PLASMA NEUTRINO EMISSION FROM A HOT, DENSE ELECTRON GAS*

Cullen L. Inman
New York University and Goddard Institute for Space Studies

AND

Malvin A. Ruderman†
New York University
Received May 13, 1964

ABSTRACT

Neutrino pair emission from coherent electron excitations (transverse plasmons) in a hot, dense stellar plasma is calculated for a regime of temperatures and densities relevant to stellar evolution. Detailed numerical results are presented for temperatures in the range 4×10^7–5×10^8 ° K and densities in the range 10^4–10^8 gm/cm^3.

I. INTRODUCTION

A coupling of electron and neutrino pairs is implied by almost all models which describe the weak Fermi interactions of elementary particles. Although there has not yet been any direct experimental detection of such an electron-neutrino coupling, it is a necessary consequence of the apparent existence (Brookhaven 1963) of an intermediate heavy charged boson in β-decay. Moreover the form and magnitude of the resulting electron-neutrino interaction are unambiguously determined. The strength of the coupling is characterized by the small Fermi constant

$$g \sim 3.08 \times 10^{-12} \frac{\hbar^3}{m_e^2 c},$$

where m_e is the electron mass; its form is analogous to the interaction of electromagnetic radiation with the electron current. An accelerated electron can then radiate a neutrino (ν)–antineutrino ($\bar{\nu}$) pair with the same matrix element as that for electromagnetic radiation but with greatly reduced probability. For example in an atomic transition of energy E the probability $R_{\nu\bar{\nu}}$ for radiation of $\nu\bar{\nu}$, relative to R_γ, that for radiating a photon, is only

$$\frac{R_{\nu\bar{\nu}}}{R_\gamma} \sim \frac{g^2 E^4}{e^2 \hbar^5 c^5} \sim 10^{-21} \left(\frac{E}{m_e c^2}\right)^4. \quad (1)$$

The probability for emitting a $\nu\bar{\nu}$ pair rises rapidly with energy (in this case, like E^7) but even an electron-positron pair will annihilate into $\nu\bar{\nu}$ rather than a pair of gamma rays only about once in 10^{22} times.

An extremely strong energy dependence is characteristic of all mechanisms for radiating $\nu\bar{\nu}$ pairs: an electron moving with frequency ω in a classical circular orbit of radius R radiates electromagnetic waves, gravitational waves, and neutrino pairs with powers, P, that follow from simple dimensional considerations:

$$P_\gamma \sim \frac{e^2}{c^3} R^2 \omega^4, \quad (2)$$

$$P_{\text{Grav}} \sim \frac{G m_e^2 R^4 \omega^6}{c^5} \qquad \text{(quadrupole)}, \quad (3)$$

* Work supported in part by the National Science Foundation.
† On leave from the University of California, Berkeley, California.

and

$$P_{\nu\bar{\nu}} \sim \frac{g^2 R^2 \omega^8}{\hbar c^8} . \tag{4}$$

At low frequencies neutrino pair processes are entirely negligible. For a double star in which each member has a different electron molecular weight, the center of electron charge rotates about the fixed center of mass, but the neutrino pair radiation is smaller by perhaps eighty orders of magnitude than the small quadrupole radiation of gravitational waves. High frequencies ($\omega \gtrsim 10^{18} \sim 1$ keV), sufficient for significant $\nu\bar{\nu}$ radiation by electrons, are available only on the microscopic scale where electrons are strongly accelerated by photons or intimate collisions.

Mechanisms for the radiation of $\nu\bar{\nu}$ pairs by electrons within a star include the following:

(a) $\gamma + e \rightarrow e + \nu + \bar{\nu}$, (photoneutrinos)
(Ritus 1962; Ida and Vahara; Chiu and Stabler 1961)

(b) $e^- + e^+ \rightarrow \nu + \bar{\nu}$, (pair annihilation neutrinos)
(Chiu and Morrison 1960; Chiu 1961)

(c) $e +$ Coulomb field $\rightarrow e + \nu + \bar{\nu}$, (neutrino bremsstrahlung)
(Gandel'man and Pineau 1960)

(d) $\gamma +$ Coulomb field $\rightarrow \nu + \bar{\nu}$, (photonuclear neutrinos)
(Rosenberg 1963; Matinyan and Tsilosani 1962)

(e) $\gamma + \gamma \rightarrow \gamma + \nu + \bar{\nu}$, (van Hieu and Shabalin 1963)

(f) plasma excitation $\rightarrow \nu + \bar{\nu}$. (plasma neutrinos)
(Adams, Ruderman, and Woo 1963).

Processes (a), (b), and (f) are the dominant ones in those regimes of density and temperature typical of stellar interiors (Reeves 1963). Photo and pair annihilation neutrino emissions have exact analogues in Compton scattering and the two-photon annihilation of an electron-positron pair. The main contribution to process (d) arises from the electron current associated with a transverse ($\boldsymbol{E}$ perpendicular to $\boldsymbol{k}$) electromagnetic wave moving through the plasma.

II. PLASMA NEUTRINOS

Plasma neutrino emission has been calculated by Adams, Ruderman, and Woo (1963) for any medium whose dielectric constants are known functions of frequency and wave-number. The relevant dielectric constants have also been evaluated for an electron gas which may be relativistic and degenerate. For a plasma frequency ω_o such that $(\hbar\omega_o)^2 \ll 4(m_e c^2)^2$ the relevant transverse dielectric constant is well approximated by

$$\epsilon^t = 1 - \frac{\omega_o^2}{\omega^2} , \tag{5}$$

with

$$\omega_o^2 = \frac{4\pi e^2}{m_e} \int dp f(p) \left(1 - \frac{p^2 c^2}{3E_p^2}\right) \frac{m_e c^2}{E_p} . \tag{6}$$

Here $f(p)$ is the momentum distribution function for the electrons and

$$E_p = (p^2 c^2 + m_e^2 c^4)^{1/2} . \tag{7}$$

The dielectric constant of equation (5) leads to a greatly simplified neutrino pair emissivity Q_t (erg/sec/cm^3):

$$Q_t = \frac{g^2 \omega_o^6}{12\pi^4 e^2 c^5} \int_0^\infty k^2 dk \, (e^{\omega\beta} - 1)^{-1}, \tag{8}$$

with $\omega = (\omega_o^2 + k^2c^2)^{1/2}$ and $\beta = (k_B T)^{-1}$. The function Q_t computed from equations (6) and (8) is given as a function of density and temperature in Section III.

The form of equation (8) can be inferred in a rather straightforward way from equation (5). For a transverse electromagnetic wave in a plasma with the dielectric constant the dispersion relation is

$$\omega^2 = k^2c^2 + (1 - \epsilon)\omega^2 = k^2c^2 + \omega_o^2 . \tag{9}$$

Therefore such waves, when quantized, behave as if they were relativistic particles of mass $\hbar\omega_o/c^2$, and they are energetically unstable against the decay into a neutrino pair. If r is the decay rate for such a particle at rest, its decay rate when moving is decreased by the usual relativistic time dilatation to $(r\omega_o/\omega)$ and its rate of production of $\nu\bar{\nu}$ becomes $(r\omega_o/\omega)\hbar\omega = r\hbar\omega_o$. Thus the total neutrino emissivity per unit volume is

$$Q_t \sim r\hbar\omega_o 8\pi \int \frac{k^2 dk}{(2\pi)^3} (e^{\beta\omega} - 1)^{-1}. \tag{10}$$

