors due to the pointing accuracy were less serious than those of instrumental broadening. For the maximum deviation of 5′, the slit of the spectrograph would not be along the diameter of the sun. However, the difference in the width of the true profile due to this maximum pointing error amounts to little more than 5% of the total width of the profile along the diameter of the sun. Furthermore, if this difference in the width of the true profile is reliably resolved, there is little difference in the final shape of the limb-darkening profiles as obtained from the aligned or displaced spectra. Also, the independent knowledge of the size of the solar disk at the focal plane of the spectrograph, while not specifically required for the Fourier method, would eliminate the trial and error involved in matching the equations in the direct numerical method, as well as provide a valuable means of checking the consistency of the results.

As a result of our analysis, we find that the determination of the instrumental-broadening profiles by taking the gradient of the reconstructed carbon-arc profiles is realiable. Also, it appears that the best method of correcting the observed solar limb-darkening profiles for instrumental broadening is the Fourier inversion method; the results are most accurate at the longer wavelengths where the instrumental profiles are the narrowest. The direct numerical methods, while being hampered by the uncertainty of the exact width of the true profile in the present case, appear to be well suited for the limb-darkening problem since a sharply defined profile corresponds well to the true radiance distribution of the sun.

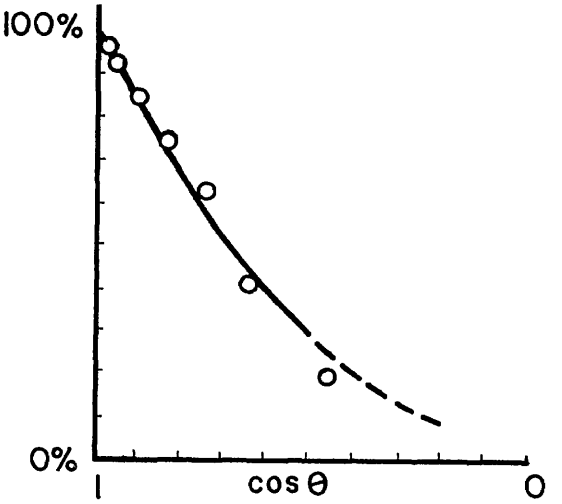

FIG. 6. Corrected solar limb-darkening profile for 2500 Å plotted as a function of cos θ.

ACKNOWLEDGMENTS

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