The integral is just the total number of "photons" of mass per unit volume in a canonical Einstein-Bose distribution. We need only calculate the $\nu\bar{\nu}$ emission $r\hbar\omega_o$ caused by the oscillating electric field with $\omega = \omega_o$ and infinite wavelength ($k = 0$). When $k = 0$ this mode is identical to the usual plasma oscillation; the accelerated electrons oscillating in phase with frequency ω_o, amplitude x, and acceleration $\omega_o^2 x$ coherently radiate neutrino pairs. But because of the finite neutrino wavelength $\lambda_\nu \sim c/\omega_o$ all of the electrons in a large volume Ω can not radiate as if they constituted a single highly charged particle. Rather only those electrons in a cube of volume $\sim(\lambda_\nu)^3$ effectively radiate coherently. Then from a large volume Ω with electron density n, the total rate of neutrino energy emission P can be estimated from equation (4):

$$P \sim \frac{g^2 \omega_o^8 x^2}{\hbar c^8} (n\lambda_\nu^3)^2 \frac{\Omega}{(\lambda_\nu)^3}, \tag{11}$$

with

$$\lambda_\nu = \frac{c}{\omega_o}. \tag{12}$$

The additional kinetic energy of the oscillating electrons is $\frac{1}{2} m\omega_o^2 x^2 n\Omega$. This energy together with an equivalent average potential energy must total $\hbar\omega_o$ when the oscillation amplitude is quantized to correspond to a single quantum of excitation (plasmon). Thus

$$m\omega_o^2 x^2 n\Omega = \hbar\omega_o . \tag{13}$$

The combination of equations (11), (12), and (13) yields P for a single plasmon at rest, i.e., $r\hbar\omega_o$:

$$\hbar\omega_o r \sim \frac{g^2}{c^5} \omega_o^4 \frac{n}{m}. \tag{14}$$

But

$$\omega_o^2 \sim 4\pi n e^2/m , \tag{15}$$

so that

$$\hbar\omega_o r \sim \frac{g^2}{e^2} \frac{\omega_o^6}{c^5}. \tag{16}$$

Equations (16) and (10) yield equation (8).

III. EXACT CALCULATION OF THE TRANSVERSE EMISSIVITY

a) Introduction

In Section II it was shown that the rate of loss of energy in neutrino pairs due to decay of transverse plasmons is

$$Q_t = 2g^2(3\pi e^2)^{-1}(2\pi)^{-3}\omega_o^6 \sum_1^\infty \int_0^\infty \exp(-n\beta\omega)k^2 dk\,. \tag{17}$$

The units are $\hbar = m_e = c = 1$. Here $g = 3.08 \times 10^{-12}$ is the weak coupling constant; ω is given by the dispersion relation for the transverse plasmons, $\omega^2 = \omega_o^2 + k^2$; and

$$\omega_o^2 = 4e^2 p_F^3/(3\pi E_F) \tag{18}$$

is the plasma frequency. Of course $\beta = 1/(k_B T)$. We take $\mu_e = 2$.

We define a function $\mathfrak{F}(x)$ of x alone by

$$\mathfrak{F}(x) = \sum_1^\infty \int_0^\infty \exp[-nx\cosh\xi]\sinh^2\xi\cosh\xi\, d\xi\,. \tag{19}$$

By considering the dispersion relation and equation (17) one sees that

$$Q_t = 2g^2(3\pi e^2)^{-1}(2\pi)^{-3}\omega_o^9\,\mathfrak{F}(\beta\omega_o)\,, \tag{20}$$

and we may note that the expression for the number density of a gas of bosons at zero chemical potential is

$$\frac{N}{V} = \frac{1}{\pi^2}\left(\frac{mc}{\hbar}\right)^3 \mathfrak{F}(\beta mc^2)\,. \tag{21}$$

Apart from care in handling the conversion of units we need then only discuss the calculation of $\mathfrak{F}(x)$.

b) The Function $\mathfrak{F}(x)$

1. *Small* ($x \leq 0.5$) *values of the argument.*—It is shown in the Appendix that for $x < 2\pi$ there exists an expansion of $x^3\mathfrak{F}(x)$; note

$$\zeta(3) = \sum_{n=1}^\infty (1/n^3)\,.$$

Truncating it,

$$\mathfrak{F}(x) \doteq \frac{1}{x^3}\Big[\!\Big[2\zeta(3) + \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}(2\ln 2 + 1)x^2 + \tfrac{1}{96}x^4 \ln x - \tfrac{1}{96}\Big\{\ln 2 - \tfrac{1}{4} + \ln 2\pi + \Big[-\frac{\zeta'(2)}{\zeta(2)}\Big]\Big\}x^4 \Big]\!\Big]\,. \tag{22}$$

For $x = 0.5$ this gives a result which differs from that obtained by summing the Hankel series (see subsection 2 below) by one part in 10^6. It will also be seen that by taking only the constant term in the square brackets, and using equation (20), we get just the equation preceding equation (28) in Adams *et al.* (1963).

2. *Intermediate and large values of the argument* ($x \geq 0.5$).—Chandrasekhar (1957) gives a result (p. 398, eq. [252] and [248]) that, omitting the minus signs and Λ, since we have bosons at zero chemical potential, reads

$$\mathfrak{F}(x) = \sum_{n=1}^{\infty} \frac{K_2(nx)}{nx} = \sum_{n=1}^{M} \frac{K_2(nx)}{nx}, \tag{23}$$

where $K_2(z)$ is the modified Bessel function of the second kind, of order 2. The criterion for the choice of M is discussed in subsection 3. The series equation (23) is a Hankel series, similar to a Dirichlet series, and converges for all x, albeit very slowly for small x: (see Greenwood [1941]) Note that for small ξ, $K_2(\xi) \approx 2/\xi^3$, so that substituting this expression in equation (23) we get again

$$\mathfrak{F}(x) \approx \frac{2}{x^3} \sum_{n=1}^{\infty} \left(\frac{1}{n^3}\right) = \frac{2\zeta(3)}{x^3},$$

for small x, just what one obtains by taking only the *zero*th-order term in braces in equation (22).

For large ξ, on the other hand, $K_2(\xi) \approx (\pi/2\xi)^{1/2} e^{-\xi}$, and if we take only the first term of equation (23), using this expression as an approximation for it, we get $\mathfrak{F}(x) \approx (\pi/2)^{1/2} x^{-3/2} e^{-x}$, for large x.

If one uses this last plus equation (20) one gets just the equation preceding equation (29) in Adams *et al.* (1963).

3. *Error terms.*—For equation (22), the small argument form, it is shown in the Appendix that the lowest order neglected term is $O(x^6 \ln x)$. The coefficient multiplying this term will be quite small.

For equation (23), the intermediate and large form, we break off the sum when

$$\frac{K_2(Mx)}{Mx} < 10^{-6} \frac{K_2(x)}{x}.$$

By approximating, for large M, the remainder term by

$$\sum_{n=M+1}^{\infty} \frac{K_2(nx)}{nx} \approx \frac{1}{x} \int_{M+1/2}^{\infty} \frac{K_2(nx)}{n} dn = \frac{1}{x} \frac{K_1[(M+\frac{1}{2})x]}{(M+\frac{1}{2})x} < \frac{1}{x} \frac{K_2(Mx)}{Mx},$$

we find that the error is less than $(1/x) \times 10^{-6}$ times the first term This is for the range 0.5–5, since for larger values we have no need of many terms.

c) Conversion of Units

The dimensions of Q_t are erg/cm^3/sec. In the units in which equation (20) is given, units of energy, length, and time are $m_e c^2$, $\hbar/m_e c$, and $\hbar/m_e c^2$, respectively. Thus Q_t in c.g.s. units is

$$Q_t = \frac{m_e c^2}{(\hbar/m_e c)^3} \frac{1}{\hbar/(m_e c^2)} \left[\frac{2}{3\pi} \times 137.04 \times (3.08 \times 10^{-12})^2 \frac{1}{(2\pi)^3}\right] \times \left(\frac{\hbar\omega_o}{m_e c^2}\right)^9 \mathfrak{F}(\beta\hbar\omega_o) = 1.228 \times 10^{22} \left(\frac{\hbar\omega_o}{m_e c^2}\right)^9 \mathfrak{F}(\beta\hbar\omega_o), \tag{24}$$

Table 1. Neutrino Emissivity by Transverse Plasmons

T	q SUB T	T	q SUB T	T	q SUB T
LCG RHO=4.00					
$.40000 \times 10^{8}$	$.30491 \times 10^{-2}$	$.20000 \times 10^{9}$	$.43162 \times 10^{0}$	$.36000 \times 10^{9}$	$.25327 \times 10^{1}$
$.60000 \times 10^{8}$	$.10963 \times 10^{-1}$	$.22000 \times 10^{9}$	$.57531 \times 10^{0}$	$.38000 \times 10^{9}$	$.29796 \times 10^{1}$
$.80000 \times 10^{8}$	$.26678 \times 10^{-1}$	$.24000 \times 10^{9}$	$.74776 \times 10^{0}$	$.40000 \times 10^{9}$	$.34762 \times 10^{1}$
$.10000 \times 10^{9}$	$.52820 \times 10^{-1}$	$.26000 \times 10^{9}$	$.95157 \times 10^{0}$	$.42000 \times 10^{9}$	$.40251 \times 10^{1}$
$.12000 \times 10^{9}$	$.92010 \times 10^{-1}$	$.28000 \times 10^{9}$	$.11894 \times 10^{1}$	$.44000 \times 10^{9}$	$.46289 \times 10^{1}$
$.14000 \times 10^{9}$	$.14687 \times 10^{0}$	$.30000 \times 10^{9}$	$.14638 \times 10^{1}$	$.46000 \times 10^{9}$	$.52902 \times 10^{1}$
$.16000 \times 10^{9}$	$.22001 \times 10^{0}$	$.32000 \times 10^{9}$	$.17774 \times 10^{1}$	$.48000 \times 10^{9}$	$.60116 \times 10^{1}$
$.18000 \times 10^{9}$	$.31405 \times 10^{0}$	$.34000 \times 10^{9}$	$.21328 \times 10^{1}$	$.50000 \times 10^{9}$	$.67958 \times 10^{1}$
LCG RHO=4.20					
$.40000 \times 10^{8}$	$.71282 \times 10^{-2}$	$.20000 \times 10^{9}$	$.10627 \times 10^{1}$	$.36000 \times 10^{9}$	$.62533 \times 10^{1}$
$.60000 \times 10^{8}$	$.26293 \times 10^{-1}$	$.22000 \times 10^{9}$	$.14174 \times 10^{1}$	$.38000 \times 10^{9}$	$.73579 \times 10^{1}$
$.80000 \times 10^{8}$	$.64689 \times 10^{-1}$	$.24000 \times 10^{9}$	$.18432 \times 10^{1}$	$.40000 \times 10^{9}$	$.85854 \times 10^{1}$
$.10000 \times 10^{9}$	$.12883 \times 10^{0}$	$.26000 \times 10^{9}$	$.23466 \times 10^{1}$	$.42000 \times 10^{9}$	$.99421 \times 10^{1}$
$.12000 \times 10^{9}$	$.22520 \times 10^{0}$	$.28000 \times 10^{9}$	$.29340 \times 10^{1}$	$.44000 \times 10^{9}$	$.11435 \times 10^{2}$
$.14000 \times 10^{9}$	$.36029 \times 10^{0}$	$.30000 \times 10^{9}$	$.36119 \times 10^{1}$	$.46000 \times 10^{9}$	$.13069 \times 10^{2}$
$.16000 \times 10^{9}$	$.54058 \times 10^{0}$	$.32000 \times 10^{9}$	$.43868 \times 10^{1}$	$.48000 \times 10^{9}$	$.14853 \times 10^{2}$
$.18000 \times 10^{9}$	$.77254 \times 10^{0}$	$.34000 \times 10^{9}$	$.52651 \times 10^{1}$	$.50000 \times 10^{9}$	$.16792 \times 10^{2}$
LCG RHO=4.40					
$.40000 \times 10^{8}$	$.16221 \times 10^{-1}$	$.20000 \times 10^{9}$	$.25971 \times 10^{1}$	$.36000 \times 10^{9}$	$.15346 \times 10^{2}$
$.60000 \times 10^{8}$	$.61954 \times 10^{-1}$	$.22000 \times 10^{9}$	$.34674 \times 10^{1}$	$.38000 \times 10^{9}$	$.18061 \times 10^{2}$
$.80000 \times 10^{8}$	$.15476 \times 10^{0}$	$.24000 \times 10^{9}$	$.45125 \times 10^{1}$	$.40000 \times 10^{9}$	$.21077 \times 10^{2}$
$.10000 \times 10^{9}$	$.31071 \times 10^{0}$	$.26000 \times 10^{9}$	$.57483 \times 10^{1}$	$.42000 \times 10^{9}$	$.24412 \times 10^{2}$
$.12000 \times 10^{9}$	$.54582 \times 10^{0}$	$.28000 \times 10^{9}$	$.71908 \times 10^{1}$	$.44000 \times 10^{9}$	$.28081 \times 10^{2}$
$.14000 \times 10^{9}$	$.87607 \times 10^{0}$	$.30000 \times 10^{9}$	$.88559 \times 10^{1}$	$.46000 \times 10^{9}$	$.32100 \times 10^{2}$
$.16000 \times 10^{9}$	$.13174 \times 10^{1}$	$.32000 \times 10^{9}$	$.10760 \times 10^{1}$	$.48000 \times 10^{9}$	$.36485 \times 10^{2}$
$.18000 \times 10^{9}$	$.18858 \times 10^{1}$	$.34000 \times 10^{9}$	$.12918 \times 10^{1}$	$.50000 \times 10^{9}$	$.41251 \times 10^{2}$

T	q SUB T	T	q SUB T	T	q SUB T
LCG RHO=4.60					
$.40000\ X10^{8}$	$.35602\ X10^{-1}$	$.20000\ X10^{9}$	$.62850\ X10^{1}$	$.36000\ X10^{9}$	$.37354\ X10^{2}$
$.60000\ X10^{8}$	$.14256\ X10^{0}$	$.22000\ X10^{9}$	$.84023\ X10^{1}$	$.38000\ X10^{9}$	$.43975\ X10^{2}$
$.80000\ X10^{8}$	$.36355\ X10^{0}$	$.24000\ X10^{9}$	$.10946\ X10^{2}$	$.40000\ X10^{9}$	$.51335\ X10^{2}$
$.10000\ X10^{9}$	$.73812\ X10^{0}$	$.26000\ X10^{9}$	$.13956\ X10^{2}$	$.42000\ X10^{9}$	$.59471\ X10^{2}$
$.12000\ X10^{9}$	$.13055\ X10^{1}$	$.28000\ X10^{9}$	$.17471\ X10^{2}$	$.44000\ X10^{9}$	$.68424\ X10^{2}$
$.14000\ X10^{9}$	$.21049\ X10^{1}$	$.30000\ X10^{9}$	$.21529\ X10^{2}$	$.46000\ X10^{9}$	$.78231\ X10^{2}$
$.16000\ X10^{9}$	$.31753\ X10^{1}$	$.32000\ X10^{9}$	$.26170\ X10^{2}$	$.48000\ X10^{9}$	$.88931\ X10^{2}$
$.18000\ X10^{9}$	$.45557\ X10^{1}$	$.34000\ X10^{9}$	$.31432\ X10^{2}$	$.50000\ X10^{9}$	$.10056\ X10^{3}$
LCG RHO=4.80					
$.40000\ X10^{8}$	$.74499\ X10^{-1}$	$.20000\ X10^{9}$	$.15011\ X10^{2}$	$.36000\ X10^{9}$	$.89949\ X10^{2}$
$.60000\ X10^{8}$	$.31782\ X10^{0}$	$.22000\ X10^{9}$	$.20105\ X10^{2}$	$.38000\ X10^{9}$	$.10594\ X10^{3}$
$.80000\ X10^{8}$	$.83355\ X10^{0}$	$.24000\ X10^{9}$	$.26232\ X10^{2}$	$.40000\ X10^{9}$	$.12372\ X10^{3}$
$.10000\ X10^{9}$	$.17184\ X10^{1}$	$.26000\ X10^{9}$	$.33485\ X10^{2}$	$.42000\ X10^{9}$	$.14338\ X10^{3}$
$.12000\ X10^{9}$	$.30679\ X10^{1}$	$.28000\ X10^{9}$	$.41959\ X10^{2}$	$.44000\ X10^{9}$	$.16501\ X10^{3}$
$.14000\ X10^{9}$	$.49773\ X10^{1}$	$.30000\ X10^{9}$	$.51749\ X10^{2}$	$.46000\ X10^{9}$	$.18871\ X10^{3}$
$.16000\ X10^{9}$	$.75411\ X10^{1}$	$.32000\ X10^{9}$	$.62947\ X10^{2}$	$.48000\ X10^{9}$	$.21457\ X10^{3}$
$.18000\ X10^{9}$	$.10854\ X10^{2}$	$.34000\ X10^{9}$	$.75649\ X10^{2}$	$.50000\ X10^{9}$	$.24269\ X10^{3}$
LCG RHO=5.00					
$.40000\ X10^{8}$	$.14647\ X10^{0}$	$.20000\ X10^{9}$	$.35234\ X10^{2}$	$.36000\ X10^{9}$	$.21359\ X10^{3}$
$.60000\ X10^{8}$	$.67973\ X10^{0}$	$.22000\ X10^{9}$	$.47315\ X10^{2}$	$.38000\ X10^{9}$	$.25169\ X10^{3}$
$.80000\ X10^{8}$	$.18510\ X10^{1}$	$.24000\ X10^{9}$	$.61862\ X10^{2}$	$.40000\ X10^{9}$	$.29409\ X10^{3}$
$.10000\ X10^{9}$	$.38953\ X10^{1}$	$.26000\ X10^{9}$	$.79101\ X10^{2}$	$.42000\ X10^{9}$	$.34098\ X10^{3}$
$.12000\ X10^{9}$	$.70433\ X10^{1}$	$.28000\ X10^{9}$	$.99257\ X10^{2}$	$.44000\ X10^{9}$	$.39259\ X10^{3}$
$.14000\ X10^{9}$	$.11524\ X10^{2}$	$.30000\ X10^{9}$	$.12256\ X10^{3}$	$.46000\ X10^{9}$	$.44914\ X10^{3}$
$.16000\ X10^{9}$	$.17565\ X10^{2}$	$.32000\ X10^{9}$	$.14922\ X10^{3}$	$.48000\ X10^{9}$	$.51087\ X10^{3}$
$.18000\ X10^{9}$	$.25392\ X10^{2}$	$.34000\ X10^{9}$	$.17948\ X10^{3}$	$.50000\ X10^{9}$	$.57800\ X10^{3}$

T	q SUB T	T	q SUB T	T	q SUB T
LCG RHO=5.20					
$.40000 \times 10^{8}$	$.26574 \times 10^{0}$	$.20000 \times 10^{9}$	$.80867 \times 10^{2}$	$.36000 \times 10^{9}$	$.49794 \times 10^{3}$
$.60000 \times 10^{8}$	$.13778 \times 10^{1}$	$.22000 \times 10^{9}$	$.10898 \times 10^{3}$	$.38000 \times 10^{9}$	$.58734 \times 10^{3}$
$.80000 \times 10^{8}$	$.39434 \times 10^{1}$	$.24000 \times 10^{9}$	$.14290 \times 10^{3}$	$.40000 \times 10^{9}$	$.68679 \times 10^{3}$
$.10000 \times 10^{9}$	$.85295 \times 10^{1}$	$.26000 \times 10^{9}$	$.18314 \times 10^{3}$	$.42000 \times 10^{9}$	$.79683 \times 10^{3}$
$.12000 \times 10^{9}$	$.15688 \times 10^{2}$	$.28000 \times 10^{9}$	$.23024 \times 10^{3}$	$.44000 \times 10^{9}$	$.91798 \times 10^{3}$
$.14000 \times 10^{9}$	$.25962 \times 10^{2}$	$.30000 \times 10^{9}$	$.28475 \times 10^{3}$	$.46000 \times 10^{9}$	$.10508 \times 10^{4}$
$.16000 \times 10^{9}$	$.39891 \times 10^{2}$	$.32000 \times 10^{9}$	$.34717 \times 10^{3}$	$.48000 \times 10^{9}$	$.11958 \times 10^{4}$
$.18000 \times 10^{9}$	$.58015 \times 10^{2}$	$.34000 \times 10^{9}$	$.41806 \times 10^{3}$	$.50000 \times 10^{9}$	$.13534 \times 10^{4}$
LCG RHO=5.40					
$.40000 \times 10^{8}$	$.43556 \times 10^{0}$	$.20000 \times 10^{9}$	$.18037 \times 10^{3}$	$.36000 \times 10^{9}$	$.11346 \times 10^{4}$
$.60000 \times 10^{8}$	$.26082 \times 10^{1}$	$.22000 \times 10^{9}$	$.24426 \times 10^{3}$	$.38000 \times 10^{9}$	$.13399 \times 10^{4}$
$.80000 \times 10^{8}$	$.79688 \times 10^{1}$	$.24000 \times 10^{9}$	$.32150 \times 10^{3}$	$.40000 \times 10^{9}$	$.15685 \times 10^{4}$
$.10000 \times 10^{9}$	$.17871 \times 10^{2}$	$.26000 \times 10^{9}$	$.41333 \times 10^{3}$	$.42000 \times 10^{9}$	$.18214 \times 10^{4}$
$.12000 \times 10^{9}$	$.33617 \times 10^{2}$	$.28000 \times 10^{9}$	$.52100 \times 10^{3}$	$.44000 \times 10^{9}$	$.21001 \times 10^{4}$
$.14000 \times 10^{9}$	$.56484 \times 10^{2}$	$.30000 \times 10^{9}$	$.64573 \times 10^{3}$	$.46000 \times 10^{9}$	$.24056 \times 10^{4}$
$.16000 \times 10^{9}$	$.87732 \times 10^{2}$	$.32000 \times 10^{9}$	$.78876 \times 10^{3}$	$.48000 \times 10^{9}$	$.27393 \times 10^{4}$
$.18000 \times 10^{9}$	$.12861 \times 10^{3}$	$.34000 \times 10^{9}$	$.95131 \times 10^{3}$	$.50000 \times 10^{9}$	$.31023 \times 10^{4}$
LOG RHO=5.60					
$.40000 \times 10^{8}$	$.62929 \times 10^{0}$	$.20000 \times 10^{9}$	$.38829 \times 10^{3}$	$.36000 \times 10^{9}$	$.25132 \times 10^{4}$
$.60000 \times 10^{8}$	$.45341 \times 10^{1}$	$.22000 \times 10^{9}$	$.52917 \times 10^{3}$	$.38000 \times 10^{9}$	$.29727 \times 10^{4}$
$.80000 \times 10^{8}$	$.15074 \times 10^{2}$	$.24000 \times 10^{9}$	$.70009 \times 10^{3}$	$.40000 \times 10^{9}$	$.34846 \times 10^{4}$
$.10000 \times 10^{9}$	$.35435 \times 10^{2}$	$.26000 \times 10^{9}$	$.90385 \times 10^{3}$	$.42000 \times 10^{9}$	$.40517 \times 10^{4}$
$.12000 \times 10^{9}$	$.68650 \times 10^{2}$	$.28000 \times 10^{9}$	$.11433 \times 10^{3}$	$.44000 \times 10^{9}$	$.46767 \times 10^{4}$
$.14000 \times 10^{9}$	$.11766 \times 10^{3}$	$.30000 \times 10^{9}$	$.14211 \times 10^{3}$	$.46000 \times 10^{9}$	$.53624 \times 10^{4}$
$.16000 \times 10^{9}$	$.18537 \times 10^{3}$	$.32000 \times 10^{9}$	$.17402 \times 10^{3}$	$.48000 \times 10^{9}$	$.61116 \times 10^{4}$
$.18000 \times 10^{9}$	$.27463 \times 10^{3}$	$.34000 \times 10^{9}$	$.21033 \times 10^{3}$	$.50000 \times 10^{9}$	$.69271 \times 10^{4}$

T	q SUB T	T	q SUB T	T	q SUB T
LOG RHO=5.80					
$.40000 \times 10^{8}$	$.78021 \times 10^{0}$	$.20000 \times 10^{9}$	$.80068 \times 10^{0}$	$.36000 \times 10^{9}$	$.53805 \times 10^{4}$
$.60000 \times 10^{8}$	$.71030 \times 10^{1}$	$.22000 \times 10^{9}$	$.11003 \times 10^{3}$	$.38000 \times 10^{9}$	$.63779 \times 10^{4}$
$.80000 \times 10^{8}$	$.26307 \times 10^{2}$	$.24000 \times 10^{9}$	$.14655 \times 10^{4}$	$.40000 \times 10^{9}$	$.74902 \times 10^{4}$
$.10000 \times 10^{9}$	$.65691 \times 10^{2}$	$.26000 \times 10^{9}$	$.19025 \times 10^{4}$	$.42000 \times 10^{9}$	$.87235 \times 10^{4}$
$.12000 \times 10^{9}$	$.13219 \times 10^{3}$	$.28000 \times 10^{9}$	$.24174 \times 10^{4}$	$.44000 \times 10^{9}$	$.10084 \times 10^{5}$
$.14000 \times 10^{9}$	$.23248 \times 10^{3}$	$.30000 \times 10^{9}$	$.30165 \times 10^{4}$	$.46000 \times 10^{9}$	$.11578 \times 10^{5}$
$.16000 \times 10^{9}$	$.37307 \times 10^{3}$	$.32000 \times 10^{9}$	$.37060 \times 10^{4}$	$.48000 \times 10^{9}$	$.13211 \times 10^{5}$
$.18000 \times 10^{9}$	$.56035 \times 10^{3}$	$.34000 \times 10^{9}$	$.44919 \times 10^{4}$	$.50000 \times 10^{9}$	$.14990 \times 10^{5}$
LOG RHO=6.00					
$.40000 \times 10^{8}$	$.80711 \times 10^{0}$	$.20000 \times 10^{9}$	$.15694 \times 10^{4}$	$.36000 \times 10^{9}$	$.11072 \times 10^{5}$
$.60000 \times 10^{8}$	$.98284 \times 10^{1}$	$.22000 \times 10^{9}$	$.21800 \times 10^{4}$	$.38000 \times 10^{9}$	$.13161 \times 10^{5}$
$.80000 \times 10^{8}$	$.41696 \times 10^{2}$	$.24000 \times 10^{9}$	$.29289 \times 10^{4}$	$.40000 \times 10^{9}$	$.15494 \times 10^{5}$
$.10000 \times 10^{9}$	$.11240 \times 10^{3}$	$.26000 \times 10^{9}$	$.38294 \times 10^{4}$	$.42000 \times 10^{9}$	$.18085 \times 10^{5}$
$.12000 \times 10^{9}$	$.23737 \times 10^{3}$	$.28000 \times 10^{9}$	$.48949 \times 10^{4}$	$.44000 \times 10^{9}$	$.20946 \times 10^{5}$
$.14000 \times 10^{9}$	$.43140 \times 10^{3}$	$.30000 \times 10^{9}$	$.61386 \times 10^{4}$	$.46000 \times 10^{9}$	$.24090 \times 10^{5}$
$.16000 \times 10^{9}$	$.70879 \times 10^{3}$	$.32000 \times 10^{9}$	$.75739 \times 10^{4}$	$.48000 \times 10^{9}$	$.27532 \times 10^{5}$
$.18000 \times 10^{9}$	$.10835 \times 10^{4}$	$.34000 \times 10^{9}$	$.92139 \times 10^{4}$	$.50000 \times 10^{9}$	$.31283 \times 10^{5}$
LOG RHO=6.20					
$.40000 \times 10^{8}$	$.67726 \times 10^{0}$	$.20000 \times 10^{9}$	$.29029 \times 10^{4}$	$.36000 \times 10^{9}$	$.21789 \times 10^{5}$
$.60000 \times 10^{8}$	$.11768 \times 10^{2}$	$.22000 \times 10^{9}$	$.40878 \times 10^{4}$	$.38000 \times 10^{9}$	$.25994 \times 10^{5}$
$.80000 \times 10^{8}$	$.59073 \times 10^{2}$	$.24000 \times 10^{9}$	$.55529 \times 10^{4}$	$.40000 \times 10^{9}$	$.30699 \times 10^{5}$
$.10000 \times 10^{9}$	$.17521 \times 10^{3}$	$.26000 \times 10^{9}$	$.73263 \times 10^{4}$	$.42000 \times 10^{9}$	$.35933 \times 10^{5}$
$.12000 \times 10^{9}$	$.39311 \times 10^{3}$	$.28000 \times 10^{9}$	$.94360 \times 10^{4}$	$.44000 \times 10^{9}$	$.41722 \times 10^{5}$
$.14000 \times 10^{9}$	$.74457 \times 10^{3}$	$.30000 \times 10^{9}$	$.11910 \times 10^{5}$	$.46000 \times 10^{9}$	$.48094 \times 10^{5}$
$.16000 \times 10^{9}$	$.12603 \times 10^{4}$	$.32000 \times 10^{9}$	$.14775 \times 10^{5}$	$.48000 \times 10^{9}$	$.55075 \times 10^{5}$
$.18000 \times 10^{9}$	$.19700 \times 10^{4}$	$.34000 \times 10^{9}$	$.18059 \times 10^{5}$	$.50000 \times 10^{9}$	$.62693 \times 10^{5}$

T	q SUB T	T	q SUB T	T	q SUB T
LOG RHO=6.40					
.40000 X10^{8}	.44834 X10^{0}	.20000 X10^{9}	.50338 X10^{4}	.36000 X10^{9}	.40850 X10^{5}
.60000 X10^{8}	.11950 X10^{2}	.22000 X10^{9}	.72106 X10^{4}	.38000 X10^{9}	.48954 X10^{5}
.80000 X10^{8}	.73637 X10^{2}	.24000 X10^{9}	.99315 X10^{4}	.40000 X10^{9}	.58048 X10^{5}
.10000 X10^{9}	.24563 X10^{3}	.26000 X10^{9}	.13254 X10^{5}	.42000 X10^{9}	.68187 X10^{5}
.12000 X10^{9}	.59358 X10^{3}	.28000 X10^{9}	.17234 X10^{5}	.44000 X10^{9}	.79425 X10^{5}
.14000 X10^{9}	.11843 X10^{4}	.30000 X10^{9}	.21928 X10^{5}	.46000 X10^{9}	.91817 X10^{5}
.16000 X10^{9}	.20801 X10^{4}	.32000 X10^{9}	.27392 X10^{5}	.48000 X10^{9}	.10901 X10^{6}
.18000 X10^{9}	.33433 X10^{4}	.34000 X10^{9}	.33681 X10^{5}	.50000 X10^{9}	.12028 X10^{6}
LOG RHO=6.60					
.40000 X10^{8}	.22771 X10^{0}	.20000 X10^{9}	.81367 X10^{4}	.36000 X10^{9}	.72753 X10^{5}
.60000 X10^{8}	.10087 X10^{2}	.22000 X10^{9}	.11905 X10^{5}	.38000 X10^{9}	.87684 X10^{5}
.80000 X10^{8}	.79523 X10^{2}	.24000 X10^{9}	.16681 X10^{5}	.40000 X10^{9}	.10450 X10^{6}
.10000 X10^{9}	.30589 X10^{3}	.26000 X10^{9}	.22579 X10^{5}	.42000 X10^{9}	.12330 X10^{6}
.12000 X10^{9}	.81050 X10^{3}	.28000 X10^{9}	.29709 X10^{5}	.44000 X10^{9}	.14420 X10^{6}
.14000 X10^{9}	.17209 X10^{4}	.30000 X10^{9}	.38183 X10^{5}	.46000 X10^{9}	.16729 X10^{6}
.16000 X10^{9}	.31636 X10^{4}	.32000 X10^{9}	.48110 X10^{5}	.48000 X10^{9}	.19270 X10^{6}
.18000 X10^{9}	.52622 X10^{4}	.34000 X10^{9}	.59597 X10^{5}	.50000 X10^{9}	.22051 X10^{6}
LOG RHO=6.80					
.40000 X10^{8}	.86165 X10^{-1}	.20000 X10^{9}	.12199 X10^{5}	.36000 X10^{9}	.12289 X10^{6}
.60000 X10^{8}	.69370 X10^{1}	.22000 X10^{9}	.18317 X10^{5}	.38000 X10^{9}	.14916 X10^{6}
.80000 X10^{8}	.73257 X10^{2}	.24000 X10^{9}	.26213 X10^{5}	.40000 X10^{9}	.17887 X10^{6}
.10000 X10^{9}	.33420 X10^{3}	.26000 X10^{9}	.36104 X10^{5}	.42000 X10^{9}	.21223 X10^{6}
.12000 X10^{9}	.98873 X10^{3}	.28000 X10^{9}	.48205 X10^{5}	.44000 X10^{9}	.24944 X10^{6}
.14000 X10^{9}	.22657 X10^{4}	.30000 X10^{9}	.62727 X10^{5}	.46000 X10^{9}	.29069 X10^{6}
.16000 X10^{9}	.44028 X10^{4}	.32000 X10^{9}	.79878 X10^{5}	.48000 X10^{9}	.33618 X10^{6}
.18000 X10^{9}	.76364 X10^{4}	.34000 X10^{9}	.99864 X10^{5}	.50000 X10^{9}	.38611 X10^{6}

T	q SUB T	T	q SUB T	T	q SUB T
LOG RHO=7.00					
.40000 X10^{8}	.23503 X10^{-1}	.20000 X10^{9}	.16879 X10^{5}	.36000 X10^{9}	.19669 X10^{6}
.60000 X10^{8}	.38023 X10^{1}	.22000 X10^{9}	.26157 X10^{5}	.38000 X10^{9}	.24081 X10^{6}
.80000 X10^{8}	.56626 X10^{2}	.24000 X10^{9}	.38406 X10^{5}	.40000 X10^{9}	.29100 X10^{6}
.10000 X10^{9}	.31620 X10^{3}	.26000 X10^{9}	.54035 X10^{5}	.42000 X10^{9}	.34763 X10^{6}
.12000 X10^{9}	.10670 X10^{4}	.28000 X10^{9}	.73445 X10^{5}	.44000 X10^{9}	.41108 X10^{6}
.14000 X10^{9}	.26795 X10^{4}	.30000 X10^{9}	.97028 X10^{5}	.46000 X10^{9}	.48169 X10^{6}
.16000 X10^{9}	.55669 X10^{4}	.32000 X10^{9}	.12518 X10^{6}	.48000 X10^{9}	.55985 X10^{6}
.18000 X10^{9}	.10157 X10^{5}	.34000 X10^{9}	.15827 X10^{6}	.50000 X10^{9}	.64591 X10^{6}
LOG RHO=7.20					
.40000 X10^{8}	.44441 X10^{-2}	.20000 X10^{9}	.21436 X10^{5}	.36000 X10^{9}	.29799 X10^{6}
.60000 X10^{8}	.16197 X10^{1}	.22000 X10^{9}	.34507 X10^{5}	.38000 X10^{9}	.36871 X10^{6}
.80000 X10^{8}	.36048 X10^{2}	.24000 X10^{9}	.52266 X10^{5}	.40000 X10^{9}	.44975 X10^{6}
.10000 X10^{9}	.25530 X10^{3}	.26000 X10^{9}	.75453 X10^{5}	.42000 X10^{9}	.54178 X10^{6}
.12000 X10^{9}	.10066 X10^{4}	.28000 X10^{9}	.10480 X10^{6}	.44000 X10^{9}	.64546 X10^{6}
.14000 X10^{9}	.28188 X10^{4}	.30000 X10^{9}	.14102 X10^{6}	.46000 X10^{9}	.76145 X10^{6}
.16000 X10^{9}	.63435 X10^{4}	.32000 X10^{9}	.18483 X10^{6}	.48000 X10^{9}	.89040 X10^{6}
.18000 X10^{9}	.12300 X10^{5}	.34000 X10^{9}	.23693 X10^{6}	.50000 X10^{9}	.10330 X10^{7}
LOG RHO=7.40					
.40000 X10^{8}	.55558 X10^{-3}	.20000 X10^{9}	.24808 X10^{5}	.36000 X10^{9}	.42668 X10^{6}
.60000 X10^{8}	.52006 X10^{0}	.22000 X10^{9}	.41804 X10^{5}	.38000 X10^{9}	.53475 X10^{6}
.80000 X10^{8}	.18484 X10^{2}	.24000 X10^{9}	.65728 X10^{5}	.40000 X10^{9}	.65975 X10^{6}
.10000 X10^{9}	.17286 X10^{3}	.26000 X10^{9}	.97880 X10^{5}	.42000 X10^{9}	.80285 X10^{6}
.12000 X10^{9}	.81841 X10^{3}	.28000 X10^{9}	.13955 X10^{6}	.44000 X10^{9}	.96524 X10^{6}
.14000 X10^{9}	.26067 X10^{3}	.30000 X10^{9}	.19201 X10^{6}	.46000 X10^{9}	.11481 X10^{7}
.16000 X10^{9}	.64504 X10^{3}	.32000 X10^{9}	.25653 X10^{6}	.48000 X10^{9}	.13525 X10^{7}
.18000 X10^{9}	.13447 X10^{5}	.34000 X10^{9}	.33435 X10^{6}	.50000 X10^{9}	.15796 X10^{7}

T	q SUB T	T	q SUB T	T	q SUB T
LOG RHO=7.60					
$.40000 \times 10^{8}$	$.43345 \times 10^{-4}$	$.20000 \times 10^{9}$	$.25926 \times 10^{5}$	$.36000 \times 10^{9}$	$.57571 \times 10^{6}$
$.60000 \times 10^{8}$	$.12127 \times 10^{0}$	$.22000 \times 10^{9}$	$.46135 \times 10^{5}$	$.38000 \times 10^{9}$	$.73281 \times 10^{6}$
$.80000 \times 10^{8}$	$.74286 \times 10^{1}$	$.24000 \times 10^{9}$	$.75856 \times 10^{5}$	$.40000 \times 10^{9}$	$.91665 \times 10^{6}$
$.10000 \times 10^{9}$	$.96082 \times 10^{2}$	$.26000 \times 10^{9}$	$.11725 \times 10^{6}$	$.42000 \times 10^{9}$	$.11293 \times 10^{7}$
$.12000 \times 10^{9}$	$.56359 \times 10^{3}$	$.28000 \times 10^{9}$	$.17250 \times 10^{6}$	$.44000 \times 10^{9}$	$.13728 \times 10^{7}$
$.14000 \times 10^{9}$	$.20884 \times 10^{4}$	$.30000 \times 10^{9}$	$.24380 \times 10^{6}$	$.46000 \times 10^{9}$	$.16491 \times 10^{7}$
$.16000 \times 10^{9}$	$.57807 \times 10^{4}$	$.32000 \times 10^{9}$	$.33333 \times 10^{6}$	$.48000 \times 10^{9}$	$.19603 \times 10^{7}$
$.18000 \times 10^{9}$	$.13133 \times 10^{5}$	$.34000 \times 10^{9}$	$.44326 \times 10^{6}$	$.50000 \times 10^{9}$	$.23084 \times 10^{7}$
LOG RHO=7.80					
$.40000 \times 10^{8}$	$.19678 \times 10^{-5}$	$.20000 \times 10^{9}$	$.24173 \times 10^{5}$	$.36000 \times 10^{9}$	$.72854 \times 10^{6}$
$.60000 \times 10^{8}$	$.19624 \times 10^{-1}$	$.22000 \times 10^{9}$	$.45895 \times 10^{5}$	$.38000 \times 10^{9}$	$.94488 \times 10^{6}$
$.80000 \times 10^{8}$	$.22635 \times 10^{1}$	$.24000 \times 10^{9}$	$.79592 \times 10^{5}$	$.40000 \times 10^{9}$	$.12017 \times 10^{7}$
$.10000 \times 10^{9}$	$.42712 \times 10^{2}$	$.26000 \times 10^{9}$	$.12862 \times 10^{6}$	$.42000 \times 10^{9}$	$.15027 \times 10^{7}$
$.12000 \times 10^{9}$	$.32175 \times 10^{3}$	$.28000 \times 10^{9}$	$.19648 \times 10^{6}$	$.44000 \times 10^{9}$	$.18513 \times 10^{7}$
$.14000 \times 10^{9}$	$.14235 \times 10^{4}$	$.30000 \times 10^{9}$	$.28677 \times 10^{6}$	$.46000 \times 10^{9}$	$.22509 \times 10^{7}$
$.16000 \times 10^{9}$	$.44951 \times 10^{4}$	$.32000 \times 10^{9}$	$.40311 \times 10^{6}$	$.48000 \times 10^{9}$	$.27051 \times 10^{7}$
$.18000 \times 10^{9}$	$.11302 \times 10^{5}$	$.34000 \times 10^{9}$	$.54916 \times 10^{6}$	$.50000 \times 10^{9}$	$.32172 \times 10^{7}$
LOG RHO=8.00					
$.40000 \times 10^{8}$	$.47800 \times 10^{-7}$	$.20000 \times 10^{9}$	$.19809 \times 10^{5}$	$.36000 \times 10^{9}$	$.85874 \times 10^{6}$
$.60000 \times 10^{8}$	$.20858 \times 10^{-2}$	$.22000 \times 10^{9}$	$.40604 \times 10^{5}$	$.38000 \times 10^{9}$	$.11391 \times 10^{7}$
$.80000 \times 10^{8}$	$.50212 \times 10^{0}$	$.24000 \times 10^{9}$	$.75007 \times 10^{5}$	$.40000 \times 10^{9}$	$.14781 \times 10^{7}$
$.10000 \times 10^{9}$	$.14708 \times 10^{2}$	$.26000 \times 10^{9}$	$.12779 \times 10^{6}$	$.42000 \times 10^{9}$	$.18817 \times 10^{7}$
$.12000 \times 10^{9}$	$.14833 \times 10^{3}$	$.28000 \times 10^{9}$	$.20417 \times 10^{6}$	$.44000 \times 10^{9}$	$.23558 \times 10^{7}$
$.14000 \times 10^{9}$	$.80734 \times 10^{3}$	$.30000 \times 10^{9}$	$.30965 \times 10^{6}$	$.46000 \times 10^{9}$	$.29063 \times 10^{7}$
$.16000 \times 10^{9}$	$.29750 \times 10^{4}$	$.32000 \times 10^{9}$	$.44998 \times 10^{6}$	$.48000 \times 10^{9}$	$.35392 \times 10^{7}$
$.18000 \times 10^{9}$	$.84264 \times 10^{4}$	$.34000 \times 10^{9}$	$.63103 \times 10^{6}$	$.50000 \times 10^{9}$	$.42601 \times 10^{7}$

where now all dimensioned quantities are in c.g.s. units. In c.g.s. units

$$p_F{}^3 = 3\pi^2 N_{\text{AVO}} \frac{1}{\mu_e} \rho = \tfrac{3}{2}\pi^2 N_{\text{AVO}}\rho \,.$$

Thus in c.g.s. units

$$\hbar^2\omega_o{}^2 = 2\pi(\hbar c)^2\left[1+\left(\frac{\hbar}{m_e c}\right)^2 (\tfrac{3}{2}\pi^2 N_{\text{AVO}}\rho)^{2/3}\right]^{-1/2} N_{\text{AVO}} \frac{e^2}{m_e c^2}\rho$$

or

$$\hbar\omega_o = 3.265 \times 10^{-11} (1 + 6.413 \times 10^{-5} \rho^{2/3})^{-1/4} \rho^{1/2} \,, \tag{25}$$

where now all dimensioned quantities are in c.g.s. units.

In c.g.s. units

$$\beta = \frac{7.244 \times 10^8}{T_7} \,, \tag{26}$$

where T_7 is the temperature in units of 10^7 ° K.

Using equations (25) and (26) in equation (24) we can compute Q_t for given values of ρ and T. Of course

$$q_t = \frac{Q_t}{\rho} \,.$$

Table 1 gives q_t as a function of ρ and T.

APPENDIX

CALCULATION OF $x^3\mathfrak{F}(x)$ FOR SMALL x

Put $\eta = x \cosh \xi$ in equation (19); then

$$x^3\mathfrak{F}(x) = \int_x^\infty \frac{1}{e^\eta - 1}\left(1 - \frac{x^2}{\eta^2}\right)^{1/2} \eta^2 d\eta \,. \tag{27}$$

By splitting the integral

$$x^3\mathfrak{F}(x) = \int_0^\infty \frac{1}{e^\eta - 1}\eta^2 d\eta - \int_0^x \frac{\eta}{e^\eta - 1}\eta d\eta - \tfrac{1}{2}x^2\int_x^\infty \frac{1}{e^\eta - 1} d\eta$$

$$+ \int_x^\infty \frac{1}{e^\eta - 1}\left[\left(1 - \frac{x^2}{\eta^2}\right)^{1/2} - 1 + \tfrac{1}{2}\frac{x^2}{\eta^2}\right]\eta^2 d\eta \,,$$

or

$$x^3\mathfrak{F}(x) = 2\zeta(3) - a_1(x) + \tfrac{1}{2}x^2 \ln(1 - e^{-x}) + a_2(x) \,. \tag{28}$$

Now

$$\begin{aligned} a_1(x) &= \int_0^x \left[1 - \tfrac{1}{2}\eta + \tfrac{1}{12}\eta^2 - \frac{B_2}{4!}\eta^4 + O(\eta^6)\right]\eta d\eta \\ &= \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{48}x^4 - O(x^6) \end{aligned} \tag{29}$$

for $x < 2\pi$.

An elementary calculation gives

$$\tfrac{1}{2}x^2 \ln(1 - e^{-x}) = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^3 + \tfrac{1}{48}x^4 - O(x^6) \,. \tag{30}$$

The calculation of $a_2(x)$ is quite lengthy; we begin by splitting the integral again:

$$a_2(x) = -\tfrac{1}{8}x^4\int_A^\infty \frac{1}{e^\eta - 1}\frac{d\eta}{\eta^2} + O(x^6)$$

$$+\int_x^A \frac{1}{e^\eta - 1}\left[\left(1-\frac{x^2}{\eta^2}\right)^{1/2} - 1 + \tfrac{1}{2}\frac{x^2}{\eta^2}\right]\eta^2 d\eta \tag{31}$$

$$= -\tfrac{1}{8}\left(\int_A^\infty \frac{1}{e^\eta - 1}\frac{d\eta}{\eta^2}\right)x^4 + a_3(x) + O(x^6).$$

Put $\eta\xi = x$; then

$$(e^\eta - 1)^{-1}\eta^2 d\eta = x^2\frac{x}{\xi}(e^{x/\xi} - 1)^{-1}\frac{d\xi}{\xi^3}$$

and

$$a_3(x) = x^2\int_{x/A}^1 \frac{x/\xi}{e^{x/\xi} - 1}[(1-\xi^2)^{1/2} - 1 + \tfrac{1}{2}\xi^2]\frac{d\xi}{\xi^3}$$

$$= x^2\int_{x/A}^1\left[1 - \tfrac{1}{2}\frac{x}{\xi} + \tfrac{1}{12}\frac{x^2}{\xi^2} - \frac{B_2}{4!}\frac{x^4}{\xi^4} + \ldots + (-)^{n+1}\frac{B_n}{(2n)!}\frac{x^{2n}}{\xi^{2n}}\right.$$

$$\left. + \ldots\right]\cdot[(1-\xi^2)^{1/2} - 1 + \tfrac{1}{2}\xi^2]\frac{d\xi}{\xi^3},$$

where we require

$$\frac{x}{\xi} < 2\pi \quad \text{or} \quad A < 2\pi .$$

Integrating term by term, we get

$$a_3(x) = x^2 b_o(x) - \tfrac{1}{2}x^3 b_1(x) + \tfrac{1}{12}x^4 b_2(x) - \frac{B_2}{4!}x^6 b_4(x) + \ldots$$

$$+ (-)^{n+1}\frac{Bn}{(2n)!}x^{2n+2}b_{2n}(x) + \ldots , \tag{32}$$

where

$$b_m(x) = \int_{x/A}^1 [(1-\xi^2)^{1/2} - 1 + \tfrac{1}{2}\xi^2]\frac{d\xi}{\xi^{3+m}}.$$

These integrals are elementary; in particular

$$b_o(x) = \tfrac{1}{4} - \tfrac{1}{2}\ln 2 + \tfrac{1}{16}(x/A)^2 + O(x^4), \tag{33}$$

$$b_1(x) = -\tfrac{1}{6} + \tfrac{1}{8}\frac{x}{A} + O(x^3), \tag{34}$$

$$b_2(x) = \tfrac{1}{8}\ln\frac{x}{A} + \tfrac{1}{32} - \tfrac{1}{8}\ln 2 + O(x^2), \tag{35}$$

$$b_4(x) = -\tfrac{1}{16}(x/A)^{-2} + O(\ln x); \tag{36}$$

and for $n > 2$

$$b_{2n}(x) = -\frac{1}{8(2n-2)}\left(\frac{x}{A}\right)^{2-2n} + O(x^{4-2n}). \tag{37}$$

Substituting equations (33)–(37) into equation (32), we get

$$a_3(x)=\left(\tfrac{1}{4}-\tfrac{1}{2}\ln 2\right)x^2+\tfrac{1}{12}x^3+\tfrac{1}{96}x^4\ln x+\Big[\tfrac{1}{16}\frac{1}{A^2}-\tfrac{1}{16}\frac{1}{A}-\tfrac{1}{96}\ln A+\tfrac{1}{12}\left(\tfrac{1}{32}-\tfrac{1}{8}\ln 2\right)+\frac{B_2}{4!\,16}A^2+\ldots+(-)^n\frac{B_n}{(2n)!\,8(2n-2)}A^{2n-2}+\ldots\Big]x^4+O(x^6\ln x). \tag{38}$$

Substituting equations (29), (30), and (31) into equation (27), and using equation (38), we get

$$x^3\mathfrak{F}(x)=2\zeta(3)+\tfrac{1}{2}x^2\ln x-\tfrac{1}{4}(2\ln 2+1)x^2+\tfrac{1}{96}x^4\ln x-\tfrac{1}{8}q(A)x^4+O(x^6\ln x)\,, \tag{39}$$

where

$$q(A)=\int_A^\infty\frac{1}{e^\eta-1}\frac{d\eta}{\eta^2}-\tfrac{1}{2}\frac{1}{A^2}+\tfrac{1}{2}\frac{1}{A}+\tfrac{1}{12}\ln A+\tfrac{1}{12}\left(\ln 2-\tfrac{1}{4}\right)-\frac{B_2}{4!\,2}A^2+\ldots+(-)^{n+1}\frac{B_n}{(2n)!\,(2n-2)}A^{2n-2}+\ldots\,; \tag{40}$$

$q(A)$ is, of course, a constant; it is, however, a bit difficult to evaluate. We will use the relations

$$\frac{1}{t}\frac{1}{e^t-1}+\tfrac{1}{2}\frac{1}{t}=\frac{1}{2t}\coth\frac{t}{2}=\sum_{n=1}^{\infty}\frac{2}{4n^2\pi^2+t^2}+\frac{1}{t^2}. \tag{41}$$

Using the first of equations (41) we find on integrating by parts

$$-\int_A^\infty \ln t\left[\frac{d}{dt}\left(\frac{1}{2t}\coth\frac{t}{2}-\frac{1}{t^2}\right)\right]dt=\int_A^\infty\frac{1}{e^t-1}\frac{dt}{t^2}-\tfrac{1}{2}\frac{1}{A^2}+1/(2A)+(1/12)\ln A+O(A^2\ln A) \tag{42}$$

$$\therefore\ q(A)=q(0)=-\int_0^\infty \ln t\left[\frac{d}{dt}\left(\frac{1}{2t}\coth\frac{t}{2}-\frac{1}{t^2}\right)\right]dt+\frac{\ln 2-\frac{1}{4}}{12}.$$

Substituting the second of equations (41) into equation (42), and using the absolute and uniform convergence of the series (Knopp 1963) we obtain

$$q(0)-\tfrac{1}{12}\left(\ln 2-\tfrac{1}{4}\right)=4\int_0^\infty t\ln t\left(\sum_{n=1}^{\infty}\frac{1}{(4n^2\pi^2+t^2)^2}\right)dt=4\sum_{n=1}^{\infty}B(2n\pi), \tag{43}$$

where

$$B(\xi)\equiv\int_0^\infty\frac{x\ln x}{(x^2+\xi^2)^2}\,dx\,.$$

Putting $x^2=y$, we get (de Haan 1963)

$$B(\xi)=\tfrac{1}{4}\int^\infty\frac{\ln y}{(\xi^2+\nu)^2}\,d\nu=\tfrac{1}{4}\frac{1}{\xi^2}\ln(\xi^2)=\frac{1}{2\xi^2}\ln\xi\,.$$

By using this in equation (43),

$$
\begin{aligned}
q(0)-\tfrac{1}{12}(\ln 2-\tfrac{1}{4}) &= 2\sum_{n=1}^{\infty}\frac{1}{(2n\pi)^2}\ln(2n\pi) \\
&= \frac{1}{2\pi^2}\left(\ln 2\pi\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{\ln n}{n^2}\right).
\end{aligned} \tag{44}
$$

But

$$
\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)=\zeta(2)=\frac{\pi^2}{6},
$$

and

$$
\sum_{n=1}^{\infty}\frac{\ln n}{n^2}=\left[-\frac{d}{dz}\sum_{n=1}^{\infty}e^{-z\ln n}\right]_{z=2}=-\zeta'(2).
$$

So

$$
q(0)=\frac{1}{12}\left(\ln 2-\tfrac{1}{4}+\ln 2\pi-\frac{\zeta'(2)}{\zeta(2)}\right). \tag{45}
$$

Using this in equation (39)

$$
\begin{aligned}
x^3\mathfrak{F}(x) &= 2\zeta(3)+\tfrac{1}{2}x^2\ln x-\tfrac{1}{4}(2\ln 2+1)x^2+\tfrac{1}{96}x^4\ln x \\
&\quad -\tfrac{1}{96}\left(\ln 2-\tfrac{1}{4}+\ln 2\pi-\frac{\zeta'(2)}{\zeta(2)}\right)x^4+O(x^6\ln x)\,.
\end{aligned} \tag{46}
$$

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